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The discontinuous oblique derivative problem for quasilinear elliptic complex equations of second order in multiply connected domains

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Abstract

In this article, we discuss the discontinuous oblique derivative boundary value problem for quasilinear uniformly elliptic complex equation of second order

$$w_{z\bar{z}}=F(z,w,w_z,\overline{w}_z,w_{zz},\overline{w}_{zz}) \text{ in } D, \tag{0.1}$$

with the discontinuous boundary conditions

$$\begin{aligned}&\mathrm{Re}[\overline{\lambda_j(t)}w_t+\sigma_1(t)w(t)+\tau_1(t)]=0,\\&\mathrm{Re}[\overline{\lambda_2(t)}\overline{w}_t+\sigma_2(t)w(t)+\tau_2(t)]=0,\end{aligned}\quad t\in\Gamma^*, \tag{0.2}$$

in a multiply connected domain, the above boundary value problem will be called Problem P. If the complex equation (0.1) satisfies the conditions similar to Condition C of (1.1), and the boundary condition (0.2) satisfies the conditions similar to (1.6) below, then we can obtain some solvability results of Problem P. The discontinuous boundary value problem possesses many applications in mechanics and physics etc.

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1 Formulation of discontinuous oblique derivative problem for complex equations

In this article, we consider the quasilinear uniformly elliptic complex equation of second order

$$\begin{cases} w_{z\bar{z}}=F(z,w,w_z,\overline{w}_z,w_{zz},\overline{w}_{zz}), F=Q_1w_{zz}+Q_2\overline{w}_{zz}+A_1w_z+A_2\overline{w}_z+A_3w+A_4, \\ Q_j=Q_j(z,w,w_z,\overline{w}_z),\ j=1,2,\ A_j=A_j(z,w,w_z,\overline{w}_z),\ j=1,...,4, \end{cases} \tag{1.1}$$

in an $N+1-$connected domain D. Denote by $\Gamma=\cup_{j=0}^{N}\Gamma_j$ the boundary contours of the domain D and let $\Gamma\in C_\mu^2\,(0<\mu<1)$. Without loss of generality, we assume that D is a circular domain in $|z|<1$, bounded by the $(N+1)$-circles $\Gamma_j:|z-z_j|=r_j,\ j=0,1,...,N$ and $\Gamma_0=\Gamma_{N+1}:|z|=1,\ z=0\in D$. In this article, the notations are as the same in References [3-12]. Suppose that (1.1) satisfies the following conditions.

Condition C 1) $Q_j(z,w,w_z,\overline{w}_z)(j=1,2), A_j(z,w,w_z,\overline{w}_z)(j=1,...,4)$ are measurable in $z\in D$ for all continuously differentiable functions $w(z)$ in D, and satisfy

$$L_p[A_j(z,w,w_z,\overline{w}_z),\overline{D}]\le k_{j-1},\ j=1,...,4, \tag{1.2}$$

in which $p,p_0\,(2<p_0\le p),k_j(j=0,1,2,3)$ are non-negative constants.

2) The above functions are continuous in $w,w_z,\overline{w}_z\in\mathbb{C}$ for almost every point $z\in D$, and $Q_j=0\,(j=1,2),\ A_j=0\,(j=1,...,4)$ for $z\in\mathbb{C}\backslash D$.

3) The complex equation (1.1) satisfies the following uniform ellipticity condition, namely for any functions $w(z)\in C^1(D)$, the inequality

$$|Q_j|\le q_j,\ j=1,2,\ q_1+q_2<1, \tag{1.3}$$

holds for almost every point $z\in D$, where $q_j\,(j=1,2)$ are all non-negative constants.

The discontinuous oblique derivative boundary value problem for the complex equation (1.1) may be formulated as follows.

Problem P Find a continuously differentiable solution $w(z)$ of complex equation (1.1) in $D^*=\overline{D}\backslash Z$ satisfying the boundary conditions

$$\begin{matrix} \mathrm{Re}[\overline{\lambda_1(z)}w_z+\sigma_1(z)w(z)+\tau_1(z)]=0, \\ \mathrm{Re}[\overline{\lambda_2(z)}\overline{w}_z+\sigma_2(z)w(z)+\tau_2(z)]=0, \end{matrix}\quad z\in\Gamma^*=\Gamma\backslash Z, \tag{1.4}$$

where $\lambda_l(z) = a_l(z) + \mathrm{i}b_l(z)$, $|\lambda_l(z)| = 1$ on $\varGamma\,(l = 1, 2)$, and $Z = \{t_1, t_2, ..., t_m\}$ are the first kind of discontinuous points of $\lambda_l(z)$ on $\varGamma$, $\hat{\varGamma}_j$ is an arc from the point t_{j-1} to t_j on $\hat{\varGamma}$, $\hat{\varGamma}_j\,(j = 1, 2, ..., m)$ does not include the end points; we can assume that $t_j \in \varGamma_0\,(j = 1, ..., m_0), t_j \in \varGamma_1\,(j = m_0 + 1, ..., m_1), ..., t_j \in \varGamma_N\,(j = m_{N-1} + 1..., m)$ are all discontinuous points of $\lambda(z)$ on $\varGamma$. Denote by $\lambda_l(t_j - 0)$ and $\lambda_l(t_j + 0)$ the left limit and right limit of $\lambda_l(z)$ as $z \to t_j\,(j = 1, 2, ..., m, l = 1, 2)$ on $\varGamma$, and

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\phi_{lj}} = \frac{\lambda_l(t_j - 0)}{\lambda_l(t_j + 0)},\ \gamma_{lj} = \frac{1}{\pi \mathrm{i}} \ln\left[\frac{\lambda_l(t_j - 0)}{\lambda_l(t_j + 0)}\right] = \frac{\phi_{lj}}{\pi} - K_{lj}, \\ &K_{lj} = \left[\frac{\phi_{lj}}{\pi}\right] + J_{lj},\ J_{lj} = 0 \text{ or } 1,\ j = 1, ..., m, l = 1, 2, \end{aligned} \tag{1.5}$$

in which $0 \le \gamma_{lj} < 1$ when $J_{lj} = 0$, and $-1 < \gamma_{lj} < 0$ when $J_{lj} = 1$, $j = 1, ..., m, l = 1, 2$. Set

$$K_l = \frac{1}{2\pi}\Delta_\Gamma \arg \lambda_l(z) = \sum_{j=1}^{m} \frac{K_{lj}}{2},\ l = 1, 2,$$

and $K = (K_1, K_2)$ is called the index of Problem P. Moreover, $\lambda_l(z), \sigma_l(z), \tau_l(z)$ $(l = 1, 2)$ satisfy the conditions

$$\begin{aligned} &C_\alpha[\lambda_l(z), \hat{\Gamma}_j] \le k_0,\ C_\alpha[|z - t_j|^{\beta_{lj}}|z - t_{j-1}|^{\beta_{lj-1}}\sigma_l(z), \hat{\Gamma}_j] \le \varepsilon k_0, \\ &C_\alpha[|z - t_j|^{\beta_{lj}}|z - t_{j-1}|^{\beta_{lj-1}}\tau_l(z), \hat{\Gamma}_j] \le k_4,\ l = 1, 2,\ j = 1, ..., m, \end{aligned} \tag{1.6}$$

in which $\alpha\,(1/2 < \alpha < 1)$ is a constant, where $\eta_j = \max(\eta_{ij}, \eta_{2j})$, $\gamma_{lj}\,(j = 1, ..., m, l = 1, 2)$ are real constants as stated in (1.5), $\tau\,(\le \min(\alpha, 1 - 2/p_0), \delta\,(\le \min[\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \tau])$ are sufficiently small positive constants, such that $\beta_{lj} + \gamma_{lj} < 1$, $j = 1, ..., m, l = 1, 2$. We require that the solution $w(z)$ possesses the property

$$\begin{aligned} &R(z)w_z, R(z)w_{\bar{z}} = C_\delta(\overline{D}),\ R(z) = \textstyle\prod_{j=1}^m |z - t_j|^{\eta_j/\tau^2},\ \eta_j = \max(\eta_{1j}, \eta_{2j}), \\ &\eta_{lj} = \begin{cases} \beta_{lj} + \tau, \text{ for } \gamma_{lj} \ge 0, \text{ and } \gamma_{lj} < 0, \beta_{lj} \ge |\gamma_{lj}|, \\ |\gamma_{lj}| + \tau, \text{ for } \gamma_{lj} < 0, \beta_{lj} < |\gamma_{lj}|, j = 1, ..., m, l = 1, 2, \end{cases} \end{aligned} \tag{1.7}$$

in the neighborhood$(\subset D)$ of $z_j\,(j = 1, ..., m)$.

In general, Problem P may not be solvable. Hence we consider its modified well posed-ness shown below.

Problem Q Find a system of continuous solutions $(U(z),V(z),w(z))$ $(w(z)\in C(\overline{D}),R(z)U(z),R(z)V(z)\in W^1_{p_0}(\overline{D}),\ 2<p_0<p)$ of the first order system of complex equations

$$\begin{aligned}&U_{\overline{z}}=F(z,w,U,V,U_z,V_z),\ F=Q_1U_z+Q_2\overline{V}_{\overline{z}}\\&+A_1U+A_2\overline{V}+A_3w+A_4\overline{w}+A_5, V_{\overline{z}}=\overline{U}_z=\overline{\rho(z)},\end{aligned}\tag{1.8}$$

satisfying the boundary conditions

$$\begin{aligned}&\mathrm{Re}[\overline{\lambda_1(z)}U(z)+\sigma_1(z)w(z)]=\tau_1(z)+h_1(z)\overline{\lambda_1(z)}X_1(z),\ z\in\Gamma^*,\\&\mathrm{Re}[\overline{\lambda_2(z)}V(z)+\sigma_1(z)w(z)]=\tau_2(z)+h_2(z)\overline{\lambda_2(z)}X_2(z),\ z\in\Gamma^*,\\&\mathrm{Im}[\overline{\lambda_1(a_j)}U(a_j)+\sigma_1(a_j)w(a_j)]=b_{lj},\ j\in J,\\&\mathrm{Im}[\overline{\lambda_2(a_j)}V(a_j)+\sigma_2(a_j)w(a_j)]=b_{2j},\ j\in J,\\&j\in J_l=\begin{cases}1,...,2K_l-N+1,\ K_l\geq N,\\N-K_l'+1,...,N-K_l'+[K_l]+1,0\leq K_l<N,\end{cases}\quad l=1,2,\end{aligned}\tag{1.9}$$

in which $[K_l]$ is denoted the integer part of the number K_l, $K_l'=[K_l+1/2]\,(l=1,2)$, $X_l(z)\,(l=1,2)$ are as stated in (1.13) below; there is in no harm assuming that the partial indexes K_l of $\lambda_l(z)$ on $\Gamma_k\,(k=0,1,...,N_0\,(\leq N))$ are integers, and the partial indexes K_l of $\lambda(z)$ on $\Gamma_k\,(k=N_0+1,...,N)$ are no integers, (if K_{N+1} of $\lambda_l(z)$ on Γ_{N+1} is no an integer, then we can similarly discuss;) $a_j\in\Gamma_k\,(k=1,...,N_0)$, $a_j\in\Gamma_0\,(j=N_0+1,...,2K_l-N+1,$ if $K_l\geq N,\ l=1,2)$ are distinct points; and when $N-K_l'+1\leq N_0$, $a_{j+N-K_l'}\in\Gamma_k\,(k=1,...,N_0-N+K_l')$, $a_j\in\Gamma_0\,(j=N_0-N+K_l'+1,...,[K_l]+1,$ if $0\leq K_l<N)$, otherwise $a_{N-K_l'+j}\in\Gamma_0\,(j=1,...,[K_l]+1,$ if $0\leq K_l<N,\ l=1,2)$ are distinct points; and

$$h_l(z)=\begin{cases}0,\ z\in\Gamma, & \text{if } K_l\geq N,\\ \left.\begin{array}{l}h_{lj},\ z\in\Gamma_j,\ k=1,...,N-K_l',\\0,z\in\Gamma_j,j=N-K_l'+1+2,...,N-K_l'+[K_l]+1\end{array}\right\} & \text{if } 0\leq K_l<N,\\ \left.\begin{array}{l}h_{lj},\ z\in\Gamma_j,\ j=1,...,N,\\ [1+(-1)^{2K_l}]h_{l0}+\mathrm{Re}\displaystyle\sum_{m=1}^{[|K_l|+1/2]-1}(h^+_{lm}+ih^-_{lm})z^m,z\in\Gamma_0\end{array}\right\} & \text{if } K_l<0,l=1,2,\end{cases}\tag{1.10}$$

where $h_{lj}\,(j=0,1,...,N)$, $h^{\pm}_{lm}\,(m=1,...,-K_l-1,K_l<0,\ l=1,2)$ are unknown real constants to be determined appropriately, and the relation

$$w(z)=w_0+\int_{a_0}^{z}[U(z)dz+\sum_{m=1}^{N}\frac{d_m}{z-z_m}dz+\overline{V(z)}d\overline{z}], \tag{1.11}$$

in which $Q_j = Q_j(z,w,U,V,U_z,V_z), j = 1,...,4, A_j = A_j(z,w,V,V), j = 1,...,7$, where $a_0 = 1$, $d_m (m = 1,...,N)$ are appropriate real constants such that the function determined by the integral in (1.11) is single-valued in D, $|\lambda_l(t)| = 1$, and $K_l = \frac{1}{2\pi}\Delta_\Gamma \lambda_l(t)\,(l = 1,2)$, and

$$\begin{aligned} Y_l(z)=&\prod_{j=1}^{m_0}(z-t_j)^{\gamma_{lj}}\prod_{s=1}^{N}(z-z_s)^{-[\tilde{K}_{ls}]}\prod_{j=m_0+1}^{m_1}\left(\frac{z-t_j}{z-z_1}\right)^{\gamma_{lj}}\cdots\prod_{j=m_{N_0-1}+1}^{m_{N_0}}\left(\frac{z-t_j}{z-z_{N_0}}\right)^{\gamma_{lj}}\\ &\times\prod_{j=m_{N_0}+1}^{m_{N_0+1}}\left(\frac{z-t_j}{z-z_{N_0+1}}\right)^{\gamma_{lj}}\left(\frac{z-t'_{N_0+1}}{z-z_{N_0+1}}\right)\cdots\prod_{j=m_{N-1}+1}^{m}\left(\frac{z-t_j}{z-z_N}\right)^{\gamma_{lj}}\left(\frac{z-t'_N}{z-z_N}\right), l=1,2, \end{aligned} \tag{1.12}$$

where $\tilde{K}_{ls} = \sum_{j=m_{s-1}+1}^{m_s} K_{lj}\,(l = 1,2)$ are denoted the partial indexes on $\Gamma_s\,(s = 1,...,N)$; and $t'_j\,(\in \Gamma_j,\, j = N_0+1,...,N)$ are fixed points, which are not the discontinuous points at Z. Similarly to (1.7)-(1.12), Chapter V, [5], we see that

$$\frac{\lambda_l(t_j-0)}{\lambda_l(t_j+0)}\overline{\left[\frac{Y_l(t_j-0)}{Y_l(t_j+0)}\right]}=\frac{\lambda_l(t_j-0)}{\lambda_l(t_j+0)}e^{-i\pi\gamma_{lj}}=\pm1,\ l=1,2$$

it only needs to charge the symbol on some arcs on Γ, then $\lambda_l(z)\overline{Y_l(z)}/|Y_l(z)|\,(l = 1,2)$ on Γ are continuous. In this case, its index

$$\kappa_l=\frac{1}{2\pi}\Delta_\Gamma[\lambda_l(z)\overline{Y_l(z)}]=K_l-\frac{N-N_0}{2},\ l=1,2$$

are an integer; and

$$\begin{aligned} &X_l(z)=\begin{cases} z^{[\kappa_l]}e^{iS_l(z)}Y_l(z),\ z\in\Gamma_0,\\ e^{i\theta_{lj}}e^{iS_l(z)}Y_l(z),\ z\in\Gamma_j, j=1,...,N, \end{cases} \quad \mathrm{Im}[\overline{\lambda_l(z)}X_l(z)]=0,\ z\in\Gamma,\\ &\mathrm{Re}S_l(z)=S_{l1}(z)-\theta_l(t),\ S_{l1}(z)=\begin{cases} \arg\lambda_l(z)-[K_l]\arg z-\arg Y_l(z),\ z\in\Gamma_0,\\ \arg\lambda_l(z)-\arg Y_l(z),\ z\in\Gamma_j, j=1,...,N, \end{cases}\\ &\theta_l(z)=\begin{cases} 0,\ z\in\Gamma_0,\\ \theta_{lj},\ z\in\Gamma_j,\ j=1,...,N, \end{cases} \quad \mathrm{Im}[S_l(1)]=0,\ l=1,2, \end{aligned} \tag{1.13}$$

in which $S_l(z)\,(l=1,2)$ are the solutions of the modified Dirichlet problem with the above boundary condition for analytic functions, $\theta_{lj}\,(j=1,...,N,\ l=1,2)$ are real constants. We assume that

$$|b_{lj}|\le k_4,\ j\in J_l,\ l=1,2,\ |w_0|\le k_4, \tag{1.14}$$

where k_4 is a real constant as before.

In this article, we first discuss the modified boundary value problem (Problem Q) for a system of first order complex equations, which corresponds to Problem P for the complex equation (1.1). We establish then the integral expression and a priori estimates of solutions for Problem Q. By the estimates and the Leray-Schauder theorem, we can prove the existence of a solution for Problem Q, and so derive the results of the solvability for Problem P for the system (1.1) with some conditions as follows.

Theorem 1.1. (The Main Theorem) *Suppose that the second order quasilinear system* (1.1) *satisfy Condition* C *and* (2.19) *below. If the constants* q_2,ε,k_1,k_2 *in* (1.2), (1.3), (1.6), (1.14) *are all sufficiently small, then Problem* P *for* (1.1) *possesses the following results on solvability:*

(1) *When the indices* $K_j=\frac{1}{2\pi}\Delta_\Gamma\arg\lambda_j(t)\ge N\,(j=1,2)$, *Problem* P *for* (1.1) *has* $2N$ *solvability conditions, and the solution depends on* $2(K_1+K_2-N+2)$ *arbitrarily real constants.*

(2) *When the indices* $0\le K_j<N\,(j=1,2)$, *the total number of the solvability conditions for Problem* P *is not greater than* $4N-[K_1+1/2]-[K_2+1/2]$ *and the solution depends on* $[K_1]+[K_2]+4$ *arbitrarily real constants.*

(3) *When* $0\le K_1<N, K_2\ge N$ *(or* $K_1\ge N, 0\le K_2<N$*), the total number of the solvability conditions for Problem* P *is not greater than* $3N-[K_1+1/2]$ *(or* $3N-[K_2+1/2]$*) and the solution depends on* $[K_1]+2K_2-N+4$ *(or* $2K_1+[K_2]-N+4$*) arbitrarily real constants.*

(4) *When* $K_1<0, K_2\ge N$ *(or* $K_1\ge N, K_2<0$*), Problem* P *has* $3N-2K_1-1$ *(or* $3N-2K_2-1$*) solvability conditions, and the solution depends on* $2K_2-N+3$ *(or* $2K_1-N+3$*) arbitrarily real constants.*

(5) *When* $K_1<0, 0\le K_2<N$ *(or* $0\le K_1<N, K_2<0$*), Problem* P *has* $4N-2K_1-[K_2+1/2]-1$ *(or* $4N-[K_1+1/2]-2K_2-1$*) solvability conditions, and the solution depends on* $[K_2]+3$ *(or* $[K_1]+3$*) arbitrarily real constants.*

(6) *When* $K_1 < 0, K_2 < 0$, *Problem* P *has* $4N - 2K_1 - 2K_2 - 2$ *solvability conditions, and the solution depends on two arbitrarily real constants.*

2 Estimates of solutions of discontinuous oblique derivative problem

In this section, we first develop some estimates of solutions of Problem Q for elliptic complex systems (1.8).

Theorem 2.1. *Suppose that Condition* C *holds and the four constants* q_2, ε, k_1, k_2 *in* (1.2), (1.3), (1.6) *are small enough. Then any solution* $[U(z), V(z), w(z)]$ *of Problem* Q *for* (1.8) *satisfies the estimates*

$$\begin{aligned} L_1 = L_1(U) = C_\delta[R(z)U(z), \bar{D}] + L_{p_0}[|RSU_{\bar{z}}| + |RSU_z|, \overline{D}] \le M_1, \\ L_2 = L_2(V) = C_\delta[R(z)V(z), \bar{D}] + L_{p_0}[|RSV_{\bar{z}}| + |RSV_z|), \overline{D}] \le M_1, \end{aligned} \tag{2.1}$$

$$S_0 = S_0(w) = C_\delta[w(z), \overline{D}] + C_\delta[R(z)w_z, \overline{D}] + C_\delta[R(z)\overline{w}_z, \overline{D}] \le M_2, \tag{2.2}$$

where

$$\begin{aligned} &R(z) = \textstyle\prod_{j=1}^m |z - t_j|^{\eta_j/\tau^2},\ S(z) = \prod_{j=1}^m |z - t_j|^{1/\tau^2}, \\ &\eta_j = \max(\eta_{1j}, \eta_{l2}),\ j = 1, ..., m, \\ &\eta_{lj} = \begin{cases} \beta_{lj} + \tau, \text{ for } \gamma_{lj} \ge 0, \text{ and } \gamma_{lj} < 0, \beta_{lj} \ge |\gamma_{lj}|, \\ |\gamma_{lj}| + \tau, \text{ for } \gamma_{lj} < 0, \beta_{lj} < |\gamma_{lj}|, j = 1, ..., m, l = 1, 2, \end{cases} \end{aligned}$$

and $\delta\,(\le \min(\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \tau)$, $\tau\,(\le \min(\alpha, 1 - 2/p_0))$, $p_0\,(2 < p_0 \le p)$, M_1 *and* M_2 *are positive constants,* $M_j = M_j(q_0, p_0, \delta, k^*, K, D)$, $j = 1, 2$, $k^* = k^*(k_0, k_3, k_4)$, *and* $K = (K_1, K_2)$.

Proof. Let the solution $[w(z), U(z), V(z)]$ of Problem Q be substituted into the system (1.8), the boundary conditions (1.9), and the relation (1.11). It is clear that (1.8) and (1.9) can be rewritten in the form

$$U_{\bar{z}} - Q_1 U_z - A_1 U = A,\ A = Q_2 V_z + A_2 V + A_3 w + A_4,\ V_{\bar{z}} = \overline{U}_z, \tag{2.3}$$

$$\begin{gathered}\mathrm{Re}[\overline{\lambda_1(z)}U(z)]=r_1(z)+h_1(z)\overline{\lambda_1(z)}X_1(z),\\ \mathrm{Re}[\overline{\lambda_2(z)}V(z)]=r_2(z)+h_2(z)\overline{\lambda_2(z)}X_2(z),\\ r_l(z)=\tau_l(z)-\mathrm{Re}[\sigma_l(z)w(z)],\ \ z\in\Gamma,\ l=1,2,\end{gathered}\tag{2.4}$$

where A and $r_l(l=1,2)$ satisfy the inequalities

$$\begin{gathered}L_{p_0}[RSA,\overline{D}]\le q_2L_{p_0}[RSV_z,\overline{D}]+L_{p_0}[A_2,\overline{D}]C[RV,\overline{D}]\\ +L_{p_0}[A_3,\overline{D}]C[w,\overline{D}]+L_{p_0}[A_4,\overline{D}]\le q_2L_2+k_1L_2+k_2S_1+k_3,\end{gathered}\tag{2.5}$$

$$C_\alpha[Rr_l,\Gamma]\le C_\alpha[R\sigma_l,\Gamma]C[w,\Gamma]+C_\alpha[R\tau_l,\Gamma]\le\varepsilon k_0S_1+k_4,\ l=1,2,\tag{2.6}$$

in which $S_1=C[w,\overline{D}]$, we mention that the some items k_2S_1,k_3 should be replaced by $k_5k_2S_1,k_5k_3$, where $k_5=C[R(z),\overline{D}]$, but for convenience we omit them.

Moreover from (2.3) and (2.4), we can obtain

$$\begin{aligned}L_1&\le M_3[(q_2+k_1)L_2+k_2S_1+k_3+\varepsilon k_0S_1+2k_4]\\ &=M_3[(q_2+k_1)L_2+(k_2+\varepsilon k_0)S_1+k_3+2k_4],\end{aligned}\tag{2.7}$$

where $M_3=M_3(q_0,p_0,\delta,k_0,K,D)$. Noting that $V(z)$ is a solution of the modified problem for $V_{\bar z}=\overline{U}_z$, we have

$$L_2\le M_3[L_1+\varepsilon k_0S_1+2k_4].\tag{2.8}$$

In addition, from (1.11), we can derive

$$\begin{gathered}d_m=\frac{i}{2\pi}\int_{\Gamma_m}[U(z)dz+\overline{V(z)}d\overline{z}],\ m=1,...,N,\\ S_1=C[w,\overline{D}]\le k_4+M_4[C(RU,\overline{D})+C(RV,\overline{D})]\le k_4+M_4(L_1+L_2),\end{gathered}\tag{2.9}$$

where $M_4=M_4(D)$.

Combining (2.7)-(2.9), we can derive that

$$\begin{aligned}L_2\le\ &M_3\{M_3[(q_2+k_1)L_2+(k_2+\varepsilon k_0)(k_4+M_4(L_1+L_2))\\ &+k_3+2k_4]+\varepsilon k_0(k_4+M_4(L_1+L_2))+2k_4\}\\ &\le M_3\{(q_2+k_1)M_3L_2+(k_2+\varepsilon k_0)(1+M_3)M_4(L_1+L_2)\\ &+k_4(k_2+\varepsilon k_0)(1+M_3)+(k_3+2k_4)(1+M_3)\}.\end{aligned}\tag{2.10}$$

Provided that the constants q_2,ε,k_1,k_2 are sufficiently small, for instance, $M_3[(q_2+k_1)M_3+(k_2+\varepsilon k_0)(1+M_3)M_4]<1/2$, we must have

$$\begin{aligned}L_2\le\ &2M_3[(k_2+\varepsilon k_0)(1+M_3)M_4L_1+k_4(k_2+\varepsilon k_0)(1+M_3)\\ &+(k_3+2k_4)(1+M_3)]=M_5L_1+M_6,\end{aligned}\tag{2.11}$$

where $M_5 = 2M_3(k_2 + \varepsilon K_0)(1 + M_3)M_4$, $M_6 = 2M_3[k_4(k_2 + \varepsilon k_0)(1 + M_3) +(k_3 + 2k_4)(1 + M_3)]$. Letting (2.11) and (2.9) be substituted into (2.7), we can obtain

$$\begin{aligned} L_1 \leq M_3[(q_2+k_1)(M_5L_1+M_6)+(k_2+\varepsilon k_0)M_4(L_1+L_2)+k_4(k_2+\varepsilon k_0) \\ +k_3 + 2k_4] \leq M_3\{[(q_2 + k_1)M_5 + (k_2 + \varepsilon k_0)M_4(1 + M_5)]L_1 \\ +(q_2 + k_1)M_6 + (k_2 + \varepsilon k_0)M_4M_6+k_4(k_2 + \varepsilon k_0) + k_3 + 2k_4\}. \end{aligned} \tag{2.12}$$

Moreover if $q_2, \varepsilon, k_1, k_2$ are small enough such that $M_3[(q_2 + k_1)M_5 + (k_2 + \varepsilon k_0)(1 + M_5)M_4] < 1/2$, then the estimates

$$L_1 \leq 2M_3[(q_2+k_1)M_6+(k_2+\varepsilon k_0)M_4M_6+k_4(k_2+\varepsilon k_0)+k_3+2k_4] = M_7 \tag{2.13}$$

is concluded, and

$$L_2 \leq M_5M_7 + M_6 \leq M_1 = \max(M_7, M_5M_7 + M_6). \tag{2.14}$$

Furthermore, from (1.11) it follows that (2.2) holds. □

From Theorem 2.1, we can derive the following result.

Theorem 2.2. *Under the same conditions in Theorem* 2.1*, any solution* $[U(z), V(z), w(z)]$ *of Problem* Q *for* (1.8) *satisfies the estimates*

$$L_1 = L_1(U) \leq M_8k,\ L_2 = L_2(V) \leq M_8k, \tag{2.15}$$

$$S_0 = S_0(w) \leq M_9k, \tag{2.16}$$

where $M_j = M_j(q_0, p_0, \delta, k_0, K, D), j = 8, 9,$ *and* $k = k_3 + 2k_4$.

Proof. We substitute the solution $[U(z), V(z), w(z)]$ of Problem Q into the system (1.8), the boundary conditions (1.9) and the relation (1.11). Similarly to the proof of Theorem 2.1, we can obtain the results as in (2.1) and (2.2), namely

$$L_1 = L_1(U) \leq M_8k,\ L_2 = L_2(V) \leq M_8k, \tag{2.17}$$

$$S_0 = S_0(w) \leq M_9k, \tag{2.18}$$

in which $k = k_3 + 2k_4$, $M_j = M_j(q_0, p_0, \delta, k_0, K, D), j = 8, 9$.

In order to prove the uniqueness of solutions of Problem Q for (1.8), we need to add the following condition: For any continuously differentiable functions

$w_j(z)(j=1,2)$ on $\overline{D}$ and any continuous functions $U(z), V(z)(\in W^1_{p_0}(\tilde{D})(2<p_0\le p)$, $\tilde{D}$ is any closed subset), there is

$$\begin{aligned}&F(z,w_1,w_{1z},\overline{w}_{1z},U_z,V_z)-F(z,w_2,w_{2z},\overline{w}_{2z}U_z,V_z)\\&=\tilde{Q}_1U_z+\tilde{Q}_2V_z+\tilde{A}_1(w_{1z}-w_{2z})+\tilde{A}_2(\overline{w}_{1z}-\overline{w}_{2z})+\tilde{A}_3(w_1-w_2),\end{aligned}\tag{2.19}$$

where $|\tilde{Q}_j|\le q_j,\ j=1,2,\ \tilde{A}_j\in L_{p_0}(\overline{D}),\ j=1,2,3.$ □

Theorem 2.3. *If Condition* C, (2.19) *hold, and* q_2,ε,k_1,k_2 *in* (1.2), (1.3), (1.6) *are small enough, then the solution* $[w(z),\ U(z),\ V(z)]$ *of Problem* Q *for* (1.8) *is unique.*

Proof. Denote by $[w_j(z),U_j(z),V_j(z)](j=1,2)$ two solutions of Problem Q for (1.8), and substitute them into (1.8),(1.9) and (1.11), we see that $[w,U,V]=[w_1(z)-w_2(z),U_1(z)-U_2(z),V_1(z)-V_2(z)]$ is a solution of the following homogeneous boundary value problem

$$U_{\bar{z}}=\tilde{Q}_1U_z+\tilde{Q}_2V_z+\tilde{A}_1U+\tilde{A}_2V+\tilde{A}w,\ V_{\bar{z}}=U_z,\ z\in D,\tag{2.20}$$

$$\begin{cases}\operatorname{Re}[\overline{\lambda_1(z)}U(z)+\sigma_1(z)w(z)]=h_1(z)\overline{\lambda_1(z)}X_1(z),\\ \operatorname{Re}[\overline{\lambda_2(z)}V(z)+\sigma_2(z)w(z)]=h_2(z)\overline{\lambda_2(z)}X_2(z),\end{cases}z\in\Gamma,\tag{2.21}$$

$$\begin{cases}\operatorname{Im}[\overline{\lambda_1(z)}U(z)+\sigma_1(z)w(z)]|_{z=a_j}=0,\ j\in J_1,\\ \operatorname{Im}[\overline{\lambda_2(z)}V(z)+\sigma_2(z)w(z)]|_{z=a_j}=0,\ j\in J_2,\end{cases}\tag{2.22}$$

$$w(z)=w_0-\int_1^z[U(z)dz-\sum_{m=1}^N\frac{d_m}{z-z_m}]dz+\overline{V(z)}d\bar{z}\ \text{ in }\ D,\tag{2.23}$$

the coefficients of which satisfy same conditions of (1.8),(1.9) and (1.11), but $k_3=k_4=0$.

On the basis of Theorem 2.2, provided q_2,k_1,k_2 and ε are sufficiently small, we can derive that $w(z)=U(z)=V(z)=0$ in $\overline{D}$, i.e. $w_1(z)=w_2(z)$, $U_1(z)=U_2(z)$, $V_1(z)=V_2(z)$ in $\overline{D}$. □

3 Solvability of discontinuous oblique derivative problem

In the following, we use the foregoing estimates of solutions and the Leray-Schauder theorem to prove the solvability of Problem Q for the nonlinear elliptic complex system (1.8).

Theorem 3.1. *Suppose that the second order quasilinear system* (1.1) *satisfy Condition* C *and* (2.19). *If the constants* $q_2, \varepsilon, k_1, k_2$ *in* (1.2), (1.3), (1.6) *are all sufficiently small, then Problem* Q *for* (1.8) *is solvable.*

Proof. First of all, we assume that $F(z, w, U, V, U_z, V_z)$ of (1.8) equal to 0 in the neighborhood D^* of the boundary Γ. The equation is denoted by

$$U_{\bar{z}} = F^*(z, w, U, V, U_z, V_z),\ V_{\bar{z}} = \overline{U}_z \text{ in } D. \tag{3.1}$$

Then we consider the system of first order equations with the parameter $t \in [0, 1]$, namely

$$U^*_{\bar{z}} = t[F^*(z, w, U, V, U^*_z, V^*_z),\ V^*_{\bar{z}} = t\overline{U^*}_z. \tag{3.2}$$

Moreover we introduce the Banach space $B = \hat{W}^1_{p_0}(D) \times \hat{W}^1_{p_0}(D) \times \hat{C}^1(\overline{D})(2 < p_0 \le p)$. Denote by B_M the set of systems of continuous functions: $\omega = [U(z), V(z), w(z)]$ satisfying the inequalities:

$$\begin{gathered} L_1(U) = C_\delta[RU, \overline{D}] + L_{p_0}[|RSU_{\bar{z}}| + |RSU_z|, \overline{D}] < M_{10},\ L_2(V) < M_{10}, \\ \hat{C}^1[w(z), \overline{D}] = C[w(z), \overline{D}] + C[Rw_z, \overline{D}] + C[R\overline{w}_z, \overline{D}] < M_{10}, \end{gathered} \tag{3.3}$$

in which $M_{10} = M_1 + M_2 + 1$, δ, M_1, M_2 are non-negative constants as stated in (2.1) and (2.2). It is evident that B_M is a bounded open set in B.

Next, we only discuss Problem Q for (3.2) and arbitrarily select a system of functions: $\omega = [U(z), V(z), w(z)] \in B_M$. Substitute it into the appropriate positions of (3.2),(1.9) and (1.11), and then consider the boundary value problem (Problem Q) with the parameter $t \in [0, 1]$:

$$U^*_{\bar{z}} = t[F^*(z, w, U, V, U_z, V_z),\ V^*_{\bar{z}} = t\overline{U}_z,\ z \in D, \tag{3.4}$$

$$\begin{cases} \operatorname{Re}[\overline{\lambda_1(z)}U^*(z) + t\sigma_1(z)w(z)] = \tau_1(z) + h_1(z)\overline{\lambda_1(z)}X_1(z), \\ \operatorname{Re}[\overline{\lambda_2(z)}V^*(z) + t\sigma_2(z)w(z)] = \tau_2(z) + h_2(z)\overline{\lambda_2(z)}X_2(z), \end{cases} z \in \Gamma, \tag{3.5}$$

$$\begin{cases} \operatorname{Im}[\overline{\lambda_1(a_j)}U^*(a_j)+t\sigma_1(a_j)w(a_j)]=b_{lj},\ j\in J_1,\\ \operatorname{Im}[\overline{\lambda_2(a_j)}V^*(a_j)+t\sigma_2(a_j)w(a_j)]=b_{2j},\ j\in J_2, \end{cases} \tag{3.6}$$

$$w^*(z)=w_0+\int_1^z[U^*(z)+\sum_{m=1}^N\frac{d_m}{z-z_m}]dz+\overline{V^*(z)}d\bar z,\ z\in D, \tag{3.7}$$

where $U(z),V(z),w(z)$ are known functions as stated before.

Noting that Problem Q consists of two modified Riemann-Hilbert problems for elliptic complex equations of first order and applying the method in the proof of Theorem 6.6, Chapter V, [4] and Theorem 3.5.3, Chapter 3, [12], we see that there exist the solutions $U^*(z),V^*(z)\in\hat W^1_{p_0}(D)(2<p_0\le p)$. From (3.7), the single-valued function $w^*(z)$ in $\overline{D}$ is determined.

Denote by $\omega^*=[U^*(z),V^*(z),w^*(z)]=T(\omega,t)(0\le t\le 1)$ the mapping from ω onto ω^*. According to Theorem 2.2, if $\omega=[U(z),V(z),w(z)]=T(\omega,t)(0\le t\le 1)$, then $\omega=[U(z),V(z),w(z)]$ satisfies the estimates in (2.1),(2.2), consequently $\omega\in B_M$. Setting $B_0=B_M\times[0,1]$, we shall verify that the mapping $\omega^*=T(\omega,t)(0\le t\le 1)$ satisfies the three conditions of the Leray-Schauder theorem:

(1) When $t=0$, by Theorem 2.2, it is evident that $\omega^*=T(\omega,0)\in B_M$.

(2) As stated before, the solution $\omega=[U(z),V(z),w(z)]$ of the functional equation $\omega=T(\omega,t)(0\le t\le 1)$ satisfies the estimates in (2.1),(2.2), which shows that $\omega=T(\omega,t)(0\le t\le 1)$ does not have any solution $\omega=[U(z),V(z),w(z)]$ on the boundary $\partial B_M=\overline{B_M}\backslash B_M$.

(3) For every $t\in[0,1]$, $\omega^*=T(\omega,t)$ continuously maps the Banach space B into itself, and is completely continuous in B_M. Besides, for $\omega\in\overline{B_M}, T(\omega,t)$ is uniformly continuous with respect to $t\in[0,1]$.

In fact, let us choose any sequence $\omega_n=[U_n(z),V_n(z),w_n(z)](n=1,2,...)$, which belongs to $\overline{B_M}$. By Theorem 2.1, it is not difficult to see that $\omega_n^*=[U_n^*,V_n^*,w_n^*]=T(\omega_n,t)(0\le t\le 1)$ satisfies the estimates

$$L_1(U_n^*)\le M_{12},\ L_2(V_n^*)\le M_{12},\ S_0(w_n^*)\le M_{13}, \tag{3.8}$$

in which $M_j=M_j(q_0,p_0,\delta,k_0,K,D,M)$, $j=12,13$, $n=1,2,....$ We can select subsequences of $\{U_n^*(z)\},\{V_n^*(z)\}$, $\{w_n^*(z)\}$, which uniformly converge to $U_0^*(z)$, $V_0^*(z),w_0^*(z)$ in $\overline{D}$, and $\{U_{nz}^*\},\{U_{n\bar z}^*\}$, $\{V_{nz}^*\},\{V_{n\bar z}^*\}$ in D weakly converge to U_{0z}^*, $U_{0\bar z}^*$, $V_{0z}^*,V_{0\bar z}^*$, respectively.

For convenience, the same notations will be used to denote the subsequences. From $\omega_n^* = T(\omega_n, t)$ and $\omega_0^* = T(\omega_0, t)(0 \le t \le 1)$, we obtain

$$\begin{aligned} &U^*_{n\bar z} - U^*_{0\bar z} = t[F(z, w_n, U_n, V_n, U^*_{nz}, V^*_{nz}) - F(z, w_n, U_n, V_n, U^*_{0z}, V^*_{0z}) + c_n],\\ &c_n = F(z, w_n, U_n, V_n, U^*_{0z}, V^*_{0z}) - F(z, w_0, U_0, V_0, U^*_{0z}, V^*_{0z}),\\ &V^*_{n\bar z} - V^*_{0\bar z} = t[\overline{U^*}_{nz} - \overline{U^*}_{0z}],\ z \in D, \end{aligned} \tag{3.9}$$

$$\begin{cases} \mathrm{Re}[\overline{\lambda_1(z)}(U_n^* - U_0^*) + t\sigma_1(z)(w_n - w_0)] = h_1(z)\overline{\lambda_1(z)}X_1(z),\\ \mathrm{Re}[\overline{\lambda_2(z)}(V_n^* - V_0^*) + t\sigma_2(z)(w_n - w_0)] = h_2(z)\overline{\lambda_2(z)}X_2(z), \end{cases} z \in \Gamma, \tag{3.10}$$

$$\begin{cases} \mathrm{Im}[\overline{\lambda_1(a_j)}[U_n^*(a_j) - U_0^*(a_j)] + t\sigma_1(a_j)[w_n(a_j) - w_0(a_j)] = 0,\ j \in J_1,\\ \mathrm{Im}[\overline{\lambda_2(a_j)}[V_n^*(a_j) - V_0^*(a_j)] + t\sigma_2(a_j)[w_n(a_j) - w_0(a_j)]] = 0,\ j \in J_2, \end{cases} \tag{3.11}$$

$$w_n^*(z) - w_0^*(z) = \int_1^z [U_n^*(z) - U_0^*(z) + \sum_{m=1}^N \frac{d_m}{z - z_m}]dz + [\overline{V_n^*(z)} - \overline{V_0^*(z)}]d\bar z. \tag{3.12}$$

By using the way in (1.2.53), Chapter 1, [11], we can prove that

$$L_{p_0}[RSc_n, \overline{D}] \to 0$$

for $n \to \infty$, since when $n \to \infty$, $\{c_n\}$ converges to 0 for almost every point $z \in D$. Because of the completeness of the Banach space B, there exists a system of functions $\omega_0 = [U_0(z), V_0(z), w_0(z)] \in B$, such that

$$L_1(U_n - U_0) \to 0,\ L_2(V_n - V_0) \to 0 \ \text{ and } \ S(w_n - w_0) \to 0 \text{ as } \ m \to \infty.$$

This shows the complete continuity of $\omega^* = T(\omega, t)(0 \le t \le 1)$ on $\overline{B_M}$. By a similar method, we can also prove that $\omega^* = T(\omega, t)(0 \le t \le 1)$ continuously maps $\overline{B_M}$ into B, and $T(\omega, t)$ is uniformly continuous with respect to $t \in [0, 1]$ for $\omega \in \overline{B_M}$.

Hence by the Leray-Schauder theorem, we see that the functional equation $\omega = T(\omega, t)(0 \le t \le 1)$ with $t = 1$, i.e. Problem Q for (1.8) has a solution. □

Finally we can cancel the assumption that $F(z, w, U, V, U_z, V_z)$ of (1.8) equal to 0 in the neighborhood D^* of the boundary Γ by the method as stated in the proof of Theorem 4.7, Chapter II, [3].

From the above theorem, the result in Theorem 1.1 can be derived.

Proof of Theorem 1.1. We first discuss the case: $0 \leq K_l < N$ $(l = 1, 2)$. Let the solution $[w(z), U(z), V(z)]$ of Problem Q for the complex system (1.8) be substituted into (1.9)–(1.11). The functions $h_l(z)(l = 1, 2)$ and the complex constants $d_m\,(m = 1, ..., N)$ are then determined. If the functions and the constants are equal to zero, namely the following equalities hold:

$$h_l(z)=h_{lj}=0,\ j=1,...,N-[K_l+1/2],\ \text{when}\ 0\leq K_l<N, l=1,2, \tag{3.13}$$

and

$$d_m = \text{Re}d_m + i\text{Im}d_m = 0,\ m = 1, ..., N, \tag{3.14}$$

then $w_z = U(z)$, $\overline{w}_z = V(z)$, $w(z)$ is a solution of Problem P for (1.1). Hence when $0 \leq K_l < N(l = 1, 2)$, Problem P for (1.1) has $4N-[K_1+1/2]-[K_2+1/2]$ solvability conditions. In addition, the real constants $b_{lj}\,(j = N - K_l' + [K_l] + 1, ..., N + 1, l = 1, 2)$ in (1.9) and the complex constant w_0 in (1.11) may be arbitrary, this shows that the general solution of Problem P $(0 \leq K_l < N, l = 1, 2)$ is dependent on $[K_1] + [K_2] + 4$ arbitrary real constants. Thus (2) is proved.

Similarly, other cases can be obtained. □

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