Theoretical Mathematics & Applications, vol.3, no.2, 2013, 49-54
ISSN: 1792-9687 (print), 1792-9709 (online)
Scienpress Ltd, 2013

A Dulac function for a quadratic system

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Abstract

In this work we find a Dulac function for a Quadratic System. We use Dulac's criterion that gives sufficient conditions for the non–existence of periodic orbits of dynamical systems in simply connected regions of the plane. Using a Dulac function we can rule out periodic orbits.

Mathematics Subject Classification: 34A34, 34C25
Keywords: Dulac functions, Quasilinear differential equations, Bendixson–Dulac criterion

1 Introduction

It is important to study in differential equations the analysis of the periodic orbits that there are in a system in the plane. It is well known certain systems

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Article Info: *Received* : April 24, 2013. *Revised* : May 31, 2013
Published online : June 25, 2013

have no limit cycles. For this should be considered: Bendixson's criteria, indices, invariant lines and critical points. See [1, 2, 3, 4, 5, 6].

In this paper we are interested in constructing a general system that not have periodic orbits using Dulac functions. We use the criterion of Bendixson–Dulac, see [7].

Theorem 1.1. (Bendixson-Dulac criterion [7]) *Let $f_1(x_1, x_2), f_2(x_1, x_2)$ and $h(x_1, x_2)$ be functions C^1 in a simply connected domain $D \subset \mathbb{R}^2$ such that $\frac{\partial(f_1 h)}{\partial x_1} + \frac{\partial(f_2 h)}{\partial x_2}$ does not change sign in D and vanishes at most on a set of measure zero. Then the system*

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \dot{x}_2 = f_2(x_1, x_2), \quad (x_1, x_2) \in D, \end{cases} \tag{1}$$

does not have periodic orbits in D.

According to this criterion, to rule out the existence of periodic orbits of the system (1) in a simply connected region D, we need to find a function $h(x_1, x_2)$ that satisfies the conditions of the theorem of Bendixson–Dulac, such function h is called a Dulac function. In Saez and Szanto [6] was constructed Lyapunov functions using Dulac functions to assure the nonexistence of periodic orbits. Our goal is to find a general dynamical system on the plane that not have periodic orbits using Dulac functions.

2 Method to obtain Dulac functions

A Dulac function for the system (1) satisfies the equation

$$f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} = h \left(c(x_1, x_2) - \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) \right) \tag{2}$$

(see [7]).

Theorem 2.1. [7] *For the system of differential equations* (1) *a solution* h *of the associated system* (2) *(for some function* c *which does not change sign and vanishes only on a subset of measure zero) is a Dulac function for* (1) *in any simply connected region* A *contained in* $D\backslash\{h^{-1}(0)\}$.

Theorem 2.2. [7] *For the system of differential equations* (1)*, if* (2) *(for some function* c *which does not change of sign and it vanishes only on a subset of measure zero) has a solution* h *on* D *such that* h *does not change sign and vanishes only on a subset of measure zero, then* h *is a Dulac function for* (1) *on* D.

3 Main Result

Theorem 3.1. *The system*

$$\begin{cases} \dot{x}_1 = A_1x_1^2 + B_1x_1x_2 + F_1, \\ \dot{x}_2 = A_2x_1^2 + C_2x_2^2 + F_2, \end{cases}$$

with $A_1, B_1, F_1, A_2, C_2, F_2$ *positive constants, does not have periodic orbits in a simply connected domain* $D \subset \mathbb{R}^2$ *where* h *is well defined.*

Proof. Taking

$$z = \alpha x_1 + \beta x_2 + \gamma$$

we obtain

$$\frac{\partial z}{\partial x_1} = \alpha, \quad \frac{\partial z}{\partial x_2} = \beta.$$

Substituting in (2) and by chain rule we have

$$[\alpha(A_1x_1^2 + B_1x_1x_2 + F_1) + \beta(A_2x_1^2 + C_2x_2^2 + F_2)]\frac{d\widetilde{h}}{dz} = \widetilde{h}(c(x_1, x_2) - divF)$$

We want

$$[\eta(\alpha x_1 + \beta x_2 + \gamma)^2 + \mu(\alpha x_1 + \beta x_2 + \gamma) + \nu]\frac{d\widetilde{h}}{dz} = \widetilde{h}(c(x_1, x_2) - divF)$$

so we have

$$[\eta z^2 + \mu z + \nu]\frac{d\widetilde{h}}{dz} = \widetilde{h}(c(x_1, x_2) - divF)$$

Namely,

$$[\alpha(A_1x_1^2 + B_1x_1x_2 + F_1) + \beta(A_2x_1^2 + C_2x_2^2 + F_2)]$$
$$= [\eta(\alpha x_1 + \beta x_2 + \gamma)^2 + \mu(\alpha x_1 + \beta x_2 + \gamma) + \nu]$$

We have

$$\begin{aligned}
&\alpha A_1 + \beta A_2 = \eta\alpha^2,\\
&\alpha B_1 = 2\eta\alpha\beta,\\
&\beta C_2 = \eta\beta^2,\\
&\alpha F_1 + \beta F_2 = \eta\gamma^2 + \mu\gamma + \nu,\\
&0 = 2\eta\alpha\gamma + \mu\alpha,\\
&0 = 2\eta\beta\gamma + \mu\beta.
\end{aligned}$$

If $\alpha \neq 0$ and $\beta \neq 0$ then $B_1 = 2\eta\beta$ and $C_2 = \eta\beta$. Also $B_1 = 2C_2$, $C_2 \neq 0$ and $\eta = \frac{C_2}{\beta}$. Also, we have $0 = \eta\alpha^2 - A_1\alpha - \beta A_2$ then by $C_2 = \eta\beta$ we have $\alpha = \frac{A_1 \pm \sqrt{{A_1}^2 + 4A_2C_2}}{2C_2}\beta$. Also we have

$$0 = 2\eta\alpha\gamma + \mu\alpha$$

then $0 = 2\eta\gamma + \mu$. From

$$0 = 2\eta\beta\gamma + \mu\beta$$

we have $0 = 2\eta\gamma + \mu$. Then $\mu = -2\eta\gamma$. We have

$$\alpha F_1 + \beta F_2 = \eta\gamma^2 + \mu\gamma + \nu.$$

Also by $\mu = -2\eta\gamma$ we have $\alpha F_1 + \beta F_2 = -\eta\gamma^2 + \nu$. From here we have $\nu = \alpha F_1 + \beta F_2 + \eta\gamma^2$. By $\mu = -2\eta\gamma$, we have that $\eta\gamma^2 - \nu = \mu^2 - 4\eta\nu < 0$. Until now we can summarize

$$\begin{aligned}
&\eta = \frac{C_2}{\beta},\\
&\mu = -2\eta\gamma,\\
&\nu = \alpha F_1 + \beta F_2 + \eta\gamma^2.
\end{aligned}$$

By initial system $divF = 2A_1x_1 + B_1x_2 + 2C_2x_2$. We can take $c(x_1, x_2) := -div^2F + divF - 1 < 0$, then $c(x_1, x_2) - divF = -(4A_1^2x_1^2 + 4A_1(B_1 + 2C_2)x_1x_2 +$

$(B_1+2C_2)^2x_2^2+1)$. We want

$$c(x_1,x_2)-divF=\eta_2 z^2+\eta_1 z+\eta_0.$$

As $z=\alpha x_1+\beta x_2+\gamma$, then $c(x_1,x_2)-divF=\eta_2(\alpha x_1+\beta x_2+\gamma)^2+\eta_1(\alpha x_1+\beta x_2+\gamma)+\eta_0$. We have to compare

$$\begin{aligned}&-(4A_1^2x_1^2+4A_1(B_1+2C_2)x_1x_2+(B_1+2C_2)^2x_2^2+1)\\&=\eta_2(\alpha x_1+\beta x_2+\gamma)^2+\eta_1(\alpha x_1+\beta x_2+\gamma)+\eta_0.\end{aligned}$$

We have

$$\begin{aligned}&\eta_2\alpha^2=-4A_1^2,\\&2\eta_2\alpha\beta=-4A_1(B_1+2C_2),\\&\eta_2\beta^2=-(B_1+2C_2)^2,\\&2\eta_2\alpha\gamma+\eta_1\alpha=0,\\&2\eta_2\beta\gamma+\eta_1\beta=0,\\&\eta_2\gamma^2+\eta_1\gamma+\eta_0=-1.\end{aligned}$$

Now, we can summarize

$$\begin{aligned}\eta_2&=\frac{-4A_1^2}{\alpha^2},\\\eta_1&=-2\eta_2\gamma,\\\eta_0&=-1-\eta_2\gamma^2-\eta_1\gamma.\end{aligned}$$

Then

$$\frac{d\tilde{h}}{\tilde{h}}=\frac{\eta_2 z^2+\eta_1 z+\eta_0}{\eta z^2+\mu z+\nu}dz.$$

Then integrating on both sides

$$\tilde{h}=e^{\int\frac{\eta_2 z^2+\eta_1 z+\eta_0}{\eta z^2+\mu z+\nu}dz}.$$

Then system does not contain periodic orbits in $\mathbb{R}^2$. □

Example 3.2. Consider system

$$\begin{cases}\dot{x}_1=2x_1^2+2x_1x_2,\\\dot{x}_2=-x_1^2+x_2^2,\end{cases}$$

then the solution of the partial differential equation (2) is $\tilde{h} = e^{-16z+1/z}$, with $z = x_1 + x_2$ and $x_1 + x_2 \neq 0$. This example provides a Dulac function $\tilde{h}$, and the system does not contain periodic orbits in $\mathbb{R}^2 \backslash \{x_1 + x_2 = 0\}$.

ACKNOWLEDGEMENTS. The authors express their deep gratitude to CONACYT-México, Programa de Mejoramiento del Profesorado (PROMEP)-México and Universidad de Cartagena for financial support.

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