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Universal inequalities for eigenvalues of elliptic operators in divergence form on domains in complete noncompact Riemannian manifolds

Yanli Li[1] and Feng Du[2]

Abstract

In this paper, we study the eigenvalue problem of elliptic operators in divergence form, and obtain some universal inequalities for eigenvalues of elliptic operators in divergence form on domains in complete simple connected noncompact Riemannian manifolds admitting special functions which include hyperbolic space. Especially, by using our universal inequalities, we can get the different universal inequalities including Yang inequality.

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[1] School of Electronic and Information Science, Jingchu University of Technology, Hubei Jingmen 448000, P.R. China.
[2] School of Mathematics and Physics Science, Jingchu University of Technology, Hubei Jingmen 448000, P.R. China.

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1 Introduction

Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Let Δ be the Laplacian operator acting on functions on M and consider the following eigenvalues problem for the Laplacian operator

$$\begin{cases} \Delta u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

it is known that this eigenvalue problem has a discrete spectrum,

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots ,$$

where each eigenvalue is repeated with its multiplicity. When $M = R^n, \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, Payne-Pólya-Weinberger [11] in 1956 proved

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{i=1}^{k} \lambda_i. \tag{1.2}$$

In 1980, Hile-Protter [9] strengthened (1.1), and proved

$$\frac{kn}{4} \leq \sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i}. \tag{1.3}$$

In 1991, Yang [13] gave the following much stronger inequality

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i. \tag{1.4}$$

From inequality (1.3), we can get a weaker but explicit form

$$\lambda_{k+1} \leq \left(1 + \frac{4}{n}\right) \frac{1}{k} \left(\sum_{i=1}^{k} \lambda_i\right). \tag{1.5}$$

These inequalities are called universal inequalities because they do not involve domain dependence.

In the following, we introduce an elliptic operator in divergence form in Riemannian manifold. Let $(M, \langle , \rangle)$ be an n-dimensional compact Riemannian manifold with boundary ∂M (possibly empty). Let $A : M \to \mathrm{End}(TM)$ be a smooth symmetric and positive definite section of the bundle of all endomorphisms of TM. Let V be a nonnegative continuous function on M. Denote by Δ and ∇ the Laplacian and the gradient operator of M respectively. Define

$$Lu = -\mathrm{div}(A\nabla u) + Vu, \tag{1.6}$$

where for a vector field X on M, $\mathrm{div}X$ denotes the divergence of X. The operator L defined in (1.5) is an elliptic operator in divergence form. It is easy to see that the Laplacian operator and Schrödinger operator are it's special cases. In 2010, do Carmo-Wang-Xia [6] considered the eigenvalue problem of the elliptic operator in divergence form with weight such that

$$\begin{cases} Lu = \lambda\rho u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where ρ is a weight function which is positive and continuous on M. They got a Yang type inequality

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4\xi_2^2\rho_2^2}{n\rho_1^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\frac{1}{\xi_1}\left(\lambda_i - \frac{V_0}{\rho_2}\right)+\frac{n^2H_0^2}{4\rho_1}\right), \tag{1.8}$$

where $\xi_1 I \leq A$ and $tr(A) \leq n\xi_2$ throughout M, $\rho_1 \leq \rho(x) \leq \rho_2$, $\forall x \in M$, I is the identity map, $\xi_1, \xi_2, \rho_1, \rho_2$ are positive constants, $H_0 = \max_M(|H|)$, $V_0 = \min_M(V)$, and $|H|$ is the mean curvature of M immersed into some Euclidean space R^N. Recently, Du-Wu-Li[7] considered the problem (1.7) on complete Riemannian manifolds, they proved the inequality

$$\begin{aligned} \sum_{i=1}^{k} f(\lambda_i) \leq\ & \frac{2\rho_2}{n\rho_1}\left(\frac{n\xi_2}{\xi_1}\sum_{i=1}^{k} g(\lambda_i)\right)^{\frac{1}{2}} \\ & \times\left(\sum_{i=1}^{k}\frac{(f(\lambda_i))^2}{(\lambda_{k+1}-\lambda_i)g(\lambda_i)}\left(\lambda_i+\frac{S}{\rho_1}\right)\right)^{\frac{1}{2}}, \end{aligned} \tag{1.9}$$

where $(f,g) \in \Im_{\lambda_{k+1}}$,$S = \sup_M\left(\frac{n^2\xi_1}{4}|\mathbf{H}|^2 - V\right)$, and $\mathbf{H}$ is the mean curvature vector field.

In this paper, we will consider the eigenvalue problem (1.7) on bounded domains in the complete simply connected noncompact Riemannian manifolds. Then we obtain.

Theorem 1.1. *Let $(M, \langle,\rangle)$ be an $n(n \geq 3)$-dimensional complete noncompact simply connected Riemannian manifold with Sectional curvature Sec satisfying $-K^2 \leq \mathrm{Sec} \leq -k^2$. For a bounded domain Ω, Let $A : \Omega \to \mathrm{End}(T\Omega)$ be a smooth symmetric and positive definite section of the bundle of all endomorphisms of $T\Omega$, and assume that $\xi_1 I \leq A \leq \xi_2 I$ throughout Ω, $\forall x \in \Omega, \rho_1 \leq \rho(x) \leq \rho_2$, here $\xi_1, \xi_2, \rho_1, \rho_2$ are positive constants. Let λ_i be the i^{th} eigenvalue of the eigenvalue problem (1.7), then for any $(f, g) \in \Im_{\lambda_{k+1}}$, we have*

$$\begin{aligned}\sum_{i=1}^{k} f(\lambda_i) \quad \leq \quad & \left\{\frac{1}{\rho_2\xi_2}\sum_{i=1}^{k} g(\lambda_i)\right\}^{\frac{1}{2}} \times \left\{\sum_{i=1}^{k}\frac{(f(\lambda_i))^2}{(\lambda_{k+1}-\lambda_i)g(\lambda_i)}\right.\\ & \left.\left(\frac{1}{\xi_1}(\rho_2\lambda_i - V_0) - (n^2-1)k^2 + 2(n-1)K^2\right)\right\}^{\frac{1}{2}} \qquad (1.10)\end{aligned}$$

where $V_0 = \min_{x\in\Omega} V(x)$.

From Theorem 1.1, we can get the following.

Corollary 1.2. *Under the assumption of Theorem 1.1, if M is a hyperbolic space $H^n(-1)$, we have*

$$\begin{aligned}\sum_{i=1}^{k} f(\lambda_i) \quad \leq \quad & \left\{\frac{1}{\rho_2\xi_2}\sum_{i=1}^{k} g(\lambda_i)\right\}^{\frac{1}{2}}\\ & \times \left\{\sum_{i=1}^{k}\frac{(f(\lambda_i))^2}{(\lambda_{k+1}-\lambda_i)g(\lambda_i)}\left(\frac{1}{\xi_1}(\rho_2\lambda_i - V_0) - (n-1)^2\right)\right\}^{\frac{1}{2}} (1,11)\end{aligned}$$

where $V_0 = \min_{x\in\Omega} V(x)$.

Remark. *Taking $(f(\lambda_i), g(\lambda_i)) = ((\lambda_{k+1}-\lambda_i)^2, (\lambda_{k+1}-\lambda_i)^2)$ in (1.11), we can get a Yang inequality that*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \quad \leq \quad \frac{1}{\rho_2\xi_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\frac{1}{\xi_1}(\rho_2\lambda_i - V_0) - (n-1)^2\right)$$

So, by choosing different $(f(\lambda_i), g(\lambda_i))$, we can get different universal inequalities.

2 Preliminaries

In this section, firstly, we shall introduce a family of couples of functions [8].

Definition 2.1. *Let $\lambda > 0$, a couple (f, g) of functions defined on $[0, \lambda]$ belongs to $\Im_\lambda$ as that*
(i) *f and g are positive,*
(ii) *f and g satisfy the following condition, for any $x, y \in [0, \lambda]$, such that $x \neq y$,*

$$\left(\frac{f(x)-f(y)}{x-y}\right)^2 + \left(\frac{(f(x))^2}{g(x)(\lambda - x)} + \frac{(f(y))^2}{g(y)(\lambda - y)}\right)\left(\frac{g(x)-g(y)}{x-y}\right) \leq 0. \tag{2.1}$$

We can easily find that $g(x)$ is a nonincreasing function.

In the following, we will introduce a lemma which is obtained by Du-Wu-Li[7].

Lemma 2.1. *Let $(M, \langle,\rangle)$ be an n-dimensional compact Riemannian manifold with boundary ∂M (possibly empty). Let λ_i be the i^{th} eigenvalue of the eigenvalue problem of elliptic operators in divergence form with weight ρ such that*

$$Lu = \lambda \rho u \quad in \quad M, \quad u = 0 \quad on \quad \partial M,$$

and u_i be the orthonormal eigenfunction corresponding to λ_i, that is,

$$Lu_i = \lambda_i \rho u_i \quad in \quad M, \qquad and \quad u_i = 0 \quad on \quad \partial M,$$

$$\int_M \rho u_i u_j = \delta_{ij}, \quad \forall\ i, j = 1, 2, \cdots$$

Then for any $h \in C^2(\overline{M})$ and $(f, g) \in \Im_{\lambda_{k+1}}$, we have

$$\sum_{i=1}^k f(\lambda_i)\|u_i \nabla h\|^2 \leq \delta \sum_{i=1}^k g(\lambda_i) \int_M u_i^2 \langle \nabla h, A\nabla h\rangle + \sum_{i=1}^k \frac{(f(\lambda_i))^2}{\delta(\lambda_{k+1}-\lambda_i)g(\lambda_i)} \left\| \frac{1}{\sqrt{\rho}}\left(\langle \nabla u_i, \nabla h\rangle + \frac{1}{2}u_i \Delta h\right)\right\|^2, \tag{2.2}$$

where δ is any positive constant and $\|f\|^2 = \int_M f^2$.

From this Lemma, if we can find a "nice" function on bounded domains in complete Riemannian manifolds and take it in (2.2) and (2.3), we can get universal inequalities in complete Riemannian manifolds. In the following, we will introduce a function on a bounded domain in complete noncompact simple connected Riemannian manifolds(for the more details about this function, we refer [5]).

Let $(M, \langle,\rangle)$ be an n-dimensional complete noncompact Riemannian manifold, and Ω be a bounded connected domain in M. $\mathrm{Sec}(M)$ is the section curvature satisfying $-K^2 \leq \mathrm{Sec}(M) \leq -k^2$, where k, K is constants and $0 \leq k \leq K$. When p is not in $\overline{\Omega}$, define the distance function $r(x) = d(x,p)$, then

$$(n-1)k\frac{\mathrm{cosh}kr}{\mathrm{sinh}kr} \leq \Delta r \leq (n-1)K\frac{\mathrm{cosh}Kr}{\mathrm{sinh}Kr}, \tag{2.3}$$

because of $\partial_r \Delta r = -|\mathrm{Hess}\ r|^2 - \mathrm{Ric}(\partial_r, \partial_r)$, so

$$-\partial_r \Delta r \leq (n-1)K^2\frac{\mathrm{cosh}Kr}{\mathrm{sinh}Kr} - (n-1)k^2, \tag{2.4}$$

where $\mathrm{Hess}, \mathrm{Ric}$ are the Hessian operator and Ricci curvature operator, respectively.

3 Proofs of the main results

In this section, we will give the proofs of Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. Taking $h = r$ in the (2.2), we can get

$$\begin{aligned}\sum_{i=1}^{k} f(\lambda_i)\|u_i\|^2 \quad &\leq \quad \delta\sum_{i=1}^{k} g(\lambda_i)\int_M u_i^2\langle\nabla r, A\nabla r\rangle + \sum_{i=1}^{k}\frac{(f(\lambda_i))^2}{\delta(\lambda_{k+1}-\lambda_i)g(\lambda_i)} \\ &\left\|\frac{1}{\sqrt{\rho}}\left(\langle\nabla u_i, \nabla r\rangle + \frac{1}{2}u_i\Delta r\right)\right\|^2,\end{aligned} \tag{3.1}$$

Because of $\rho_1 \le \rho(x) \le \rho_2, \xi_1 I \le A \le \xi_2 I$, and (2.1), we have

$$\left\|\frac{1}{\sqrt{\rho}}u_i\right\|^2 \ge \frac{1}{\rho_2} \tag{3.2}$$

$$\xi_1 \le \langle \nabla r, A\nabla r\rangle \le \xi_2. \tag{3.3}$$

Take (3.2) and (3.3) into (3.1), we have

$$\begin{aligned}\frac{1}{\rho_2}\sum_{i=1}^{k} f(\lambda_i) \quad \le \quad & \frac{\xi_2}{\rho_1}\delta\sum_{i=1}^{k} g(\lambda_i) + \sum_{i=1}^{k}\frac{(f(\lambda_i))^2}{4\delta(\lambda_{k+1}-\lambda_i)g(\lambda_i)} \\ & \left\|\frac{1}{\sqrt{\rho}}\left(2\langle\nabla u_i, \nabla r\rangle + u_i\Delta r\right)\right\|^2 \end{aligned} \tag{3.4}$$

It is known that

$$\begin{aligned}\lambda_i = \int_M u_i L(u_i) \quad &= \quad \int_M u_i\left(-\mathrm{div}(A\nabla u_i) + Vu_i\right) \\ &= \quad \int_M \left(\langle Au_i, u_i\rangle + Vu_i^2\right) \\ &\ge \quad \int_M \left(\xi_1|\nabla u_i|^2 + Vu_i^2\right),\end{aligned}$$

which yields

$$\int_M |\nabla u_i|^2 \le \frac{1}{\xi_1}\left(\lambda_i - \frac{V_0}{\rho_2}\right). \tag{3.5}$$

From (2.4) and (2.5), we can get

$$\begin{aligned} & \left\|\frac{1}{\sqrt{\rho}}\left(2\langle\nabla u_i, \nabla r\rangle + u_i\Delta r\right)\right\|^2 \\ = \quad & \int_\Omega \frac{1}{\rho}\left(2\langle\nabla r, \nabla u_i\rangle + u_i\Delta r\right)^2 \\ \le \quad & \frac{1}{\rho_1}\left(4\int_\Omega \langle\nabla r, \nabla u_i\rangle^2 - \int_\Omega u_i^2(\Delta r)^2 - 2\int_\Omega u_i^2\langle\nabla r, \nabla\Delta r\rangle\right) \\ \le \quad & \frac{1}{\rho_1}\left(\int_\Omega |\nabla u_i|^2 - \int_\Omega u_i^2(\Delta r)^2 - 2\int_\Omega u_i^2\partial_r(\Delta r)\right) \\ \le \quad & \frac{1}{\rho_1\xi_1}\left(\lambda_i - \frac{V_0}{\rho_2}\right) - \frac{(n-1)^2k^2}{\rho_1}\int_\Omega u_i^2\frac{\cosh^2 kr}{\sinh^2 kr} \\ & + \frac{2(n-1)K^2}{\rho_1}\int_\Omega u_i^2\frac{\cosh^2 Kr}{\sinh^2 Kr} - \frac{2(n-1)k^2}{\rho_1\rho_2} \\ \le \quad & \frac{1}{\rho_1\xi_1}\left(\lambda_i - \frac{V_0}{\rho_2}\right) - \frac{(n^2-1)k^2}{\rho_1\rho_2} + \frac{2(n-1)K^2}{\rho_1\rho_2} \\ & - \frac{(n-1)^2}{\rho_1}\int_\Omega u_i^2\frac{k^2}{\sinh^2 kr} + \frac{2(n-1)}{\rho_1}\int_\Omega u_i^2\frac{K^2}{\sinh^2 Kr} \end{aligned} \tag{3.6}$$

Since $K \geq k \geq 0, r > 0$, we obtain

$$\frac{K}{\sinh Kr} \leq \frac{k}{\sinh kr}, \tag{3.7}$$

Since $n \geq 3$, we get

$$(n-1)^2\frac{k^2}{\sinh^2 kr} - 2(n-1)\frac{K^2}{\sinh^2 Kr} \geq (n-1)(n-3)\frac{k^2}{\sinh^2 kr} \geq 0, \tag{3.8}$$

so, by (3.7)-(3.9), we can get

$$\begin{aligned} & \left\| \frac{1}{\rho}\left(u_i \Delta r + 2\langle \nabla r, \nabla u_i\rangle\right) \right\|^2 \\ \leq \; & \frac{1}{\rho_1\xi_1}\left(\lambda_i - \frac{V_0}{\rho_2}\right) - \frac{(n^2-1)k^2}{\rho_1\rho_2} + \frac{2(n-1)K^2}{\rho_1\rho_2} \end{aligned} \tag{3.9}$$

Taking (3.10) into (3.4), we obtain

$$\begin{aligned} \frac{1}{\rho_2}\sum_{i=1}^k f(\lambda_i) \; \leq \; & \frac{\xi_2}{\rho_1}\delta\sum_{i=1}^k g(\lambda_i) + \sum_{i=1}^k \frac{(f(\lambda_i))^2}{4\delta(\lambda_{k+1}-\lambda_i)g(\lambda_i)} \\ & \times\left(\frac{1}{\rho_1\xi_1}\left(\lambda_i - \frac{V_0}{\rho_2}\right) - \frac{(n^2-1)k^2}{\rho_1\rho_2} + \frac{2(n-1)K^2}{\rho_1\rho_2}\right) \end{aligned} \tag{3.10}$$

above inequality implies that

$$\begin{aligned} \sum_{i=1}^k f(\lambda_i) \; \leq \; & \frac{\rho_2\xi_2}{\rho_1}\delta\sum_{i=1}^k g(\lambda_i) + \sum_{i=1}^k \frac{(f(\lambda_i))^2}{4\delta(\lambda_{k+1}-\lambda_i)g(\lambda_i)} \\ & \times\frac{1}{\rho_1}\left(\frac{1}{\xi_1}(\rho_2\lambda_i - V_0) - (n^2-1)k^2 + 2(n-1)K^2\right) \end{aligned} \tag{3.11}$$

$$\delta = \left\{\frac{1}{4\rho_2\xi_2}\frac{\sum_{i=1}^k \frac{(f(\lambda_i))^2}{(\lambda_{k+1}-\lambda_i)g(\lambda_i)}}{\sum_{i=1}^k g(\lambda_i)}\left(\frac{1}{\xi_1}(\rho_2\lambda_i - V_0) - (n^2-1)k^2 + 2(n-1)K^2\right)\right\}^{\frac{1}{2}}$$

(3.11) becomes

$$\begin{aligned} \sum_{i=1}^k f(\lambda_i) \; \leq \; & \left\{\frac{1}{\rho_2\xi_2}\sum_{i=1}^k g(\lambda_i)\right\}^{\frac{1}{2}} \times \left\{\sum_{i=1}^k \frac{(f(\lambda_i))^2}{(\lambda_{k+1}-\lambda_i)g(\lambda_i)}\right. \\ & \left.\left(\frac{1}{\xi_1}(\rho_2\lambda_i - V_0) - (n^2-1)k^2 + 2(n-1)K^2\right)\right\}^{\frac{1}{2}} \end{aligned} \tag{3.12}$$

This completes the proof of Theorem 1.1. $\square$

Proof of Corollary 1.2. Taking $K = k = 1$ in (3.12), we immediately obtain (1.11). $\square$

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