Universal inequalities for eigenvalues of elliptic operators in divergence form on domains in complete noncompact Riemannian manifolds

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Abstract

In this paper, we study the eigenvalue problem of elliptic operators in divergence form, and obtain some universal inequalities for eigenvalues of elliptic operators in divergence form on domains in complete simple connected noncompact Riemannian manifolds admitting special functions which include hyperbolic space. Especially, by using our universal inequalities, we can get the different universal inequalities including Yang inequality.

Mathematics Subject Classification: 35P15, 53C42, 58G25

Keywords: Elliptic operators in divergence form, eigenvalues, universal inequalities, complete noncompact Riemannian manifolds, hyperbolic space

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Article Info: Received: April 20, 2013. Revised: May 30, 2013

Published online: June 25, 2013
1 Introduction

Let $\Omega$ be a bounded domain in an $n$-dimensional complete Riemannian manifold $M$. Let $\Delta$ be the Laplacian operator acting on functions on $M$ and consider the following eigenvalues problem for the Laplacian operator

$$\begin{cases}
\Delta u = -\lambda u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}$$

(1.1)

it is known that this eigenvalue problem has a discrete spectrum,

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots,$$

where each eigenvalue is repeated with its multiplicity. When $M = \mathbb{R}^n, \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, Payne-Pólya-Weinberger [11] in 1956 proved

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{i=1}^k \lambda_i.$$  

(1.2)

In 1980, Hile-Protter [9] strengthened (1.1), and proved

$$\frac{kn}{4} \leq \sum_{i=1}^k \frac{\lambda_i}{\lambda_{k+1} - \lambda_i}.$$  

(1.3)

In 1991, Yang [13] gave the following much stronger inequality

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i.$$  

(1.4)

From inequality (1.3), we can get a weaker but explicit form

$$\lambda_{k+1} \leq \left(1 + \frac{4}{n}\right) \frac{1}{k} \left(\sum_{i=1}^k \lambda_i\right).$$  

(1.5)

These inequalities are called universal inequalities because they do not involve domain dependence.

In the following, we introduce an elliptic operator in divergence form in Riemannian manifold. Let $(M, \langle \cdot, \cdot \rangle)$ be an $n$-dimensional compact Riemannian manifold with boundary $\partial M$ (possibly empty). Let $A : M \to \text{End}(TM)$ be a
smooth symmetric and positive definite section of the bundle of all endomorphisms of $T M$. Let $V$ be a nonnegative continuous function on $M$. Denote by $\Delta$ and $\nabla$ the Laplacian and the gradient operator of $M$ respectively. Define

$$Lu = -\text{div}(A\nabla u) + Vu,$$  

where for a vector field $X$ on $M$, $\text{div}X$ denotes the divergence of $X$. The operator $L$ defined in (1.5) is an elliptic operator in divergence form. It is easy to see that the Laplacian operator and Schrödinger operator are its special cases. In 2010, do Carmo-Wang-Xia [6] considered the eigenvalue problem of the elliptic operator in divergence form with weight such that

$$Lu = \lambda \rho u, \quad \text{in } \Omega,$$  

$$u = 0, \quad \text{on } \partial \Omega,$$  

where $\rho$ is a weight function which is positive and continuous on $M$. They got a Yang type inequality

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4c_2^3 \rho_2}{n \rho_1^2} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left( \frac{1}{\xi_1} \left( \lambda_i - \frac{V_0}{\rho_2} \right) + \frac{n^2 H_0^2}{4 \rho_1} \right),$$  

where $\xi_1 I \leq A$ and $\text{tr}(A) \leq n \xi_2$ throughout $M$, $\rho_1 \leq \rho(x) \leq \rho_2$, $\forall x \in M$, $I$ is the identity map, $\xi_1, \xi_2, \rho_1, \rho_2$ are positive constants, $H_0 = \max_M(|H|)$, $V_0 = \min_M(V)$, and $|H|$ is the mean curvature of $M$ immersed into some Euclidean space $R^N$. Recently, Du-Wu-Li[7] considered the problem (1.7) on complete Riemannian manifolds, they proved the inequality

$$\sum_{i=1}^{k} f(\lambda_i) \leq \frac{2 \rho_2}{n \rho_1} \left( \frac{n \xi_2}{\xi_1} \sum_{i=1}^{k} g(\lambda_i) \right)^{\frac{1}{2}}$$  

$$\times \left( \sum_{i=1}^{k} \frac{(f(\lambda_i))^2}{(\lambda_{k+1} - \lambda_i)g(\lambda_i)} \left( \lambda_i + \frac{S}{\rho_1} \right)^{\frac{1}{2}} \right),$$  

where $(f, g) \in \mathcal{Z}_{\lambda_{k+1}}$, $S = \sup_M \left( \frac{n^2 \xi_2}{4} |H|^2 - V \right)$, and $H$ is the mean curvature vector field.

In this paper, we will consider the eigenvalue problem (1.7) on bounded domains in the complete simply connected noncompact Riemannian manifolds. Then we obtain.
Theorem 1.1. Let \((M, \langle \cdot, \cdot \rangle)\) be an \(n(n \geq 3)\)-dimensional complete noncompact simply connected Riemannian manifold with Sectional curvature \(\text{Sec}\) satisfying
\[-K^2 \leq \text{Sec} \leq -k^2.\]
For a bounded domain \(\Omega\), let \(A : \Omega \to \text{End}(T\Omega)\) be a smooth symmetric and positive definite section of the bundle of all endomorphisms of \(T\Omega\), and assume that \(\xi_1 I \leq A \leq \xi_2 I\) throughout \(\Omega\), \(\forall x \in \Omega, \rho_1 \leq \rho(x) \leq \rho_2\), here \(\xi_1, \xi_2, \rho_1, \rho_2\) are positive constants. Let \(\lambda_i\) be the \(i^{th}\) eigenvalue of the eigenvalue problem (1.7), then for any \((f, g) \in \mathcal{S}_{\lambda_{k+1}}\), we have
\[
\sum_{i=1}^{k} f(\lambda_i) \leq \left\{ \frac{1}{\rho_2 \xi_2} \sum_{i=1}^{k} g(\lambda_i) \right\}^{\frac{1}{2}} \times \left\{ \sum_{i=1}^{k} \frac{(f(\lambda_i))^2}{(\lambda_{k+1} - \lambda_i) g(\lambda_i)} \left( \frac{1}{\xi_1} (\rho_2 \lambda_i - V_0) - (n - 1)^2 \right) \right\}^{\frac{1}{2}} \tag{1.10}
\]
where \(V_0 = \min_{x \in \Omega} V(x)\).
From Theorem 1.1, we can get the following.

Corollary 1.2. Under the assumption of Theorem 1.1, if \(M\) is a hyperbolic space \(H^n(-1)\), we have
\[
\sum_{i=1}^{k} f(\lambda_i) \leq \left\{ \frac{1}{\rho_2 \xi_2} \sum_{i=1}^{k} g(\lambda_i) \right\}^{\frac{1}{2}} \times \left\{ \sum_{i=1}^{k} \frac{(f(\lambda_i))^2}{(\lambda_{k+1} - \lambda_i) g(\lambda_i)} \left( \frac{1}{\xi_1} (\rho_2 \lambda_i - V_0) - (n - 1)^2 \right) \right\}^{\frac{1}{2}} \tag{1.11}
\]
where \(V_0 = \min_{x \in \Omega} V(x)\).

Remark. Taking \((f(\lambda_i), g(\lambda_i)) = ((\lambda_{k+1} - \lambda_i)^2, (\lambda_{k+1} - \lambda_i)^2)\) in (1.11), we can get a Yang inequality that
\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{1}{\rho_2 \xi_2} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left( \frac{1}{\xi_1} (\rho_2 \lambda_i - V_0) - (n - 1)^2 \right)
\]
So, by choosing different \((f(\lambda_i), g(\lambda_i))\), we can get different universal inequalities.
2 Preliminaries

In this section, firstly, we shall introduce a family of couples of functions [8].

**Definition 2.1.** Let $\lambda > 0$, a couple $(f, g)$ of functions defined on $[0, \lambda]$ belongs to $\mathcal{I}_\lambda$ as that

(i) $f$ and $g$ are positive,

(ii) $f$ and $g$ satisfy the following condition, for any $x, y \in [0, \lambda]$, such that $x \neq y$,

$$
\left( \frac{f(x) - f(y)}{x - y} \right)^2 + \left( \frac{(f(x))^2}{g(x)(\lambda - x)} + \frac{(f(y))^2}{g(y)(\lambda - y)} \right) \left( \frac{g(x) - g(y)}{x - y} \right) \leq 0.
$$

We can easily find that $g(x)$ is a nonincreasing function.

In the following, we will introduce a lemma which is obtained by Du-Wu-Li[7].

**Lemma 2.1.** Let $(M, \langle \cdot, \cdot \rangle)$ be an $n$-dimensional compact Riemannian manifold with boundary $\partial M$ (possibly empty). Let $\lambda_i$ be the $i^{th}$ eigenvalue of the eigenvalue problem of elliptic operators in divergence form with weight $\rho$ such that

$$
Lu = \lambda \rho u \text{ in } M, \quad u = 0 \text{ on } \partial M,
$$

and $u_i$ be the orthonormal eigenfunction corresponding to $\lambda_i$, that is,

$$
Lu_i = \lambda_i \rho u_i \text{ in } M, \quad \text{and } u_i = 0 \text{ on } \partial M,
$$

$$
\int_M \rho u_i u_j = \delta_{ij}, \quad \forall \ i, j = 1, 2, \cdots
$$

Then for any $h \in C^2(M)$ and $(f, g) \in \mathcal{I}_{\lambda_{k+1}}$, we have

$$
\sum_{i=1}^{k} f(\lambda_i) \|u_i \nabla h\|^2 \leq \delta \sum_{i=1}^{k} g(\lambda_i) \int_M u_i^2 \langle \nabla h, A \nabla h \rangle + \sum_{i=1}^{k} \frac{(f(\lambda_i))^2}{\delta(\lambda_{k+1} - \lambda_i)g(\lambda_i)} \left\| \frac{1}{\sqrt{\rho}} \left( \langle \nabla u_i, \nabla h \rangle + \frac{1}{2} u_i \Delta h \right) \right\|^2,
$$

(2.2)
where $\delta$ is any positive constant and $\|f\|^2 = \int_M f^2$.

From this Lemma, if we can find a “nice” function on bounded domains in complete Riemannian manifolds and take it in (2.2) and (2.3), we can get universal inequalities in complete Riemannian manifolds. In the following, we will introduce a function on a bounded domain in complete noncompact simple connected Riemannian manifolds(for the more details about this function, we refer [5]).

Let $(M, \langle, \rangle)$ be an $n$-dimensional complete noncompact Riemannian manifold, and $\Omega$ be a bounded connected domain in $M$. Sec($M$) is the section curvature satisfying $-K^2 \leq \text{Sec}(M) \leq -k^2$, where $k, K$ is constants and $0 \leq k \leq K$. When $p$ is not in $\overline{\Omega}$, define the distance function $r(x) = d(x, p)$, then

$$(n-1)k \frac{\cosh kr}{\sinh kr} \leq \Delta r \leq (n-1)K \frac{\cosh Kr}{\sinh Kr},$$

because of $\partial_r \Delta r = -|\text{Hess} \ r|^2 - \text{Ric}(\partial_r, \partial_r)$, so

$$-\partial_r \Delta r \leq (n-1)K^2 \frac{\cosh Kr}{\sinh Kr} - (n-1)k^2,$$

where Hess, Ric are the Hessian operator and Ricci curvature operator, respectively.

3 Proofs of the main results

In this section, we will give the proofs of Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. Taking $h = r$ in the (2.2), we can get

$$\sum_{i=1}^{k} f(\lambda_i) \|u_i\|^2 \leq \delta \sum_{i=1}^{k} g(\lambda_i) \int_M u_i^2 \langle \nabla r, A \nabla r \rangle + \sum_{i=1}^{k} \frac{(f(\lambda_i))^2}{\delta(\lambda_{k+1} - \lambda_i)g(\lambda_i)} \left\| \frac{1}{\sqrt{p}} \left( \langle \nabla u_i, \nabla r \rangle + \frac{1}{2} u_i \Delta r \right) \right\|^2,$$  

(3.1)
Because of \( \rho_1 \leq \rho(x) \leq \rho_2, \xi_1 I \leq A \leq \xi_2 I, \) and (2.1), we have
\[
\frac{1}{\sqrt{\rho}} \left\| u_i \right\|^2 \geq \frac{1}{\rho_2} \tag{3.2}
\]
\[
\xi_1 \leq \langle \nabla r, A \nabla r \rangle \leq \xi_2. \tag{3.3}
\]

Take (3.2) and (3.3) into (3.1), we have
\[
\left\| \frac{1}{\sqrt{\rho}} (2 \langle \nabla u_i, \nabla r \rangle + u_i \Delta r) \right\|^2 \tag{3.4}
\]

It is known that
\[
\lambda_i = \int_M u_i L(u_i) = \int_M u_i (-\text{div}(A \nabla u_i) + V u_i)
\]
\[
= \int_M (\langle Au_i, u_i \rangle + V u_i^2)
\]
\[
\geq \int_M (\xi_1 |\nabla u_i|^2 + V u_i^2),
\]
which yields
\[
\int_M |\nabla u_i|^2 \leq \frac{1}{\xi_1} \left( \lambda_i - \frac{V_0}{\rho_2} \right). \tag{3.5}
\]

From (2.4) and (2.5), we can get
\[
\left\| \frac{1}{\sqrt{\rho}} (2 \langle \nabla u_i, \nabla r \rangle + u_i \Delta r) \right\|^2
\]
\[
= \int_\Omega \frac{1}{\rho} (2 \langle \nabla r, \nabla u_i \rangle + u_i \Delta r)^2
\]
\[
\leq \frac{1}{\rho_1} \left( 4 \int_\Omega \langle \nabla r, \nabla u_i \rangle^2 - \int_\Omega u_i^2 (\Delta r)^2 - 2 \int_\Omega u_i^2 \langle \nabla r, \nabla \Delta r \rangle \right)
\]
\[
\leq \frac{1}{\rho_1} \left( \int_\Omega |\nabla u_i|^2 - \int_\Omega u_i^2 (\Delta r)^2 - 2 \int_\Omega u_i^2 \partial_\rho (\Delta r) \right)
\]
\[
\leq \frac{1}{\rho_1 \xi_1} \left( \lambda_i - \frac{V_0}{\rho_2} \right) - \frac{(n-1)^2 k^2}{\rho_1} \int_\Omega u_i^2 \frac{\cosh^2 k r}{\sinh^2 k r}
\]
\[
+ \frac{2(n-1) K^2}{\rho_1} \int_\Omega u_i^2 \frac{\cosh^2 K r}{\sinh^2 K r} - \frac{2(n-1) k^2}{\rho_1 \rho_2}
\]
\[
\leq \frac{1}{\rho_1 \xi_1} \left( \lambda_i - \frac{V_0}{\rho_2} \right) - \frac{(n^2-1) k^2}{\rho_1 \rho_2} + \frac{2(n-1) K^2}{\rho_1 \rho_2}
\]
\[
- \frac{(n-1)^2}{\rho_1} \int_\Omega u_i^2 \frac{k^2}{\sinh^2 k r} + \frac{2(n-1)}{\rho_1} \int_\Omega u_i^2 \frac{K^2}{\sinh^2 K r}. \tag{3.6}
\]
Since \( K \geq k \geq 0, r > 0 \), we obtain
\[
\frac{K}{\sinh Kr} \leq \frac{k}{\sinh kr},
\]
(3.7)

Since \( n \geq 3 \), we get
\[
(n - 1)^2 \frac{k^2}{\sinh^2 kr} - 2(n - 1) \frac{K^2}{\sinh^2 Kr} \geq (n - 1)(n - 3) \frac{k^2}{\sinh^2 kr} \geq 0,
\]
(3.8)

so, by (3.7)-(3.9), we can get
\[
\left\| \frac{1}{\rho} (u_i \Delta r + 2(\nabla r, \nabla u_i)) \right\|^2 \leq \frac{1}{\rho_1 \xi_1} \left( \lambda_i - \frac{V_0}{\rho_2} \right) - \frac{(n^2 - 1)k^2}{\rho_1 \rho_2} + \frac{2(n - 1)K^2}{\rho_1 \rho_2}
\]
(3.9)

Taking (3.10) into (3.4), we obtain
\[
\frac{1}{\rho_2} \sum_{i=1}^{k} f(\lambda_i) \leq \frac{\xi_2}{\rho_1} \delta \sum_{i=1}^{k} g(\lambda_i) + \sum_{i=1}^{k} \frac{(f(\lambda_i))^2}{4\delta(\lambda_{k+1} - \lambda_i)g(\lambda_i)}
\times \left( \frac{1}{\rho_1 \xi_1} \left( \lambda_i - \frac{V_0}{\rho_2} \right) - \frac{(n^2 - 1)k^2}{\rho_1 \rho_2} + \frac{2(n - 1)K^2}{\rho_1 \rho_2} \right)
\]
(3.10)

above inequality implies that
\[
\sum_{i=1}^{k} f(\lambda_i) \leq \frac{\rho_2 \xi_2}{\rho_1} \delta \sum_{i=1}^{k} g(\lambda_i) + \sum_{i=1}^{k} \frac{(f(\lambda_i))^2}{4\delta(\lambda_{k+1} - \lambda_i)g(\lambda_i)}
\times \frac{1}{\rho_1} \left( \frac{1}{\xi_1} (\rho_2 \lambda_i - V_0) - (n^2 - 1)k^2 + 2(n - 1)K^2 \right)
\]
(3.11)

\[
\delta = \left\{ \frac{1}{4\rho_2 \xi_2} \frac{\sum_{i=1}^{k} (f(\lambda_i))^2}{\sum_{i=1}^{k} g(\lambda_i)} \left( \frac{1}{\xi_1} (\rho_2 \lambda_i - V_0) - (n^2 - 1)k^2 + 2(n - 1)K^2 \right) \right\}^{\frac{1}{2}}
\]

(3.11) becomes
\[
\sum_{i=1}^{k} f(\lambda_i) \leq \left\{ \frac{1}{\rho_2 \xi_2} \sum_{i=1}^{k} g(\lambda_i) \right\}^{\frac{1}{2}} \times \left\{ \sum_{i=1}^{k} \frac{(f(\lambda_i))^2}{(\lambda_{k+1} - \lambda_i)g(\lambda_i)} \right\}^{\frac{1}{2}} \left( \frac{1}{\xi_1} (\rho_2 \lambda_i - V_0) - (n^2 - 1)k^2 + 2(n - 1)K^2 \right)^{\frac{1}{2}}
\]
(3.12)

This completes the proof of Theorem 1.1. □
Proof of Corollary 1.2. Taking $K = k = 1$ in (3.12), we immediately obtain (1.11). □

ACKNOWLEDGEMENTS. The research work is supported by Key Laboratory of Applied Mathematics of Hubei Province and The research project of Jingchu University of Technology.

References


