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On α-uniformly close-to-convex and quasi-convex functions with negative coefficients

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Abstract

In this paper we study a class of α-uniformly starlike functions with negative coefficients, a class of α-uniformly convex functions with negative coefficients, a class of α-uniformly close-to-convex functions with negative coefficients and a class of quasi-convex functions with negative coefficients.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U,

$$A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\} \tag{1}$$

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and $S = \{f \in A : f$ is univalent in $U\}$.

In [3] the subfamily T of S consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ \ a_j \geq 0, j = 2, 3, ...,\ \ z \in U \tag{2}$$

was introduced.

Let $T(n, p)$ denote the class of functions of the form

$$f(z) = z^p - \sum_{p=j}^{\infty} a_p + pz^{l+p},\ \ , a_{l+p} \geq 0,\ p, j \in \mathbb{N} = \{1, 2, ...\}, \tag{3}$$

which are analytic in U. We have $T(1,1) = T$.

The purpose of this paper is to define a class of α-uniformly close-to-convex and quasi-convex functions with negative coefficients. For this, we make use of the following well known results, which are taken from literature.

2 Preliminary Results

We begin with the assertions concerning the starlike functions with negative coefficients (e.g. Theorem 2.1), we continue with the operator $I_{c+\delta}$ (see (4)) and we end by recalling some known results from [5] and [6] that we use forward in our study. The methods used to prove our results are taken from literature.

Theorem 2.1. [2] *If* $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j,\ a_j \geq 0,\ j = 2, 3, ...,\ z \in U$ *then the next assertions are equivalent:*

(i) $\sum\limits_{j=2}^{\infty} j a_j \leq 1$

(ii) $f \in T$

(iii) $f \in T^*$, *where* $T^* = T \bigcap S^*$ *and* S^* *is the well-known class of starlike functions.*

Definition 2.1. [2] *Let* $\alpha \in [0, 1)$ *and* $n \in \mathbb{N}$, *then*

$$S_n(\alpha) = \left\{ f \in A : Re \frac{D^{n+1} f(z)}{D^n f(z)} > \alpha, z \in U \right\}$$

is the set of n-starlike functions of order α.
Also, we denote $T_n(\alpha) = T \bigcap S_n(\alpha)$.

In [1] is defined the integral operator:
$I_{c+\delta} : A \to A$, $c < u \leq 1$, $1 \leq \delta < \infty$, $0 < c < \infty$, with

$$f(z) = I_{c+\delta}(F(z)) = (c+\delta) \int_0^1 u^{c+\delta-2} F(uz) du. \tag{4}$$

Remark 2.1. *If $F(z) = z + \sum_{j=2}^{\infty} c_j z^j$, the*

$$f(z) = I_{c+\delta}(F(z)) = z + \sum_{j=2}^{\infty} \frac{c+\delta}{c+j+\delta-1} a_j z^j.$$

Also we notice that $0 < \dfrac{c+\delta}{c+j+\delta-1} < 1$, where $c \in (0,\infty), j \geq 2$, $\delta \in [1,\infty)$.

Remark 2.2. *It is easy to prove that for $F(z) \in T$ and $f(z) = I_{c+\delta}(F(z))$ we have $f(z) \in T$, where $I_{c+\delta}$ is the integral operator defined by (4).*

In [5] are presented the following classes of analytic functions:

Definition 2.2. [5] *Let C_S^* denote the class of functions in S satisfying the following inequality:*

$$Re\left\{\frac{(zf'(z))'}{f'(z)+f'(-z)}\right\} > 0, \quad (z \in U). \tag{5}$$

Definition 2.3. [5] *Let $UST^{(k)}(\alpha,\beta)$ denote the class of functions in T satisfying the following inequality:*

$$Re\left\{\frac{zf'(z)}{f_k(z)}\right\} > \alpha \left|\frac{zf'(z)}{f_k(z)} - 1\right| + \beta, \quad (z \in U), \tag{6}$$

where $\alpha \geq 0$, $0 \leq \beta < 1$, $k \geq 1$ is a fixed positive integer and $f_k(z)$ are defined by the following equality:

$$f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^{\nu} z), \quad (\varepsilon^k = 1, \; z \in U). \tag{7}$$

If $k = 1$, then the class $UST^{(k)}(\alpha, \beta)$ reduces to the class of α-uniformly starlike functions of order β. If $k = 2$, $\alpha = 0$ and $\beta = 0$, then the class $UST^{(k)}(\alpha, \beta)$ reduces to the class S_S^* of starlike functions with respect to symmetric points.

From [4] we know that if $f(z) \in S$,

$$Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0,\ z \in U\,. \tag{8}$$

Definition 2.4. [5] *Let $UCV^{(k)}(\alpha, \beta)$ denote the class of functions in T satisfying the following inequality:*

$$Re\left\{\frac{(zf'(z))'}{f_k'(z)}\right\} > \alpha\left|\frac{(zf'(z))'}{f_k'(z)} - 1\right| + \beta,\ \ (z \in U), \tag{9}$$

where $\alpha \geq 0$, $0 \leq \beta < 1$, $k \geq 1$ is a fixed positive integer and $f_k(z)$ are defined by (7).

If $k = 1$, then the class $UCV^{(k)}(\alpha, \beta)$ reduces to the class of α-uniformly convex functions of order β. If $k = 2$, $\alpha = 0$ and $\beta = 0$, then the class $UCV^{(k)}(\alpha, \beta)$ reduces to the class C_S^*.

Theorem 2.2. [5] *Let $\alpha \geq 0$, $0 \leq \beta < 1$, $k \geq 1$ be a fixed positive integer and $f(z) \in T$. Then $f(z) \in UST^{(k)}(\alpha, \beta)$ iff*

$$\sum_{j=1}^{\infty}[(1+\alpha)(jk+1) - (\alpha+\beta)] \cdot a_{jk+1} + \tag{10}$$

$$\sum_{j=2,\, j\neq lk+1}^{\infty}(1+\alpha)ja_j < 1 - \beta.$$

Theorem 2.3. [6] *Let $\alpha \geq 0$, $0 \leq \beta < 1$, $k \geq 1$ be a fixed positive integer and $f(z) \in T$. Then $f(z) \in UCV^{(k)}(\alpha, \beta)$ if and only if*

$$\sum_{j=1}^{\infty}(jk+1)[(1+\alpha)(jk+1) - (\alpha+\beta)] \cdot a_{jk+1} + \tag{11}$$

$$\sum_{j=2,\, j\neq lk+1}^{\infty}(1+\alpha)j^2a_j < 1 - \beta.$$

Definition 2.5. [6] *Let $C^{(k)}(\lambda,\alpha)$ denote the class of functions in A satisfying the following inequality:*

$$Re\left\{\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f_k(z)+\lambda z f_k'(z)}\right\} > \alpha,\ \ (z \in U), \tag{12}$$

where $0 \le \alpha < 1$, $0 \le \lambda \le 1$, $k \ge 2$ is a fixed positive integer and $f_k(z)$ is defined by equality (7).

Definition 2.6. [6] *Let $QC^{(k)}(\lambda,\alpha)$ denote the class of functions in A satisfying the following inequality:*

$$Re\left\{z\cdot\frac{\lambda z^2 f'''(z)+(2\lambda+1)zf''(z)+f'(z)}{\lambda z^2 f_k''(z)+zf_k'(z)}\right\} > \alpha,\ \ (z \in U), \tag{13}$$

where $0 \le \alpha < 1$, $0 \le \lambda \le 1$, $k \ge 2$ is a fixed positive integer and $f_k(z)$ is defined by equality (7).

For convenience we write $C^{(k)}(\lambda,\alpha)\cap T$ as $C_T^{(k)}(\lambda,\alpha)$ and $QC^{(k)}(\lambda,\alpha)\cap T$ as $QC_T^{(k)}(\lambda,\alpha)$.

Theorem 2.4. [6] *Let $0 \le \alpha < 1$, $0 \le \lambda < 1$, $k \ge 2$ be a fixed positive integer and $f(z) \in T$, then $f(z) \in C_T^{(k)}(\lambda,\alpha)$ iff*

$$\sum_{j=1}^{\infty}(1+\lambda jk)(jk+1-\alpha)\cdot a_{jk+1}+ \tag{14}$$

$$\sum_{j=2,\, j\neq lk+1}^{\infty}[1+\lambda(j-1)]\cdot ja_j \le 1-\alpha.$$

Theorem 2.5. [6] *Let $0 \le \alpha < 1$, $0 \le \lambda < 1$, $k \ge 2$ be a fixed positive integer and $f(z) \in T$, then $f(z) \in QC_T^{(k)}(\lambda,\alpha)$ if and only if*

$$\sum_{j=1}^{\infty}(jk+1)(1+\lambda jk)(jk+1-\alpha)\cdot |a_{jk+1}|+ \tag{15}$$

$$\sum_{j=2,\, j\neq lk+1}^{\infty}[1+\lambda(j-1)]\cdot j^2|a_j| \le 1-\alpha.$$

3 Main results

We firstly apply the operator $I_{c+\delta}$ (see (4)) on a α-uniformly starlike function of order β with negative coefficients and we prove that the resulting function conserves in the same class of α-uniformly starlike functions of order β with negative coefficients.

Theorem 3.1. *Let* $F(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \geq 0,\ j \geq 2,\ F(z) \in UST^{(k)}(\alpha, \beta),$ $\alpha \geq 0,\ 0 \leq \beta < 1,\ k \geq 1$ *be a fixed positive integer. Then* $f(z) = I_{c+\delta}(F(z)) \in UST^{(k)}(\alpha, \beta)$, *where* $I_{c+\delta}$ *is the integral operator defined by (4).*

Proof. From Remark 2.2 we obtain $f(z) = I_{c+\delta}(F(z)) \in T$. From Remark 2.1 we have: $f(z) = z - \sum_{j=2}^{\infty} \frac{c+\delta}{c+j+\delta-1} \cdot a_j z^j$, where $0 < c < \infty,\ j \geq 2,$ $1 \leq \delta < \infty$.
From $F(z) \in UST^{(k)}(\alpha, \beta)$, by using Theorem 2.2, we have:

$$\sum_{j=1}^{\infty} [(1+\alpha)(jk+1) - (\alpha+\beta)] \cdot a_{jk+1} + \tag{16}$$

$$\sum_{j=2,\, j \neq lk+1}^{\infty} (1+\alpha) j a_j < 1 - \beta.$$

Using again Theorem 2.2 we observe that it is sufficient to prove that:

$$\sum_{j=1}^{\infty} [(1+\alpha)(jk+1) - (\alpha+\beta)] \cdot \frac{c+\delta}{c+jk+\delta} + \tag{17}$$

$$\sum_{j=2,\, j \neq lk+1}^{\infty} (1+\alpha) j \cdot \frac{c+\delta}{c+j+\delta-1} < 1 - \beta.$$

From hypothesis we have

$$0 < \frac{c+\delta}{c+jk+\delta} < 1 \quad and \quad 0 < \frac{c+\delta}{c+j+\delta-1} < 1. \tag{18}$$

Thus, we see that, by using (16) and (18), the condition (17) holds. This means that $f(z) \in UST^{(k)}(\alpha, \beta)$. □

Using a similar method as in Theorem 3.1, we apply the operator $I_{c+\delta}$ (see (4)) on a α-uniformly convex function of order β with negative coefficients and we prove that the resulting function conserves in the same class of α-uniformly convex functions of order β with negative coefficients.

Theorem 3.2. *Let* $F(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \geq 0,\ j \geq 2,\ F(z) \in UCV^{(k)}(\alpha, \beta)$, $\alpha \geq 0,\ 0 \leq \beta < 1$, $k \geq 1$ *be a fixed positive integer. Then* $f(z) = I_{c+\delta}(F(z)) \in UCV^{(k)}(\alpha, \beta)$, *where* $I_{c+\delta}$ *is the integral operator defined by (4).*

Next, we apply the operator $I_{c+\delta}$ (see (4)) on a α-uniformly close to convex function of order β with negative coefficients and we prove that the resulting function conserves in the same class of α-uniformly close to convex functions of order β with negative coefficients.

Theorem 3.3. *Let* $F(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \geq 0,\ j \geq 2,\ F(z) \in C_T^{(k)}(\alpha, \beta)$, $\alpha \geq 0,\ 0 \leq \beta < 1$, $k \geq 1$ *be a fixed positive integer. Then* $f(z) = I_{c+\delta}(F(z)) \in C_T^{(k)}(\alpha, \beta)$, *where* $I_{c+\delta}$ *is the integral operator defined by (4).*

Proof. From Remark 2.2 we have $f(z) = I_{c+\delta}(F(z)) \in T$. From Remark 2.1 we have: $f(z) = z - \sum_{j=2}^{\infty} \frac{c+\delta}{c+j+\delta-1} \cdot a_j z^j$, where $0 < c < \infty$, $j \geq 2$, $1 \leq \delta < \infty$.
From $F(z) \in C_T^{(k)}(\alpha, \beta)$, by using Theorem 2.4, we have:

$$\sum_{j=1}^{\infty}(1+\lambda jk)(jk+1-\alpha) \cdot a_{jk+1} + \tag{19}$$

$$\sum_{j=2,\, j\neq lk+1}^{\infty} [1+\lambda(j-1)]ja_j \leq 1-\alpha.$$

Using again Theorem 2.4 we notice that it is sufficient to prove that:

$$\sum_{j=1}^{\infty}(1+\lambda jk)(jk+1-\alpha) \cdot \frac{c+\delta}{c+jk+\delta} + \tag{20}$$

$$\sum_{j=2,\, j\neq lk+1}^{\infty} [1+\lambda(j-1)]j \cdot \frac{c+\delta}{c+j+\delta-1} \leq 1-\alpha.$$

From hypothesis we have

$$0 < \frac{c+\delta}{c+jk+\delta} < 1 \quad and \quad 0 < \frac{c+\delta}{c+j+\delta-1} < 1. \tag{21}$$

Thus, we obtain, by using (19) and (21), that the condition (20) holds. This means that $f(z) \in C_T^{(k)}(\alpha, \beta)$. □

We end our research by taking into account a similar method as in Theorem 3.3, where we apply the operator $I_{c+\delta}$ (see (4)) on a quasi-convex function of order β with negative coefficients and we prove that the resulting function conserves in the same class of quasi-convex functions of order β with negative coefficients.

Theorem 3.4. *Let* $F(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \geq 0,\ j \geq 2,\ F(z) \in QC_T^{(k)}(\alpha, \beta)$, $\alpha \geq 0,\ 0 \leq \beta < 1,\ k \geq 1$ *be a fixed positive integer. Then* $f(z) = I_{c+\delta}(F(z)) \in QC_T^{(k)}(\alpha, \beta)$, *where* $I_{c+\delta}$ *is the integral operator defined by (4).*

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