Theoretical Mathematics & Applications, vol.3, no.2, 2013, 15-28
ISSN: 1792-9687 (print), 1792-9709 (online)
Scienpress Ltd, 2013

Estimates eigenvalues of fourth-order weighted polynomial operator on a hyperbolic space

Feng Du[1] **and Yanli Li**[2]

Abstract

In this paper, we consider the eigenvalue problem of fourth-order weighted polynomial operator on bounded domains in a hyperbolic space, and get a general inequality. By using this inequality, we obtain some universal inequalities of the eigenvalues. Moreover, by these universal inequalities, we can get some results for the biharmonic operator.

Mathematics Subject Classification: 35P15, 53C40
Keywords: Fourth-order weighted polynomial operator, eigenvalues, universal inequalities, hyperbolic space

[1] School of Mathematics and Physics Science, Jingchu University of Technology, Hubei Jingmen 448000, P.R. China.
[2] School of Electronic and Information Science, Jingchu University of Technology, Hubei Jingmen 448000, P.R. China.

Article Info: *Received* : April 20, 2013. *Revised* : May 30, 2013
Published online : June 25, 2013

1 Introduction

Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Let Δ be the Laplacian operator acting on functions on M and consider the following eigenvalue problem for the biharmonic operator

$$\begin{cases} \Delta^2 u = -\lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν denotes the outward unit normal vector field of $\partial\Omega$. It is known that this eigenvalue problem has a discrete spectrum

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots ,$$

where each eigenvalue is repeated with its multiplicity. When $M = R^n, \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, Payne-Pólya-Weinberger [9] in 1956 proved

$$\lambda_{k+1} - \lambda_k \leq \frac{8n+2}{n^2} \sum_{i=1}^k \lambda_i. \tag{1.2}$$

In 1984, Hile and Yeh [6] strengthened (1.2), and proved

$$\frac{n^2 k^{\frac{3}{2}}}{8n+2} \left(\sum_{i=1}^k \lambda_i \right)^{\frac{1}{2}} \leq \sum_{i=1}^k \frac{\lambda_i^{\frac{1}{2}}}{\lambda_{k+1} - \lambda_i}. \tag{1.3}$$

In 2006, Cheng-Yang [3] gave the following much stronger inequality

$$\lambda_{k+1} \leq \frac{1}{k} \sum_{i=1}^k \lambda_i + \left(\frac{8(n+2)}{n^2} \right)^{\frac{1}{2}} \frac{1}{k} \left(\sum_{i=1}^k \lambda_i (\lambda_{k+1} - \lambda_i) \right). \tag{1.4}$$

These inequalities are called universal inequalities because they do not involve domain dependence.

When M is a hyperbolic $H^n(-1)$, Cheng-Yang[4] have proved the following inequality

$$\begin{aligned} & \sum_{i,j=1}^k (\lambda_{k+1} - \lambda_i)^2 \\ \leq \; & 24 \sum_{i,j=1}^k (\lambda_{k+1} - \lambda_i) \left(\lambda_i^{\frac{1}{2}} - \frac{(n-1)^2}{4} \right) \left(\lambda_j^{\frac{1}{2}} - \frac{(n-1)^2}{6} \right). \end{aligned} \tag{1.5}$$

In this paper, we consider the following eigenvalue problem of fourth-order weighted polynomial operator on a bounded domains Ω in the hyperbolic space $H^n(-1)$ such that

$$\begin{cases} \left(\Delta^2 - a\Delta + b\right) u = \lambda \rho u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where ρ is a positive and continuous function on Ω, and the constants $a, b \geq 0$. Then we obtain.

Theorem 1.1. *Let Ω be a bounded domain in n-dimensional hyperbolic $H^n(-1)$ and let λ_i be the i^{th} eigenvalue of the eigenvalue problem (1.6). If $\forall x \in \Omega, \rho_1 \leq \rho(x) \leq \rho_2$, then we have*

$$\begin{aligned} &\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \\ \leq\ & \left\{ \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \left(6\rho_2 A_i - 2(n-1)^2 + \frac{\rho_2((n-1)^2 + a)}{\rho_1} \right) \right\}^{\frac{1}{2}} \\ & \times \left\{ \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \frac{1}{\rho_1} \left(4\rho_2 A_i - (n-1)^2 \right) \right\}^{\frac{1}{2}}. \end{aligned} \tag{1.7}$$

where $A_i = \frac{-a+\sqrt{a^2+4\left(\lambda_i - \frac{b}{\rho_2}\right)}}{2\rho_1}$.

From Theorem 1.1, we can get the following weaker but more explicit inequality.

Corollary 1.2. *Under the assumption of Theorem 1.1, if $\rho \equiv 1$, we have*

$$\begin{aligned} &\sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \\ \leq\ & 24 \sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i) \left(A_i - \frac{(n-1)^2}{4} \right) \left(A_j - \frac{(n-1)^2 - a}{6} \right). \end{aligned} \tag{1.8}$$

where $A_i = \frac{-a+\sqrt{a^2+4\left(\lambda_i - \frac{b}{\rho_2}\right)}}{2\rho_1}$, $A_j = \frac{-a+\sqrt{a^2+4\left(\lambda_j - \frac{b}{\rho_2}\right)}}{2\rho_1}$.

Remark. By (1.9), when $a = b = 0$, (1.8) becomes (1.5), in fact, problem (1.1) is the special case of problem (1.6).

2 A key lemma

In this section, we will introduce a lemma which play a key role in the proofs of the main results of this paper.

Lemma 2.1. *Let $(M, \langle , \rangle)$ be an n-dimensional compact Riemannian manifold with boundary ∂M (possibly empty). Let λ_i be the i^{th} eigenvalue of the eigenvalue problem of fourth-order weighted polynomial operator with weight ρ such that*

$$\begin{cases} \left(\Delta^2 - a\Delta + b\right) u = \lambda \rho u, & in \ \ M, \\ u = \frac{\partial u}{\partial \nu} = 0, & on \ \ \partial M, \end{cases}$$

and u_i be the orthonormal eigenfunction corresponding to λ_i, that is,

$$\begin{cases} \left(\Delta^2 + a\Delta + b\right) u_i = \lambda_i \rho u_i, & in \ \ M, \\ u_i = 0, & on \ \ \partial\Omega, \\ \int_M \rho u_i u_j = \delta_{ij}, & \forall \ i, j = 1, 2, \cdots \end{cases}$$

Then for any $h \in C^4(\overline{M})$, we have

$$\begin{aligned} & \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \int_M u_i^2 |\nabla h|^2 \\ \leq & \sum_{i=1}^{k}\delta(\lambda_{k+1} - \lambda_i)^2 \int_M h u_i p_i \\ & + \sum_{i=1}^{k} \frac{\lambda_{k+1} - \lambda_i}{\delta} \int_M \frac{1}{\rho}\left(\langle \nabla h, \nabla u_i \rangle + \frac{u_i \Delta h}{2}\right)^2, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} p_i \ = \ & 2\langle \nabla h, \nabla(\Delta u_i)\rangle + \Delta h \Delta u_i + 2\Delta\left(\langle \nabla h, \nabla u_i \rangle\right) \\ & + \Delta\left(u_i \Delta h\right) - 2a\langle \nabla h, \nabla u_i\rangle - a u_i \Delta h. \end{aligned}$$

Proof. Let $\varphi_i = hu_i - \sum_{j=1}^{k} a_{ij}u_j$ for any integer $k \geq 1$, where

$$a_{ij} = \sum_{j=1}^{k} \int_M \rho h u_i u_j = a_{ji},$$

then we have

$$\varphi_i|_{\partial M} = 0, \quad \text{and} \int_M \rho \varphi_i u_j = 0, \forall i, j = 1, \cdots, k,$$

from the Rayleigh-Ritz inequality, we get

$$\lambda_{k+1} \int_M \rho \varphi_i^2 \leq \int_M \varphi_i(\Delta^2 + a\Delta + b)\varphi_i. \tag{2.2}$$

By directly computation, we have

$$\Delta(hu_i) = h\Delta u_i + 2\langle \nabla h, \nabla u_i \rangle + u_i \Delta h, \tag{2.3}$$

and

$$\begin{aligned} \Delta^2(hu_i) &= \Delta\left(h\Delta u_i + 2\langle \nabla h, \nabla u_i \rangle + u_i \Delta h, \right) \\ &= h\Delta^2 u_i + 2\langle \nabla h, \nabla(\Delta u_i) \rangle \\ &\quad + \Delta h \Delta u_i + 2\Delta\left(\langle \nabla h, \nabla u_i \rangle\right) + \Delta\left(u_i \Delta h\right). \end{aligned} \tag{2.4}$$

By (2.3) and (2.4), we have

$$(\Delta^2 + a\Delta + b)(hu_i) = \lambda_i \rho h u_i + p_i, \tag{2.5}$$

where

$$\begin{aligned} p_i &= 2\langle \nabla h, \nabla(\Delta u_i) \rangle + \Delta h \Delta u_i + 2\Delta\left(\langle \nabla h, \nabla u_i \rangle\right) \\ &\quad + \Delta\left(u_i \Delta h\right) - 2a\langle \nabla h, \nabla u_i \rangle - a u_i \Delta h. \end{aligned}$$

Because of $\int_M \rho \varphi_i u_j = 0$, we can get

$$\begin{aligned} \int_M \varphi_i(\Delta^2 + a\Delta + b)\varphi_i &= \int_M \varphi_i(\Delta^2 + a\Delta + b)(hu_i) \\ &= \lambda_i \int_M \varphi_i \rho h u_i + \int_M \varphi_i p_i \\ &= \lambda_i \int_M \rho \varphi_i^2 + \int_M h u_i p_i - \sum_{j=1}^{k} a_{ij} b_{ij}, \end{aligned} \tag{2.6}$$

where $b_{ij} = \int_M p_i u_j$.

By (2.2) and (2.6), we have

$$(\lambda_{k+1} - \lambda_i) \int_M \rho \varphi_i^2 \le \int_M h u_i p_i - \sum_{j=1}^{k} a_{ij} b_{ij}, \tag{2.7}$$

Using integration by parts, we have

$$\begin{aligned}
& \int_M \Delta u_j \langle \nabla h, \nabla u_i \rangle - \int_M \Delta u_i \langle \nabla h, \nabla u_j \rangle \\
= & -\int_M h \mathrm{div}(\Delta u_j \nabla u_i) + \int_M h \mathrm{div}(\Delta u_i \nabla u_j) \\
= & -\int_M h \langle \nabla(\Delta u_j), \nabla u_i \rangle + \int_M h \Delta u_j \Delta u_i \\
& + \int_M h \langle \nabla(\Delta u_i), \nabla u_j \rangle - \int_M h \Delta u_i \Delta u_j \\
= & -\int_M h \langle \nabla(\Delta u_j), \nabla u_i \rangle + \int_M h \langle \nabla(\Delta u_i), \nabla u_j \rangle \\
= & \int_M u_i \mathrm{div}(h \nabla(\Delta u_j)) - \int_M u_j \mathrm{div}(h \nabla(\Delta u_i)) \\
= & \int_M h u_i \Delta^2 u_j + \int_M u_i \langle \nabla h, \nabla(\Delta u_j) \rangle - \int_M h u_j \Delta^2 u_i - \int_M u_j \langle \nabla h, \nabla(\Delta u_i) \rangle \\
= & -\int_M h u_i \Delta^2 u_j + \int_M h u_j \Delta^2 u_i - \int_M \Delta u_j \left(\langle \nabla u_i, \nabla h \rangle - u_i \Delta h \right) + \\
& \int_M \Delta u_i \left(\langle \nabla u_j, \nabla h \rangle - u_j \Delta h \right),
\end{aligned} \tag{2.8}$$

which implies that

$$\begin{aligned}
& 2 \int_M \Delta u_j \langle \nabla h, \nabla u_i \rangle - 2 \int_M \Delta u_i \langle \nabla h, \nabla u_j \rangle \\
= & -\int_M h u_i \Delta^2 u_j + \int_M h u_j \Delta^2 u_i + \int_M u_i \Delta u_j \Delta h - \int_M u_j \Delta u_i \Delta h.
\end{aligned} \tag{2.9}$$

We also have

$$\begin{aligned}
& \int_M u_j \Delta \langle \nabla h, \nabla u_i \rangle + \int_M u_j \langle \nabla h, \nabla(\Delta u_i) \rangle \\
= & \int_M \Delta u_j \langle \nabla h, \nabla u_i \rangle - \int_M \Delta u_i \langle \nabla h, \nabla u_j \rangle + \int_M u_j \Delta u_i \Delta h,
\end{aligned} \tag{2.10}$$

$$\int_M u_j \Delta(u_i \Delta h) = \int_M u_i \Delta u_j \Delta h, \tag{2.11}$$

and

$$\begin{aligned}
& \int_M u_j \left\{-2\langle\nabla h, \nabla u_i\rangle + u_i\Delta h\right\} \\
= & \int_M 2h\langle\nabla u_j, \nabla u_i\rangle - \int_M 2hu_j\Delta u_i + \int_M h\Delta(u_iu_j) \\
= & \int_M hu_i\Delta u_j - \int_M hu_j\Delta u_i. \qquad (2.12)
\end{aligned}$$

Combining (2.9)-(2.12), we get

$$\begin{aligned}
b_{ij} &= \int_M p_iu_j \\
&= \int_M u_j \left\{2\langle\nabla h, \nabla(\Delta u_i)\rangle + \Delta h\Delta u_i + 2\Delta\left(\langle\nabla h, \nabla u_i\rangle\right)\right\} \\
&\quad + \int_M u_j \left\{\Delta\left(u_i\Delta h\right) - 2a\langle\nabla h, \nabla u_i\rangle - au_i\Delta h\right\} \\
&= -\int_M hu_i\Delta^2u_j - \int_M hu_j\Delta^2u_i + \int_M ahu_i\Delta u_j - \int_M ahu_j\Delta u_i \\
&= \int_M hu_i(\Delta^2 + a\Delta + b)(u_j) - \int_M hu_j(\Delta^2 + a\Delta + b)(u_i) \\
&= (\lambda_j - \lambda_i)a_{ij}. \qquad (2.13)
\end{aligned}$$

It follows from (2.7) and (2.13) that

$$(\lambda_{k+1} - \lambda_i)\int_M \rho\varphi_i^2 \le \int_M hu_ip_i - \sum_{j=1}^{k}(\lambda_j - \lambda_i)a_{ij}^2. \qquad (2.14)$$

Setting $t_{ij} = \int_M u_j \left(\langle\nabla h, \nabla u_i\rangle + \frac{u_i\Delta h}{2}\right)$, then $t_{ij} = -t_{ji}$ and

$$\begin{aligned}
& \int_M -2\varphi_i\left(\langle\nabla h, \nabla u_i\rangle + \frac{u_i\Delta h}{2}\right) \\
= & \int_M \left(-2hu_i\langle\nabla h, \nabla u_i\rangle - hu_i^2\Delta h\right) + 2\sum_{j=1}^{k} a_{ij}t_{ij} \\
= & \int_M u_i^2|\nabla h|^2 + 2\sum_{j=1}^{k} a_{ij}t_{ij}. \qquad (2.15)
\end{aligned}$$

By (2.14), (2.15) and Schwartz inequality, we get

$$\begin{aligned}
&(\lambda_{k+1}-\lambda_i)^2\left(\int_M u_i^2|\nabla h|^2+2\sum_{j=1}^k a_{ij}t_{ij}\right)\\
=\ &(\lambda_{k+1}-\lambda_i)^2\int_M -2\sqrt{\rho}\varphi_i\left(\frac{1}{\sqrt{\rho}}\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}\right)-\sum_{j=1}^k t_{ij}\sqrt{\rho}u_j\right)\\
\leq\ &\delta(\lambda_{k+1}-\lambda_i)^3\int_M\rho\varphi_i^2\\
&+\frac{\lambda_{k+1}-\lambda_i}{\delta}\int_M\left(\frac{1}{\sqrt{\rho}}\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}\right)-\sum_{j=1}^k t_{ij}\sqrt{\rho}u_j\right)^2\\
\leq\ &\delta(\lambda_{k+1}-\lambda_i)^2\left(\int_M hu_ip_i-\sum_{j=1}^k(\lambda_j-\lambda_i)a_{ij}^2\right)\\
&+\frac{\lambda_{k+1}-\lambda_i}{\delta}\left(\int_M\frac{1}{\rho}\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}\right)^2-\sum_{j=1}^k t_{ij}^2\right),
\end{aligned}\tag{2.16}$$

where δ is any positive constant. Summing over i from 1 to k in (2.16) and noticing $a_{ij}=a_{ji}, t_{ij}=-t_{ji}$, we have

$$\begin{aligned}
&\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_M u_i^2|\nabla h|^2-2\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)a_{ij}t_{ij}\\
\leq\ &\sum_{i=1}^k\delta(\lambda_{k+1}-\lambda_i)^2\int_M hu_ip_i+\sum_{i=1}^k\frac{\lambda_{k+1}-\lambda_i}{\delta}\int_M\frac{1}{\rho}\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}\right)^2\\
&-\sum_{i,j=1}^k\delta(\lambda_{k+1}-\lambda_i)(\lambda_j-\lambda_i)^2a_{ij}^2-\sum_{i,j=1}^k\frac{\lambda_{k+1}-\lambda_i}{\delta}t_{ij}^2,
\end{aligned}\tag{2.17}$$

which gives

$$\begin{aligned}
&\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_M u_i^2|\nabla h|^2\\
\leq\ &\sum_{i=1}^k\delta(\lambda_{k+1}-\lambda_i)^2\int_M hu_ip_i+\sum_{i=1}^k\frac{\lambda_{k+1}-\lambda_i}{\delta}\int_M\frac{1}{\rho}\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}\right)^2
\end{aligned}$$

Hence (2.1) is true, this completes the proof of Lemma 2.1. □

3 Proofs of the main results

In this section, we will give the proofs of Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. Using the upper half-space model, $H^n(-1)$ is given by

$$\mathbf{R}^n_+ = \{(x_1, x_2, \cdots, x_n) | x_n > 0\}$$

with the standard metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

In this case , by a simple computation, we have the Laplacian in $H^n(-1)$:

$$\Delta = x_n^2 \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + (2-n)x_n^2 \frac{\partial}{\partial x_n}$$

Set $f = \ln x_n$, we can get $|\nabla f| = 1, \Delta f = 1 - n$.

Taking $h = f$ in (2.1), and noticing $|\nabla f| = 1$, we can get

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \int_\Omega u_i^2 \;\; &\leq \;\; \sum_{i=1}^{k} \delta(\lambda_{k+1}-\lambda_i)^2 \int_\Omega f u_i p_i \\ &+ \sum_{i=1}^{k} \frac{\lambda_{k+1}-\lambda_i}{\delta} \int_\Omega \frac{1}{\rho} \left(\langle \nabla f, \nabla u_i \rangle + \frac{u_i \Delta f}{2} \right)^2 \end{aligned} \quad (3.1)$$

Because of $\rho_1 \leq \rho(x) \leq \rho_2$ and $\int_\Omega \rho u_i^2 = 1$, we have

$$\int_M u_i^2 \geq \frac{1}{\rho_2} \quad (3.2)$$

Taking (3.2) into (3.1), we have

$$\begin{aligned}\frac{1}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \;\; &\leq \;\; \sum_{i=1}^{k} \delta(\lambda_{k+1}-\lambda_i)^2 \int_\Omega f u_i p_i \\ &+ \sum_{i=1}^{k} \frac{\lambda_{k+1}-\lambda_i}{\delta} \int_\Omega \frac{1}{\rho} \left(\langle \nabla f, \nabla u_i \rangle + \frac{u_i \Delta f}{2} \right)^2 . \end{aligned} \quad (3.3)$$

Since $(\Delta^2 - a\Delta + b)(u_i) = \lambda_i \rho u_i$, then we have

$$\begin{aligned}\int_\Omega u_i \Delta^2 u_i - a\int_\Omega u_i \Delta u_i + b\int_\Omega u_i^2 &= \int_\Omega u_i(\Delta^2 - a\Delta + b)(u_i) \\ &= \lambda_i \int_\Omega \rho u_i^2 = \lambda_i, \end{aligned} \tag{3.4}$$

and by Schwartz inequality, we have

$$\int_\Omega u_i \Delta u_i \leq \left(\int_\Omega (\Delta u_i)^2 \int_\Omega u_i^2\right)^{\frac{1}{2}} \leq \left(\int_\Omega \frac{1}{\rho_1}(\Delta u_i)^2\right)^{\frac{1}{2}} = \left(\int_\Omega \frac{1}{\rho_1} u_i \Delta^2 u_i\right)^{\frac{1}{2}} \tag{3.5}$$

by (3.5) and (3.6), we can get

$$\lambda_i \geq \rho_1 \left(\int_\Omega u_i \Delta u_i\right)^2 - a\int_\Omega u_i \Delta u_i + \frac{b}{\rho_2},$$

this is a quadratic inequality of $\int_\Omega u_i \Delta u_i$, solving it, we obtain

$$\frac{a - \sqrt{a^2 + 4\left(\lambda_i - \frac{b}{\rho_2}\right)}}{2\rho_1} \leq \int_\Omega u_i \Delta u_i \leq \frac{a + \sqrt{a^2 + 4\left(\lambda_i - \frac{b}{\rho_2}\right)}}{2\rho_1},$$

setting

$$A_i = \frac{-a + \sqrt{a^2 + 4\left(\lambda_i - \frac{b}{\rho_2}\right)}}{2\rho_1},$$

which imply that

$$-\int_\Omega u_i \Delta u_i \leq A_i. \tag{3.6}$$

Since $\int_\Omega u_i \Delta u_i = -\int_\Omega |\nabla u_i|^2$, we have

$$\int_\Omega |\nabla u_i|^2 \leq A_i. \tag{3.7}$$

From $|\nabla f| = 1, \Delta f = n-1$, (3.6)-(3.7) and by the definition of p_i, we can get

$$\begin{aligned}
& \int_\Omega f u_i p_i \\
= & \int_\Omega f u_i \{2\langle \nabla f, \nabla(\Delta u_i)\rangle + \Delta f \Delta u_i + 2\Delta\left(\langle \nabla f, \nabla u_i\rangle\right) \\
& + \Delta\left(u_i \Delta f\right) - 2a\langle \nabla f, \nabla u_i\rangle - a u_i \Delta f\} \\
= & \int_\Omega -2\{u_i \Delta u_i \langle \nabla f, \nabla f\rangle + f\Delta u_i \langle \nabla u_i, \nabla f\rangle + f u_i \Delta f \Delta u_i\} \\
& + \int_\Omega f u_i \Delta f \Delta u_i + \int_\Omega \{\Delta f u_i + f\Delta u_i + 2\langle \nabla f, \nabla u_i\rangle\} \\
& \times\{2\langle \nabla f, \nabla u_i\rangle + u_i \Delta f\} + \int_\Omega -2af u_i \langle \nabla f, \nabla u_i\rangle \\
& + \int_\Omega 2af u_i \langle \nabla f, \nabla u_i\rangle + \int_\Omega a u_i^2 \langle \nabla f, \nabla f\rangle \\
= & -\int_\Omega 2u_i \Delta u_i \langle \nabla f, \nabla f\rangle + \int_\Omega 4\langle \nabla f, \nabla u_i\rangle^2 + \int_\Omega 4u_i \Delta f \langle \nabla f, \nabla u_i\rangle \\
& + \int_\Omega (u_i \Delta f)^2 + \int_\Omega a u_i^2 \langle \nabla f, \nabla f\rangle \\
\leq & -\int_\Omega 2u_i \Delta u_i + \int_\Omega 4|\nabla f|^2 |\nabla u_i|^2 + \int_\Omega 4u_i (n-1)\langle \nabla f, \nabla u_i\rangle \\
& + \int_\Omega ((n-1)u_i)^2 + \int_\Omega a u_i^2 \\
\leq & \ 6A_i - \frac{2(n-1)^2}{\rho_2} + \frac{(n-1)^2 + a}{\rho_1}.
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
& \int_\Omega \frac{1}{\rho}\left(\langle \nabla f, \nabla u_i\rangle + \frac{1}{2}u_i \Delta f\right)^2 \\
= & \int_\Omega \frac{1}{\rho}\left(\langle \nabla f, \nabla u_i\rangle + \frac{n-1}{2}u_i\right)^2 \\
\leq & \ \frac{1}{\rho_1}\left(\int_\Omega \langle \nabla f, \nabla u_i\rangle^2 - \frac{(n-1)^2}{4}\int_\Omega u_i^2\right) \\
\leq & \ \frac{1}{\rho_1}\left(\int_\Omega |\nabla f|^2 |\nabla u_i|^2 - \frac{(n-1)^2}{4}\int_\Omega u_i^2\right) \\
\leq & \ \frac{1}{\rho_1}\left(A_i - \frac{(n-1)^2}{4\rho_2}\right),
\end{aligned} \tag{3.9}$$

Taking (3.8) and (3.9) into (3.3), we obtain

$$\begin{aligned}\frac{1}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \ \leq\ & \sum_{i=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^2\left(6A_i-\frac{2(n-1)^2}{\rho_2}+\frac{(n-1)^2+a}{\rho_1}\right)\\ &+\sum_{i=1}^{k}\frac{\lambda_{k+1}-\lambda_i}{\delta}\frac{1}{\rho_1}\left(A_i-\frac{(n-1)^2}{4\rho_2}\right). \qquad (3.10)\end{aligned}$$

In (3.10), taking

$$\delta=\frac{\left(\frac{1}{\rho_1}\left(A_i-\frac{(n-1)^2}{4\rho_2}\right)\right)^{\frac{1}{2}}}{\left(6A_i-\frac{2(n-1)^2}{\rho_2}+\frac{(n-1)^2+a}{\rho_1}\right)^{\frac{1}{2}}}, \qquad (3.11)$$

we can get

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\\ \leq\ & \left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(6\rho_2A_i-2(n-1)^2+\frac{\rho_2((n-1)^2+a)}{\rho_1}\right)\right\}^{\frac{1}{2}}\\ &\times\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\frac{1}{\rho_1}\left(4\rho_2A_i-(n-1)^2\right)\right\}^{\frac{1}{2}}. \qquad (3.12)\end{aligned}$$

This completes the proof of Theorem 1.1. □

We introduce the following lemma to complete the proof of Corollary 1.2.

Lemma 3.1. (Reverse Chebyshev inequality [5]). *Suppose* $\{a_i\}_{i=1}^k$ *and* $\{b_i\}_{i=1}^k$ *are two real sequences with* $\{a_i\}$ *increasing and* $\{b_i\}$ *decreasing, then we have*

$$\sum_{i=1}^{k}a_ib_i\leq\frac{1}{k}\left(\sum_{i=1}^{k}a_i\right)\left(\sum_{i=1}^{k}b_i\right) \qquad (3.13)$$

Proof of Corollary 1.2. Taking $\rho_1 = \rho_2 = 1$ in (3.12), we obtain

$$\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\right\}^2 \le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(6A_i-(n-1)^2+a\right) \times\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4A_i-(n-1)^2\right). \quad (3.14)$$

Since $\{\lambda_{k+1}-\lambda_i\}_{i=1}^k$ is decreasing and $\{6A_i-(n-1)^2+a\}_{i=1}^k$ is increasing, it follows from (3.13) that

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(6A_i-(n-1)^2+a\right) \le \frac{1}{k}\left(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\right)\left(\sum_{j=1}^{k}\left(6A_j-(n-1)^2+a\right)\right). \quad (3.15)$$

By (3.14) and (3.15), we can get

$$\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4A_i-(n-1)^2\right)\left(6A_j-(n-1)^2+a\right).$$

This completes the proof of Corollary 1.1. □

ACKNOWLEDGEMENTS. The research work is supported by Key Laboratory of Applied Mathematics of Hubei Province and The research project of Jingchu University of Technology.

References

[1] M.S. Ashbaugh, *Isoperimetric and unversal inequalities for eigenvalues*, In: Davies, E.B., Safalov, Y.(eds.) *Spectral theory and geometry* (Edinburgh, 1998), London Math. Soc. Lecture Notes, Cambridge University Press, Cambridge 1999, pp. 95-139.

[2] D. Chen and Q.M. Cheng, Extrinsic estimates for eigenvalues of the Laplace operator, *J. Math. Soc. Jpn.*, **60**, (2008), 325-339.

[3] Q.M. Cheng and H.C. Yang, Inequalities for eigenvalues of a clamped plate problem, *Trans. Amer. Math. Soc.*, **262**(3), (2006), 663-675.

[4] Q.M. Cheng and H.C. Yang, Inequalities for eigenvalues of a clamped plate problem on a hyperbolic space, *Proc. Am. Math. Soc.*, **139**(2), (2011), 461-471.

[5] G. Hardy, J.E. Littlewood and G. Pólya, *Inequality*, 2nd edn., Cambridge University Press, Cambridge, 1994.

[6] G.N. Hile and R.Z. Yeh, Inequalities for eigenvalues of the biharmonic operator, *Pacific. J. Math.*, **112**, (1984), 115-133.

[7] G.N. Hile and M.H. Protter, Inequalities for eigenvalues of the Laplacian, *Indiana Univ. Math. J.*, **29**, (1980), 523-538.

[8] E.M. Harrel and J. Stubbe, On trace inequalities and the universal eigenvalue estimates for some partial differntial operators, *Trans. Am. Math. Soc.*, **349**, (1997), 1797-1809.

[9] L.E. Payne, G. Pólya and H.F. Weinberger, On the ratio of consecutive eigenvalues, *J. Math. Phys.*, **35**, (1956), 289-298.

[10] Q. Wang and C. Xia, Inequalities for eigenvalues of a clamped plate problem. *Calc. Var. PDE.*, **40**(1-2), (2011), 273-289.

[11] H.C. Yang, An estimate of the difference between consecutive eigenvalues, preprint *IC/91/60 of ICTP*, Trieste, 1991.