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Lagrange's equations of motion for oscillating central-force field

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Abstract

A body undergoing a rotational motion under the influence of an attractive force may equally oscillate vertically about its own axis of rotation. The up and down vertical oscillation will certainly cause the body to possess another different generalized coordinates in addition to the rotating coordinate. We have shown analytically and qualitatively in this work, the effect of the vertical oscillating motion of a body caused by the vibrational effect of the attractive central force. The total energy possess by the body is now the sum of the radial energy and the oscillating energy. The results show that the total energy is negative and highly attractive.

Keywords: Elliptical plane, vertical oscillation, critical velocity

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1 Introduction

A central force is a conservative force [1]. It is a force directed always toward or away from a fixed center O, and whose magnitude is a function only of the distance from O. In spherical coordinates, with O as origin, a central force is given by $F = f(r)\hat{r}$. Physically, such a force represents an attraction if $\left(f(r) \prec 0\right)$ and repulsion if $\left(f(r) \succ 0\right)$, from a fixed point located at the origin $r = 0$.

Examples of attractive central forces are the gravitational force acting on a planet due to the sun. Nuclear forces binding electrons to an atom undoubtedly have a central character. The force between a proton or an alpha particle and another nucleus is a repulsive central force.

The relevance of the Central - force motion in the macroscopic and microscopic frames warrants a detailed study of the theoretical mechanics associated with it. So far, researchers have only considered central - force motion, as motion only in the translational and rotational plane with coordinates (r,θ), for example, see Keplerian orbits [2, 3]. However, the theoretical knowledge advanced by these researchers in line with this type of motion is scientifically restricted as several possibilities are equally applicable.

There exist four standard formulations of classical mechanics: (i) Isaac Newton’s formulation – Newtonian mechanics (ii) Lagrange’s formulation – Lagrange’s mechanics (iii) Hamilton’s formulation – Hamiltonian mechanics (iv) De Alambert’s formulation – De Alambertian mechanics. All these formulations are utilized in the theory of mechanics where applicable.

Some of the conditions satisfied by a body undergoing a Central - force motion is as follows: (i) the motion of the body can be translational and rotational in the elliptical plane with polar coordinates (r,θ), (ii) the body can be rotating and revolving about its own axis in the elliptic plane (r,θ), (iii) the body can be translating and rotating in the elliptical plane (r,θ), at the same time, oscillating up and down about its own axis (iv) the body can be translating and rotating in the elliptic plane (r, θ), at the same time, oscillating up and down above the axis of rotation but not below the axis of rotation (v) the combination of any of these conditions form another class of a central - force motion.

In order to make the mechanics of a Central - force motion sufficiently meaningful, we have in this work extended the theory which has only been that of translational and rotational in the elliptical plane with polar coordinates (r, θ), by including spin oscillation.

Under this circumstance, we shall be contending with a total of 6 - generalized coordinates or degrees of freedom; 2 from the translational and rotational motion in the elliptical plane (r, θ), 2 from the orbital spin oscillations (β, α) and 2 from the tangential spin oscillations (μ, ϕ). Consequently, these parameters form the basis of our classical theory of 6-dimensional motion.

The number of independent ways in which a mechanical system can move without violating any constraints which may be imposed is called the number of degrees of freedom of the system. The number of degrees of freedom is the number of quantities which must be specified in order to determine the velocities of all particles in the system for any motion which does not violate the constraints [4].

There is a single source producing the force that depends only on distance in the theory of central-force motion and the force law is symmetric [5]. If this is the case, then, there can be no torques present in the system as there would have to be a preferred axis about which the torques acts.

In this work, we are solving the problem of oscillating central force motion in a resistive non-symmetric system. That is, the upward displacement is not equal to the downward displacement in the tangential spin oscillating phase. Consequently, the radii distances from the central point are not equal. This however, causes torques thereby making the system under study non-spherically symmetric.

Meanwhile, I hereby request the permission of the reader to excuse the lack of intensive references to the current literature. I don't know of other current authors who have studied these questions before now. I believe this is the first time this work is under investigation.

This paper is outlined as follows. Section 1, illustrates the basic concept of the work under study. The mathematical theory is presented in section 2. While in section 3, we present the analytical discussion of the results obtained. The conclusion of this work is shown in section 4 and this is immediately followed by appendix and list of references.

2 Mathematical theory

2.1 Evaluation of the velocity and acceleration

We have elaborately shown in (A. 6) in the appendix that the position vector $\vec{r}$ of a body whose motion is translational and rotational in a plane polar orbit as well as oscillating about a given equilibrium position in a central-force motion is given by the equation

$$\vec{r} = r\,\hat{r} = r\,\hat{r}(\theta, \beta, \mu, \alpha, \phi) \tag{2.1}$$

$$v = \frac{d\vec{r}}{dt} = \frac{d\,r}{d\,t}\hat{r} + r\left(\frac{d\,\hat{r}}{d\,\theta}\frac{d\theta}{dt} + \frac{d\hat{r}}{d\beta}\frac{d\beta}{dt} + \frac{d\,\hat{r}}{d\,\mu}\frac{d\mu}{dt} + \frac{d\hat{r}}{d\alpha}\frac{d\alpha}{dt} + \frac{d\,\hat{r}}{d\phi}\frac{d\,\phi}{d\,t}\right) \tag{2.2}$$

$$v = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta} + r\,\dot{\beta}\,\hat{\beta} + r\dot{\mu}\,\hat{\mu} + r\dot{\alpha}\,\hat{\alpha} + r\dot{\phi}\,\hat{\phi} \tag{2.3}$$

$$a=\frac{d^2\vec{r}}{dt^2}=\frac{dv}{dt}=\ddot{r}\,\hat{r}+\dot{r}\left(\frac{d\,\hat{r}}{d\,\theta}\frac{d\theta}{dt}+\frac{d\hat{r}}{d\beta}\frac{d\beta}{dt}+\frac{d\,\hat{r}}{d\,\mu}\frac{d\mu}{dt}+\frac{d\hat{r}}{d\alpha}\frac{d\alpha}{dt}+\frac{d\,\hat{r}}{d\varphi}\frac{d\,\varphi}{d\,t}\right)$$
$$+\dot{r}\,\dot{\theta}\,\hat{\theta}+r\,\ddot{\theta}\,\hat{\theta}+r\,\dot{\theta}^2\left(\frac{d\hat{\theta}}{d\theta}\right)+\dot{r}\,\dot{\beta}\,\hat{\beta}+r\ddot{\beta}\,\hat{\beta}+r\,\dot{\beta}^2\left(\frac{d\hat{\beta}}{d\beta}\right)+\dot{r}\,\dot{\mu}\,\hat{\mu} \qquad (2.4)$$
$$+r\,\ddot{\mu}\,\hat{\mu}+r\,\dot{\mu}^2\left(\frac{d\,\hat{\mu}}{d\,\mu}\right)+\dot{r}\dot{\alpha}\,\hat{\alpha}+r\ddot{\alpha}\hat{\alpha}+r\dot{\alpha}^2\left(\frac{d\hat{\alpha}}{d\alpha}\right)+\dot{r}\dot{\varphi}\hat{\varphi}+r\ddot{\varphi}\hat{\varphi}+r\dot{\varphi}^2\left(\frac{d\hat{\varphi}}{d\varphi}\right)$$

$$a=\left(\ddot{r}-r\dot{\theta}^2\right)\hat{r}+\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\hat{\theta}+\left(r\ddot{\beta}+2\dot{r}\dot{\beta}-r\dot{\beta}^2\tan\beta\right)\hat{\beta}$$
$$+\left(r\ddot{\mu}+2\dot{r}\dot{\mu}-r\dot{\mu}^2\tan\mu-2r\dot{\mu}^2\cot\mu\right)\hat{\mu} \qquad (2.5)$$
$$+\left(r\ddot{\alpha}+2\dot{r}\dot{\alpha}-r\dot{\alpha}^2\tan\alpha\right)\hat{\alpha}+\left(r\ddot{\varphi}+2\dot{r}\dot{\varphi}-r\dot{\varphi}^2\tan\varphi-2r\dot{\varphi}^2\cot\varphi\right)\hat{\varphi}$$

while the symbols appearing in (2.1) - (2.5) have been clearly defined in the appendix. However, β is the upper radial orbital oscillating angle and α is the lower radial orbital oscillating angle. Note that both of them are projections of the tangential oscillating plane onto the orbital elliptical plane.

However, let us disengage the acceleration equation in (2.5) with the view that the 5^{th} and the 8^{th} terms have the elements of angular momentum and the orbital oscillating phases. Thus

$$a=\left(\ddot{r}-r\dot{\theta}^2\right)\hat{r}+\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\hat{\theta}+\left(r\ddot{\beta}+2\dot{r}\dot{\beta}\right)\hat{\beta}-\left(r\dot{\beta}^2\tan\beta\right)\hat{\beta}$$
$$+\left(r\ddot{\mu}+2\dot{r}\dot{\mu}-r\dot{\mu}^2\tan\mu-2r\dot{\mu}^2\cot\mu\right)\hat{\mu}$$
$$+\left(r\ddot{\alpha}+2\dot{r}\dot{\alpha}\right)\hat{\alpha}-\left(r\dot{\alpha}^2\tan\alpha\right)\hat{\alpha} \qquad (2.6)$$
$$+\left(r\ddot{\varphi}+2\dot{r}\dot{\varphi}-r\dot{\varphi}^2\tan\varphi-2r\dot{\varphi}^2\cot\varphi\right)\hat{\varphi}$$

Equation (2.6) is now the new acceleration equation which governs the motion of a body undergoing a central-force motion when the effect of vertical oscillation is added.

2.2 Evaluation of the central- force field

In classical mechanics, a central force is a force whose magnitude only depends on the distance r, of the body from the origin and is directed along the line joining them [5]. Thus, from the analytical geometry of the central-force motion shown in Figure A. 1, in the appendix, permits us to write in terms of vector algebra that

$$F(r) = f(\|r\|)(\hat{r};\hat{\beta},\hat{\alpha}) = f(\|r\|)\,\hat{r} + f(\|r\|)\,\hat{\beta} + f(\|r\|)\,\hat{\alpha} = m\,a \qquad (2.7)$$

where F is a vector valued force function, f is a scalar valued force function, r is the position vector, $\|r\|$ is its length, and $\hat{r} = r/\|r\|$, is the corresponding unit vector.

We can convert (2.6) to force by simply multiplying it by the mass m of the body and equate the resulting expression to (2.7). Note that we are utilizing the orbital oscillating phase in (2.6), which is acting radially in the directions of $\hat{\beta}$ and $\hat{\alpha}$ in our calculation. Once this is done, we obtain the following sets of canonical equations of motion.

$$f(r) = m\left\{\left(\ddot{r} - r\dot{\theta}^2\right) - \left(r\dot{\beta}^2 \tan\beta + r\dot{\alpha}^2 \tan\alpha\right)\right\} \qquad (2.8)$$

$$m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0 \qquad (2.9)$$

$$m\left(r\ddot{\mu} + 2\dot{r}\dot{\mu}\right) = 0 \qquad (2.10)$$

$$m\left(r\ddot{\beta} + 2\dot{r}\dot{\beta}\right) = 0 \qquad (2.11)$$

$$m\left(r\ddot{\alpha} + 2\dot{r}\dot{\alpha}\right) = 0 \qquad (2.12)$$

$$m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) = 0 \qquad (2.13)$$

The sets of canonical equation (2.9)-(2.13) determines the angular momentum which are constants of the motion acting in the directions of increasing coordinate, $\theta, \beta, \mu, \alpha$ and ϕ. Equation (2.8) is the required new central-force field which we have developed in this study. It governs the motion of a body under a central-force when the effect of vertical spin oscillation is added.

2.3 Evaluation of the oscillating energy E_{osc} in the tangential phase

There is no force acting in the direction of the orbital angular acceleration, the lower and upper tangential oscillating phase. Since the force acting is a central force, it is always in the direction of the radial acceleration. The orbital angular acceleration, the lower and upper tangential oscillating phase are perpendicular to the line OP and it is in the increasing order of θ, μ and ϕ. As a result, by converting the tangential oscillating phase of the acceleration to force and equate the result to zero, we get

$$m\left(r\ddot{\mu}+2\dot{r}\dot{\mu}-r\dot{\mu}^2\tan\mu-2r\dot{\mu}^2\cot\mu\right)=0 \tag{2.14}$$

$$m\left(r\ddot{\phi}+2\dot{r}\dot{\phi}-r\dot{\phi}^2\tan\phi-2r\dot{\phi}^2\cot\phi\right)=0 \tag{2.15}$$

Because of the similarity in the two equations, we shall only solve (2.14) and assume the same result for the other one. Hence from (2.14)

$$\frac{m}{r}\left(r^2\,\ddot{\mu}+2r\,\dot{r}\,\dot{\mu}-r^2\dot{\mu}^2\tan\mu-2r^2\dot{\mu}^2\cot\mu\right)=0 \tag{2.16}$$

$$\frac{d}{dt}\left(r^2\,\dot{\mu}\right)-\frac{d}{dt}\int\left(r^2\dot{\mu}^2\tan\mu+2r^2\dot{\mu}^2\cot\mu\right)=0\,; \qquad \left(m/r\neq 0\right) \tag{2.17}$$

$$\frac{d}{dt}\left\{\left(r^2\,\dot{\mu}\right)-\int\left(r^2\dot{\mu}^2\tan\mu+2r^2\dot{\mu}^2\cot\mu\right)\right\}=0 \tag{2.18}$$

$$r^2\,\dot{\mu}-\int\left(r^2\dot{\mu}^2\tan\mu+2r^2\dot{\mu}^2\cot\mu\right)=E_I \tag{2.19}$$

and with a similar equation for (2.15) in frame II as

$$r^2\,\dot{\phi}-\int\left(r^2\dot{\phi}^2\tan\phi+2r^2\dot{\phi}^2\cot\phi\right)=E_{II} \tag{2.20}$$

$$E_{osc}=E_I+E_{II} \tag{2.21}$$

$$E_{osc}=r^2\left(\dot{\mu}+\dot{\phi}\right)-\int r^2\left(\dot{\mu}^2\tan\mu+\dot{\phi}^2\tan\phi\right)-2\int r^2\left(\dot{\mu}^2\cot\mu+\dot{\phi}^2\cot\phi\right) \tag{2.22}$$

The oscillating energy is a function of the radius vector and it increases negatively as the vertical oscillating angles are increased. Hence, the oscillating energy posses by the body in terms of μ and ϕ in the oscillating phase is given by (2.22). This equation determines how energy is conveyed up and down in the vertical oscillating phase.

2.4 Relationship between the radial velocity and the tangential oscillating angles

To determine the tangential oscillating angles we consider (2.14) and assume possibly that for $m \neq 0$

$$\left(r\ddot{\mu} + 2\dot{r}\dot{\mu} - r\dot{\mu}^2 \tan\mu - 2r\dot{\mu}^2 \cot\mu\right) = 0 \tag{2.23}$$

$$\dot{\mu} = \frac{2\dot{r} \pm \sqrt{4\dot{r}^2 + 4r^2\ddot{\mu}\left(\tan\mu + 2\cot\mu\right)}}{2r\left(\tan\mu + 2\cot\mu\right)} \tag{2.24}$$

$$\dot{\mu} = \frac{2\dot{r} \pm 2\dot{r}\sqrt{\left(1 + \frac{r^2}{\dot{r}^2}\ddot{\mu}\left(\tan\mu + 2\cot\mu\right)\right)}}{2r\left(\tan\mu + 2\cot\mu\right)} \tag{2.25}$$

The discriminate of (2.25) is zero provided

$$\dot{r}^2 = -r^2\ddot{\mu}\left(\tan\mu + 2\cot\mu\right) \tag{2.26}$$

$$\left|\dot{r}\right| = r\sqrt{\ddot{\mu}\left(\tan\mu + 2\cot\mu\right)} \tag{2.27}$$

Similarly, by following the same algebraic procedure for (2.15), we obtain

$$\dot{\phi} = \frac{2\dot{r} \pm 2\dot{r}\sqrt{\left(1 + \frac{r^2}{\dot{r}^2}\ddot{\phi}\left(\tan\phi + 2\cot\phi\right)\right)}}{2r\left(\tan\phi + 2\cot\phi\right)} \tag{2.28}$$

and

$$\left|\dot{r}\right| = r\sqrt{\ddot{\phi}\left(\tan\phi + 2\cot\phi\right)} \tag{2.29}$$

Thus the radial velocity is directly proportional to the radius vector and directly proportional to the square root of the vertical oscillating angles. The radial velocity decreases as the vertical spin oscillating angles are increased.

2.5 Evaluation of the Lagrange's equations of motion

From equation (2.3) we realize that the kinetic energy T of the body can be written as

$$T=\frac{1}{2}m\,v^2=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2+r^2\dot{\beta}^2+r^2\dot{\mu}^2+r^2\dot{\alpha}^2+r^2\dot{\phi}^2\right) \tag{2.30}$$

$$L=T-V(r) \tag{2.31}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k}=0 \tag{2.32}$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_k}\big(T-V(r)\big)\right)-\frac{\partial}{\partial q_k}\big(T-V(r)\big)=0 \tag{2.33}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}-\frac{\partial V(r)}{\partial \dot{q}}\right)-\frac{\partial T}{\partial q_k}+\frac{\partial V(r)}{\partial q_k}=0 \tag{2.34}$$

$$\frac{\partial}{\partial \dot{q}_k}\big(V(r)\big)=0 \quad ; \quad \frac{\partial V(r)}{\partial q_k}=\frac{dV}{dq_k} \tag{2.35}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}+\frac{dV(r)}{dq_k}=0 \tag{2.36}$$

$$q_k=\left(r,\,\theta,\,\beta,\,\mu,\,\alpha,\,\phi\right) \quad ; \quad \dot{q}_k=\left(\dot{r},\,\dot{\theta},\,\dot{\beta},\,\dot{\mu},\,\dot{\alpha},\,\dot{\phi}\right) \tag{2.37}$$

where q_k are the generalized coordinates, $\dot{q}_k$ are the associated velocity counterparts, $\partial L/\partial q_k$ generalized velocity, $\partial L/\partial \dot{q}_k$ generalized momentum. Remember that the only requirement for the generalized coordinates is that they span the space of the motion and be linearly independent.

Since the force is radially symmetric, let us evaluate (2.36) first with respect to the generalized coordinate $q_k=r$.

Then the Lagrange's equations of motion is

$$\frac{d}{dt}(m\dot{r})-m\,r\left(\dot{\theta}^2+\dot{\beta}^2+\dot{\mu}^2+\dot{\alpha}^2+\dot{\phi}^2\right)+\frac{dV(r)}{dr}=0 \tag{2.38}$$

$$\frac{d}{dt}(m\dot{r}) - mr\left(\dot{\theta}^2 + \dot{\beta}^2 + \dot{\mu}^2 + \dot{\alpha}^2 + \dot{\phi}^2\right) - f(r) = 0 \tag{2.39}$$

Also from (2.36), since L and $V(r)$ are not functions of the generalized coordinates q_k, then we have

$$\frac{\partial T}{\partial q_k} = \frac{dV(r)}{dq_k} = 0 \tag{2.40}$$

and as a result (2.36) becomes,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = 0 \tag{2.41}$$

Hence, after some straightforward algebra we obtain the following generalized momenta.

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \quad ; \quad \left(mr^2\dot{\theta}\right) = l \quad ; \quad \dot{\theta} = \frac{l}{mr^2} \tag{2.42}$$

$$\frac{d}{dt}\left(mr^2\dot{\beta}\right) = 0 \quad ; \quad \left(mr^2\dot{\beta}\right) = l \quad ; \quad \dot{\beta} = \frac{l}{mr^2} \tag{2.43}$$

$$\frac{d}{dt}\left(mr^2\dot{\mu}\right) = 0 \quad ; \quad \left(mr^2\dot{\mu}\right) = l \quad ; \quad \dot{\mu} = \frac{l}{mr^2} \tag{2.44}$$

$$\frac{d}{dt}\left(mr^2\dot{\alpha}\right) = 0 \quad ; \quad \left(mr^2\dot{\alpha}\right) = l \quad ; \quad \dot{\alpha} = \frac{l}{mr^2} \tag{2.45}$$

$$\frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0 \quad ; \quad \left(mr^2\dot{\phi}\right) = l \quad ; \quad \dot{\phi} = \frac{l}{mr^2} \tag{2.46}$$

The canonical set of equations given by (2.42) – (2.46) are referred to as the Lagrange's equations of motion for the body of mass m. Suppose we now replace (2.42) – (2.46) into (2.39) so that we realize

$$\frac{d}{dt}(m\dot{r}) - mr\left(\frac{l^2}{m^2r^4} + \frac{l^2}{m^2r^4} + \frac{l^2}{m^2r^4} + \frac{l^2}{m^2r^4} + \frac{l^2}{m^2r^4}\right) - f(r) = 0 \tag{2.47}$$

$$\frac{d}{dt}(m\dot{r}) - 5mr\left(\frac{l}{mr^2}\right)^2 - f(r) = 0 \tag{2.48}$$

$$\frac{d}{dt}(m\dot{r})-\frac{5l^2}{mr^3}-f(r)=0 \tag{2.49}$$

$$\frac{d}{dt}(m\dot{r})\dot{r}-\frac{5l^2}{mr^3}\,\dot{r}-f(r)\,\dot{r}=0 \tag{2.50}$$

$$d\left(m\dot{r}^2\right)-\frac{5l^2}{mr^3}\,dr-f(r)\,dr=0\,dt \tag{2.51}$$

$$\int d\left(\frac{1}{2}m\dot{r}^2\right)-\frac{5l^2}{m}\int\frac{dr}{r^3}-\int f(r)\,dr=\int 0\,dt \tag{2.52}$$

$$\left(\frac{1}{2}m\dot{r}^2\right)+\frac{5l^2}{2mr^2}-\int f(r)\,dr=E_r \tag{2.53}$$

Equation (2.53) gives the radial energy posses by the body as it oscillates tangentially and translates rotationally round the central point. The equation provides the energy of the body in terms of the translational radial velocity, the angular momentum and the radial force. The reader is directed to (A.22) in the appendix, where the factor of half which appears in (2.52) is discussed. However, for a conservative field

$$V(r)=-\int f(r)\,dr \tag{2.54}$$

$$\dot{r}=\sqrt{\frac{2}{m}\left(E-V(r)-\frac{5l^2}{2mr^2}\right)} \tag{2.55}$$

from which

$$l^2=\frac{2mr^2}{5}\left(E-V(r)-\frac{m}{2}\dot{r}^2\right) \tag{2.56}$$

It is evident from (2.55) that the translational radial velocity of the body depends only upon the radius vector. This of course defines the third property of central force motion. The translational radial velocity is determined by the energy as a constant of the motion, the effective potential and the angular momentum.

Therefore, the total energy E_t possess by the body is now the sum of the radial energy E_r and the oscillating energy E_{osc} .

$$E_t = \left(\frac{1}{2} m\dot{r}^2\right) + \frac{5l^2}{2mr^2} - \int f(r)\,dr + \tag{2.57}$$

$$r^2\left(\dot{\mu} + \dot{\phi}\right) - \int r^2\left(\dot{\mu}^2 \tan\mu + \dot{\phi}^2 \tan\phi\right) - 2\int r^2\left(\dot{\mu}^2 \cot\mu + \dot{\phi}^2 \cot\phi\right) \tag{2.58}$$

3 Discussion of results

The oscillating energy E_{osc} is made up of three independent generalized coordinates and two major parts. The first part is the vertical spin oscillating velocities which is perpendicular to the direction of the radius vector. The second part in (2.22) is the unbounded oscillating phase. The unrestricted nature of the integrals of E_{osc}, means that the oscillating phase has several possibilities of oscillation. However, the second term of E_{osc} increases as the vertical spin oscillating angles is increased. Whereas, the third term decreases as the vertical spin oscillating angles is increased.

The total energy E_t comprises of the radial and the oscillating part. The angular momentum part of E_t has a higher appreciable value compared to the usual equation of central-force motion and a negative effective potential. The last integrand of E_t becomes negatively small and negligible as the vertical spin oscillating angles is increased.

4 Conclusion

In general, we have in this study solved the problem of the motion of a body in a plane polar coordinate system that is subject to a central attractive force which is known and, in addition, a drag oscillating force which acts tangentially. The oscillating energy E_{osc} which determines how energy is conveyed up and down in the oscillating phase is relatively determined by the vertical spin oscillating angles.

The new force law now comprises of the radial and the tangential oscillating parts which reduces the strength of the attractive central force field. The knowledge of this type of central force motion which we have investigated in this work can be extended from plane polar coordinate system to that of spherical and cylindrical polar coordinate systems.

Appendix

Let us consider the rotational motion of a body of mass m about a fixed origin say, O, in an elliptic polar coordinate (r, θ) system. Suppose the body is also oscillating up and down about its equilibrium position as it translates rotationally round the fixed origin. The body thus possesses translational and rotational elliptical motion with polar coordinates (r, θ) and tangential spin oscillating motion described by the vertical displacement $C \rightarrow D \rightarrow C \rightarrow B \rightarrow C$ and repeatedly in the y-direction. The geometry of the analytical requirements is shown in Figure A. 1.

The reader should take note that the oscillation of the body is not entirely out of the elliptical orbit of rotation. Rather the displacement D and B above and below C is very small. The oscillation is still within the limits of the axis of rotation C. We have only decided to stretch D and B above and below C considerably enough in order to reveal the geometrical concept required for the analytical calculation.

There are six possible degrees of freedom or generalized coordinates exhibited by the motion body under this circumstance: (i) translational and rotational in the elliptical plane (r, θ), (ii) the plane of upward oscillations (β, μ) and (iii) the plane of downward oscillations (α, ϕ).

We shall compute separately the tangential spin oscillating motions in both oscillating frames and eventually combine the result with the orbital elliptical plane motion. In this study, we assume that the angular displacements in the tangential spin oscillating frames are not equal and so the system under study is not radially symmetric. Consequently, there is the existence of torque due to the non uniformity of the radii distances.

Accordingly, we can now develop relationships between the various areas indicated on Figure A. 1, with the goal to find the formula for the area swept out by the elliptical plane polar motion, and the result obtained from this is then added to the tangential oscillating triangle sections $D\hat{O}C$ and $C\hat{O}B$ respectively.

From the figure, P and Q are very small upward and downward displacements from the equilibrium axis of rotation C, that is, regions in the upper and lower triangular swept segments of the upper and lower elliptical plane. Our first task would be to connect all these oscillating spin angular degrees of freedom into an expression in terms of P and Q.

For clarity of purpose, let us define the various symbols which we may encounter in our calculations : (i) the elliptical radius r (ii) the plane of upward oscillations (β, μ), that is subtended from the upper elliptical plane (iii) the plane of downward oscillations (α, ϕ), that is subtended from the bottom or the lower part of the elliptical plane (iv) the elliptical orbital angle θ (vi) the upper tangential oscillating spin angle μ (v) the lower tangential oscillating spin angle ϕ (vi) the upper and lower orbital spin oscillating angles β and α

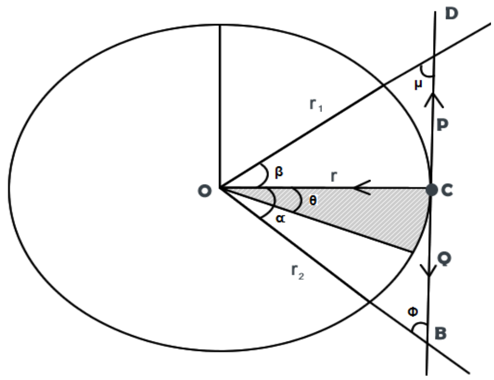

Figure A.1: Represents the elliptical and oscillating motion of a body in a central-force field. The body is oscillating up and down about the axis of rotation C. Where $\Delta D\hat{O}C$ (frame I) and $\Delta C\hat{O}B$ (frame II) are the upper and lower projections onto the plane of the ellipse, line $\overline{DC}$ (P) and $\overline{CB}$ (Q) are very small displacements from the axis C, we have only stretched them to make the geometry of the figure clear enough for the calculation. However, β is the upper orbital oscillating angle and α is the lower orbital oscillating angle. Note that both of them are projections of the tangential oscillating plane onto the orbital plane of the ellipse.

In frame I : we obtain from $\Delta D\hat{O}C$

$$r_1 = r\cos ec\mu \quad ; \quad P = r_1 \sin\beta = r\sin\beta\cos ec\mu \tag{A.1}$$

In frame II : we obtain from $\Delta C\hat{O}B$

$$r_2 = r\cos ec\phi \quad ; \quad Q = r_2 \sin\alpha = r\sin\alpha\cos ec\phi \tag{A.2}$$

In the orbital plane of rotational and translational motion, the position vector $\vec{r}$ of the body is given by

$$\vec{r} = x\,i = r\cos\theta\, i \tag{A.3}$$

However, the combination of the rotational and translational motion, with the vertical spin oscillating frames (acting in the y-direction), will yield

$$\vec{r} = x\,i + y\,j = x\,i + Pj^{\uparrow} + Q\,j^{\downarrow} = r\left(\cos\theta\, i + \sin\beta\cos ec\mu\, j^{\uparrow} + \sin\alpha\cos ec\varphi\, j^{\downarrow}\right) \quad \text{(A.4)}$$

$$\hat{r} = \frac{\partial \vec{r}}{\partial r} = \left(\cos\theta\, i + \sin\beta\cos ec\mu\, j^{\uparrow} + \sin\alpha\cos ec\phi\, j^{\downarrow}\right) \quad \text{(A.5)}$$

$$\vec{r} = r\,\hat{r} = r\,\hat{r}(\theta,\, \beta,\, \mu,\, \alpha,\, \phi) \quad \text{(A.6)}$$

$$\hat{\theta} = \frac{\partial \hat{r}}{\partial \theta} = -\sin\theta\, i \quad ; \quad \frac{\partial \hat{\theta}}{\partial \theta} = -\cos\theta\, i = -\hat{r} \quad \text{(A.7)}$$

$$\hat{\beta} = \frac{\partial \hat{r}}{\partial \beta} = \cos\beta\cos ec\mu\, j^{\uparrow} \;;$$

$$\frac{\partial \hat{\beta}}{\partial \beta} = -\sin\beta\cos ec\mu\, j^{\uparrow} = \frac{\cos\beta}{\cos\beta}\left(-\sin\beta\cos ec\mu\, j^{\uparrow}\right) \quad \text{(A.8)}$$

$$\frac{\partial \hat{\beta}}{\partial \beta} = \frac{\sin\beta}{\cos\beta}\left(-\cos\beta\cos ec\mu\, j^{\uparrow}\right) = -\tan\beta\,\hat{\beta} \quad \text{(A.9)}$$

$$\hat{\mu} = \frac{\partial \hat{r}}{\partial \mu} = -\sin\beta\cos\mu\cos ec^{2}\mu\, j^{\uparrow} \quad \text{(A.10)}$$

$$\frac{\partial \hat{\mu}}{\partial \mu} = \sin\beta\sin\mu\cos ec^{2}\mu\, j^{\uparrow} + 2\sin\beta\cos\mu\cot\mu\cos ec^{2}\mu\, j^{\uparrow} \quad \text{(A.11)}$$

$$\frac{\partial \hat{\mu}}{\partial \mu} = \frac{\cos\mu}{\cos\mu}\left(\sin\beta\sin\mu\cos ec^{2}\mu\, j^{\uparrow}\right) + 2\cot\mu\left(\sin\beta\cos\mu\cos ec^{2}\mu\, j^{\uparrow}\right) \quad \text{(A.12)}$$

$$\frac{\partial \hat{\mu}}{\partial \mu} = \frac{\sin\mu}{\cos\mu}\left(\sin\beta\cos\mu\cos ec^{2}\mu\, j^{\uparrow}\right) + 2\cot\mu\left(\sin\beta\cos\mu\cos ec^{2}\mu\, j^{\uparrow}\right) \quad \text{(A.13)}$$

$$\frac{\partial \hat{\mu}}{\partial \mu} = -\tan\mu\,\hat{\mu} - 2\cot\mu\,\hat{\mu} \quad \text{(A.14)}$$

$$\hat{\alpha} = \frac{\partial \hat{r}}{\partial \alpha} = \cos\alpha\cos ec\phi\, j^{\downarrow} \;;$$

$$\frac{\partial \hat{\alpha}}{\partial \alpha} = -\sin\alpha\cos ec\phi\, j^{\downarrow} = \frac{\cos\alpha}{\cos\alpha}\left(-\sin\alpha\cos ec\phi\, j^{\downarrow}\right) \quad \text{(A.15)}$$

$$\frac{\partial \hat{\alpha}}{\partial \alpha} = \frac{\sin\alpha}{\cos\alpha}\left(-\cos\alpha\cos ec\phi\, j^{\downarrow}\right) = -\tan\alpha\,\hat{\alpha} \quad \text{(A.16)}$$

$$\hat{\phi} = \frac{\partial \hat{r}}{\partial \phi} = -\sin\alpha\cos\phi\cos ec^{2}\phi\, j^{\downarrow} \quad \text{(A.17)}$$

$$\frac{\partial\hat{\phi}}{\partial\phi}=\sin\alpha\sin\phi\cos ec^{2}\phi\, j^{\downarrow}+2\sin\alpha\cos\phi\cot\phi\cos ec^{2}\phi\, j^{\downarrow} \tag{A.18}$$

$$\frac{\partial\hat{\phi}}{\partial\phi}=\frac{\cos\phi}{\cos\phi}\left(\sin\alpha\sin\phi\cos ec^{2}\phi\, j^{\downarrow}\right) \tag{A.19}$$

$$\frac{\partial\hat{\phi}}{\partial\phi}=\frac{\sin\phi}{\cos\phi}\left(\sin\alpha\cos\phi\cos ec^{2}\phi\, j^{\downarrow}\right)+2\cot\phi\left(\sin\alpha\cos\phi\cos ec^{2}\phi\, j^{\downarrow}\right) \tag{A.20}$$

$$\frac{\partial\hat{\phi}}{\partial\phi}=-\tan\phi\,\hat{\phi}-2\cot\phi\,\hat{\phi} \tag{A.21}$$

We also know from the rule of differentiation that

$$\begin{aligned} m\frac{d}{dt}\left(r^{2}\right)=m\frac{d}{dt}\left(r.r\right)=m\left(\frac{dr}{dt}r+r\frac{dr}{dt}\right)=m\left(2r\frac{dr}{dt}\right)\\ =m\left(2\frac{dr^{2}}{dt}\right)=\frac{d}{dt}\left(\frac{1}{2}mr^{2}\right)\end{aligned} \tag{A.22}$$

Hence, in order to remove the factor of 2 which appears in (A.22), usually a factor of half is introduced.

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