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Improving Strong Convergence Results for Hierarchical Optimization

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Abstract

In this paper, an hierarchical circularly iterative method is introduced for solving a system of variational circularly inequalities with set of fixed points of strongly quasi-nonexpansive mapping problems. Under suitable conditions, strong convergence results are proved in the setting of Hilbert spaces. Our scheme can be regarded as a more general variant of the algorithm proposed by Maingé.

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1 Introduction

The concept of variational inequalities plays an important role in structural analysis, mechanics and economics. Recently, the hierarchical variational in-

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equalities and hierarchical iterative sequence problems have attached many authors' attention(see[1]-[7], [9]-[11]).

Inspired by these results in the literature, a circularly iterative method in this paper is introduced for solving a system of variational inequalities with fixed-point set constraints. Under suitable conditions, strong convergence results are proved in the setting of Hilbert spaces. Our scheme can be regarded as a more general variant of the algorithm proposed by Maingé. The results presented in the paper improve and extend the corresponding results in [11].

2 Preliminaries

For the sake of convenience, we first recall some definitions and lemmas for our main results. We assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm$\| \cdot \|$. C is a nonempty closed convex subset of H and $Fix(T) = \{x \in C; Tx = x\}$ is the set of fixed points of a mapping $T : D \to D$. In the sequel, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. It is well-known that, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$P_C(x) = \inf_{y \in H} \|x - y\|, \qquad \forall x \in H.$$

Such a mapping P_C from H onto C is called the metric projection.

Lemma 2.1. (see [8]) *The metric projection $P_C : H \to C$ has the following basic properties:*

(1) P_C is firmly nonexpansive, i.e.,

$$\langle P_C(x) - P_C(y), x - y \rangle \geq \|P_C(x) - P_C(y)\|^2, \qquad \forall x, y \in H,$$

and so P_C is nonexpansive.

(2) $\langle x - P_C x, y - P_C x \rangle \leq 0$, for all $x \in H$ and $y \in C$.

Definition 2.2. *(1) A mapping $T : H \to H$ is said to be α-inverse-strongly monotone if there exists $\alpha > 0$ such that*

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|, \qquad \forall x, y \in H.$$

(2) A mapping $T : H \to H$ is said to be α-Lipschitzian if

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \qquad \forall x, y \in H.$$

(3) A mapping $T : H \to H$ is said to be quasi-nonexpansive if $Fix(T) \neq \Phi$ and

$$\|Tx - p\| \leq \|x - p\|, \qquad \forall x \in H, \quad p \in Fix(T).$$

(4) A mapping $T : H \to H$ is said to be strongly quasi-nonexpansive if T is quasi-nonexpansive and $x_n - Tx_n \to 0$, whenever $\{x_n\}$ is a bounded sequence in H and $\|x_n - p\| - \|Tx_n - p\| \to 0$ for some $p \in Fix(T)$.

(5) (see[12])A mapping $T : H \to H$ is said to be ω-demicontractive if $Fix(T) \neq \Phi$ and

$$\langle x - Tx, x - p\rangle \geq \frac{1-\omega}{2}\|x - Tx\|^2, \qquad \forall x \in H\, quadp \in Fix(T).$$

Obviously, the above inequality is equivalent to

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \omega\|x - Tx\|^2,$$

and it is clear from the preceding definitions, that every quasi-nonexpansive mapping is 0-demicontractive.

Lemma 2.3. (see [13]) *For $x, y \in H$ and $\omega \in [0, 1]$, we have the following statements:*

(a) $|\langle x, y\rangle| \leq \|x\|\|y\|$;

(b) $\|x + y\| \leq \|x\|^2 + 2\langle y, y + x\rangle$;

(c) $\|(1 - \omega)x + \omega y\|^2 = (1 - \omega)\|x\|^2 + \omega\|y\|^2 - \omega(1 - \omega)\|x - y\|^2$.

For prove our result, we give the following lemma about the existence and uniqueness of solutions of some related hierarchical optimization problems.

Lemma 2.4. ([11]) *Let* $\{\alpha_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $\alpha_{n_i} \le \alpha_{n_i+1}$ *for all* $i \in N$. *Then there exists a nondecreasing* $\{m_k\} \subset N$, *such that* $m_k \to \infty$ *and the following properties are satisfied for all(sufficiently large) numbers sequence* $k \subset N$:

$$\alpha_{m_k} \le \alpha_{m_k+1} \quad \text{and} \quad \alpha_k \le \alpha_{m_k+1}.$$

In fact, $m_k = \max\{j \le k : \alpha_j \le \alpha_{j+1}\}$.

Lemma 2.5. ([11]) *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \gamma_n\delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*
(a) $\lim_{n\to\infty}\gamma_n = 0$, $\Sigma_{n=1}^{\infty}\gamma_n = \infty$,
(b) $\limsup_{n\to\infty}\delta_n \le 0$.
Then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 2.6. ([11]) *Let* $\{a_n\} \subset [0,\infty)$, $\{\alpha_n\} \subset [0,1]$, $\{b_n\} \subset (-\infty,+\infty)$ *and* $\lambda \in [0,1]$, *such that*

- $\{a_n\}$ *is a bounded sequence;*
- $a_{n+1} \le (1-\alpha_n)^2 a_n + 2\alpha_n\lambda\sqrt{a_n}\sqrt{a_{n+1}} + \alpha_n b_n$, *for all* $n \in N$;
- *whenever* $\{a_{n_k}\}$ *is a subsequence of* $\{a_n\}$ *satisfying* $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \ge 0$, *it follows that* $\limsup_{k\to\infty} b_{n_k} \le 0$;
- $\lim_{n\to\infty}\alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

In [11], the existence and uniqueness of solutions of some related hierarchical optimization problems had been discussed.

Theorem 2.7. ([11]) *Let $S_1, S_2 : H \to H$ be quasi-nonexpansive mappings and $f_1, f_2 : H \to H$ be contractions. Then there exists a unique element $(p, q) \in Fix(S_1) \times Fix(S_2)$ such that the following two inequalities,*

$$\begin{cases} \langle p - f_1(q), x - p\rangle \geq 0, & \forall x \in Fix(S_1), \\ \langle q - f_2(p), y - q\rangle \geq 0, & \forall y \in Fix(S_2). \end{cases} \tag{1}$$

At the same time, Maingé define two iterative sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{cases} x_0, y_0 \in H, \\ x_{n+1} = (1 - \alpha_n)S_1x_n + \alpha_n f_1(S_2y_n), \\ y_{n+1} = (1 - \alpha_n)S_2y_n + \alpha_n f_2(S_1x_n), \end{cases} \tag{2}$$

where $\alpha_n \in [0, 1]$ satisfy $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=o}^{\infty}\alpha_n = \infty$. Then, he proved that the results as follows.

Theorem 2.8. *Let $S_1, S_2 : H \to H$ be strongly quasi-nonexpansive mappings such that $I - s_i((i = 1, 2)$ are demiclosed at zero and let $f_i((i = 1, 2)$ be contractions with the coefficient $\hat{\alpha}$. Then the iterative sequences$\{x_n\}$ and $\{y_n\}$ by (2) strong converge to (p, q), respectively, where (p, q) is the unique element in $Fix(S_1) \times Fix(S_2)$ verifying (1).*

3 Main results

First, we discuss the existence and uniqueness of solutions of some related hierarchical optimization problems.

Theorem 3.1. *Let $S_1, S_2, S_3 : H \to H$ be quasi-nonexpansive mappings and $f_1, f_2, f_3 : H \to H$ be contractions. Then there exists a unique element $(p, q, r) \in Fix(S_1) \times Fix(S_2) \times Fix(S_3)$ such that the following inequalities,*

$$\begin{cases} \langle p - f_1(q), x - p\rangle \geq 0, & \forall x \in Fix(S_1), \\ \langle q - f_2(r), y - q\rangle \geq 0, & \forall y \in Fix(S_2), \\ \langle r - f_3(p), z - r\rangle \geq 0, & \forall z \in Fix(S_3). \end{cases} \tag{3}$$

Proof. The proof is a consequence of the well-known *Banach's* contraction principle but it is given here for the sake of completeness. It is known that both sets $Fix(S_i)(i = 1, 2, 3)$ are closed and convex, and hence the projections $P_{Fix(S_i)}(i = 1, 2, 3)$ are well defined. It is clear that the mapping

$$P_{Fix(S_1)} \bullet f_1 \bullet P_{Fix(S_2)} \bullet f_2 \bullet P_{Fix(S_3)} \bullet f_3$$

is a contraction. Hence, there exists a unique element $p \in H$ such that

$$p = (P_{Fix(S_1)} \bullet f_1 \bullet P_{Fix(S_2)} \bullet f_2 \bullet P_{Fix(S_3)} \bullet f_3)p.$$

Put $r = P_{Fix(S_3)} f_3 p$ and $q = P_{Fix(S_2)} f_2 r$. Then $q \in P_{Fix(S_2)}$, $r \in P_{Fix(S_3)}$ and $p = P_{Fix(S_1)} f_1 q$.

Suppose that there is an element$(p^*, q^*, r^*) \in Fix(S_1) \times Fix(S_2) \times Fix(S_3)$ such that the following inequalities,

$$\begin{cases} \langle p^* - f_1(q^*), x - p^* \rangle \geq 0, & \forall x \in Fix(S_1), \\ \langle q^* - f_2(r^*), y - q^* \rangle \geq 0, & \forall y \in Fix(S_2), \\ \langle r^* - f_3(p^*), z - r^* \rangle \geq 0, & \forall z \in Fix(S_3). \end{cases}$$

Then $r^* = P_{Fix(S_3)} f_3 p^*, q^* = P_{Fix(S_2)} f_2 r^*$ and $p^* = P_{Fix(S_1)} f_1 q^*$. Hence, $p^* = (P_{Fix(S_1)} \bullet f_1 \bullet P_{Fix(S_2)} \bullet f_2 \bullet P_{Fix(S_3)} \bullet f_3)p^*$. This implies that $p = p^*$ and hence $q = q^*, r = r^*$. This completes the proof. □

For mappings $S_i, f_i : H \to H$ $(i = 1, 2, 3)$, we define the iterative sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1 - \alpha_n) S_1 x_n + \alpha_n f_1(S_2 y_n), \\ y_{n+1} = (1 - \alpha_n) S_2 y_n + \alpha_n f_2(S_3 z_n), \\ z_{n+1} = (1 - \alpha_n) S_3 z_n + \alpha_n f_3(S_1 x_n), \end{cases} \tag{4}$$

where $\alpha_n \in [0, 1]$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=o}^{\infty} \alpha_n = \infty$.

Theorem 3.2. *Let $S_1, S_2, S_3 : H \to H$ be strongly quasi-nonexpansive mappings such that $I - s_i((i = 1, 2, 3)$ are demiclosed at zero and let $f_i((i = 1, 2, 3)$*

be contractions with the coefficient $\hat{\alpha}$. Then the iterative sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by (4) strong converge to (p, q, r), respectively, where (p, q, r) is the unique element in $Fix(S_1) \times Fix(S_2) \times Fix(S_3)$ verifying (3).

Recall that a mapping $T : H \to H$ is demiclosed at zero if $Tx = 0$ whenever $x_n \rightharpoonup x$ and $Tx_n \to 0$. We split the proof of Theorem 3.2 into the following lemmas.

Lemma 3.3. *The sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded.*

Proof. Since $S_1, S_2, S_3 : H \to H$ be strongly quasi-nonexpansive mappings, $f_i((i = 1, 2, 3)$ be contractions with the coefficient $\hat{\alpha}$. Then we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1-\alpha_n)\|S_1x_n - p\| + \alpha_n\|f_1(S_2y_n) - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|f_1(S_2y_n) - f_1(q)\| + \alpha_n\|f_1(q) - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\hat{\alpha}\|S_2y_n - q\| + \alpha_n\|f_1(q) - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\hat{\alpha}\|y_n - q\| + \alpha_n\|f_1(q) - p\|.\end{aligned}$$

Similarly, we also have

$$\begin{aligned}&\|y_{n+1} - q\| \leq (1-\alpha_n)\|y_n - q\| + \alpha_n\hat{\alpha}\|z_n - r\| + \alpha_n\|f_2(r) - q\|, \\ &\|z_{n+1} - r\| \leq (1-\alpha_n)\|z_n - q\| + \alpha_n\hat{\alpha}\|x_n - p\| + \alpha_n\|f_3(p) - r\|.\end{aligned}$$

It implies that

$$\begin{aligned}&\|x_{n+1} - p\| + \|y_{n+1} - q\| + \|z_{n+1} - r\| \\ &\leq [1 - (1-\hat{\alpha})\alpha_n](\|x_n - p\| + \|y_n - q\| + \|z_n - r\|) \\ &\qquad\qquad +\alpha_n(\|f_1(q) - p\| + \|f_2(r) - q\| + \|f_3(p) - r\|) \\ &\leq \max\{\|x_n - p\| + \|y_n - q\| + \|z_n - r\|, \\ &\qquad\qquad \frac{\|f_1(q) - p\| + \|f_2(r) - q\| + \|f_3(p) - r\|}{1-\hat{\alpha}}\}.\end{aligned}$$

By induction, we have

$$\begin{aligned}&\|x_{n+1} - p\| + \|y_{n+1} - q\| + \|z_{n+1} - r\| \\ &\leq \max\{\|x_0 - p\| + \|y_0 - q\| + \|z_0 - r\|, \\ &\qquad\qquad \frac{\|f_1(q) - p\| + \|f_2(r) - q\| + \|f_3(p) - r\|}{1-\hat{\alpha}}\},\end{aligned}$$

for all $n \in N$. In particular, sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded. Consequently, the sequences $\{S_1x_n\}$, $\{S_2y_n\}$ and $\{S_3z_n\}$ are also bounded. $\square$

Lemma 3.4. *For each $n \in N$, the following inequality holds:*

$$\begin{cases} \|x_{n+1}-p\|^2 \leq (1-\alpha_n)^2\|(x_n-p)\|^2 + 2\alpha_n\hat{\alpha}\|y_n-q\|\|x_{n+1}-p\| \\ \qquad\qquad\qquad\qquad\qquad +2\alpha_n\langle f_1(q)-p, x_{n+1}-p\rangle, \\ \|y_{n+1}-q\|^2 \leq (1-\alpha_n)^2\|(y_n-q)\|^2 + 2\alpha_n\hat{\alpha}\|z_n-r\|\|y_{n+1}-q\| \\ \qquad\qquad\qquad\qquad\qquad +2\alpha_n\langle f_2(r)-q, y_{n+1}-q\rangle, \\ \|z_{n+1}-r\|^2 \leq (1-\alpha_n)^2\|(z_n-r)\|^2 + 2\alpha_n\hat{\alpha}\|x_n-p\|\|z_{n+1}-r\| \\ \qquad\qquad\qquad\qquad\qquad +2\alpha_n\langle f_1(p)-r, z_{n+1}-r\rangle. \end{cases} \tag{5}$$

Proof. Since

$$\begin{aligned} & \|x_{n+1}-p\|^2 \\ = \; & \|(1-\alpha_n)(S_1x_n-p)+\alpha_n(f_1(S_2y_n)-p)\|^2 \\ \leq \; & \|(1-\alpha_n)(S_1x_n-p)\|^2 + 2\langle\alpha_n(f_1(S_2y_n)-p), x_{n+1}-p\rangle \\ \leq \; & (1-\alpha_n)^2\|(S_1x_n-p)\|^2 + 2\alpha_n\langle f_1(S_2y_n)-f_1(q), x_{n+1}-p\rangle \\ & \qquad +2\alpha_n\langle f_1(q)-p, x_{n+1}-p\rangle \\ \leq \; & (1-\alpha_n)^2\|(S_1x_n-p)\|^2 + 2\alpha_n\|f_1(S_2y_n)-f_1(q)\|\|x_{n+1}-p\| \\ & \qquad +2\alpha_n\langle f_1(q)-p, x_{n+1}-p\rangle \\ \leq \; & (1-\alpha_n)^2\|(x_n-p)\|^2 + 2\alpha_n\hat{\alpha}\|S_2y_n-q\|\|x_{n+1}-p\| \\ & \qquad +2\alpha_n\langle f_1(q)-p, x_{n+1}-p\rangle \\ \leq \; & (1-\alpha_n)^2\|(x_n-p)\|^2 + 2\alpha_n\hat{\alpha}\|y_n-q\|\|x_{n+1}-p\| \\ & \qquad +2\alpha_n\langle f_1(q)-p, x_{n+1}-p\rangle, \end{aligned}$$

we have

$$\begin{cases} \|y_{n+1}-q\|^2 \leq (1-\alpha_n)^2\|(y_n-q)\|^2 + 2\alpha_n\hat{\alpha}\|z_n-r\|\|y_{n+1}-q\| \\ \qquad\qquad\qquad\qquad\qquad +2\alpha_n\langle f_2(r)-q, y_{n+1}-q\rangle, \\ \|z_{n+1}-r\|^2 \leq (1-\alpha_n)^2\|(z_n-r)\|^2 + 2\alpha_n\hat{\alpha}\|x_n-p\|\|z_{n+1}-r\| \\ \qquad\qquad\qquad\qquad\qquad +2\alpha_n\langle f_1(p)-r, z_{n+1}-r\rangle. \end{cases}$$

By Lemma 3.3, we give following result,

$$\begin{aligned}
&\|x_{n+1}-p\|^2+\|y_{n+1}-q\|^2+\|z_{n+1}-r\|^2\\
&\leq (1-\alpha_n)^2(\|(x_n-p)\|^2+\|(y_n-q)\|^2+\|(z_n-r)\|^2)\\
&+2\alpha_n\hat{\alpha}(\|y_n-q\|\|x_{n+1}-p\|+\|z_n-r\|\|y_{n+1}-q\|+\|x_n-p\|\|z_{n+1}-r\|)\\
&+2\alpha_n(\langle f_1(q)-p,x_{n+1}-p\rangle+\langle f_2(r)-q,y_{n+1}-q\rangle\\
&+\langle f_1(p)-r,z_{n+1}-r\rangle). \quad (6)
\end{aligned}$$

□

Lemma 3.5. *If there exists a subsequence* $\{n_k\}$ *of* $\{n\}$ *such that*

$$\begin{aligned}
\liminf_{k\to\infty}(&\|x_{n_k+1}-p\|^2+\|y_{n_k+1}-q\|^2+\|z_{n_k+1}-r\|^2-\|x_{n_k}-p\|^2\\
&-\|y_{n_k}-q\|^2-\|z_{n_k}-r\|^2)\geq 0,
\end{aligned}$$

then

$$\limsup_{k\to\infty}(\langle f_1(q)-p,x_{n_k+1}-p\rangle+\langle f_2(r)-q,y_{n_k+1}-q\rangle+\langle f_3(p)-r,z_{n_k+1}-r\rangle \leq 0. \quad (7)$$

Proof. In fact, we first consider the following assertion:

$$\begin{aligned}
0&\leq \liminf_{k\to\infty}(\|x_{n_k+1}-p\|^2+\|y_{n_k+1}-q\|^2\|z_{n_k+1}-r\|^2-\|x_{n_k}-p\|^2\\
&\quad-\|y_{n_k}-q\|^2-\|z_{n_k}-r\|^2)\\
&\leq \liminf_{k\to\infty}[(1-\alpha_{n_k})\|S_1x_{n_k}-p\|^2+\alpha_{n_k}\|f_1(S_2y_{n_k})-q\|^2\\
&\quad+(1-\alpha_{n_k})\|S_2y_{n_k}-q\|^2+\alpha_{n_k}\|f_2(S_3z_{n_k})-r\|^2\\
&\quad+(1-\alpha_{n_k})\|S_3z_{n_k}-r\|^2+\alpha_{n_k}\|f_3(S_1z_{n_k})-p\|^2\\
&\quad-\|x_{n_k}-p\|^2-\|y_{n_k}-q\|^2-\|z_{n_k}-r\|^2]\\
&=\liminf_{k\to\infty}[(\|S_1x_{n_k}-p\|^2-\|x_{n_k}-p\|^2)+(\|S_2y_{n_k}-q\|^2-\|y_{n_k}-q\|^2)\\
&\quad+(\|S_3z_{n_k}-r\|^2-\|z_{n_k}-r\|^2)]\\
&\leq \limsup_{k\to\infty}[(\|S_1x_{n_k}-p\|^2-\|x_{n_k}-p\|^2)+(\|S_2y_{n_k}-q\|^2-\|y_{n_k}-q\|^2)\\
&\quad+(\|S_3z_{n_k}-r\|^2-\|z_{n_k}-r\|^2)]\\
&\leq 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\lim_{k\to\infty}(\|S_1x_{n_k}-p\|^2-\|x_{n_k}-p\|^2)\\
&=\lim_{k\to\infty}(\|S_2y_{n_k}-q\|^2-\|y_{n_k}-q\|^2)\\
&=\lim_{k\to\infty}(\|S_3z_{n_k}-r\|^2-\|z_{n_k}-r\|^2)\\
&=0.
\end{aligned}$$

By Lemma 3.3, the sequences $\{\|S_1x_{n_k}-p\|+\|x_{n_k}-p\|\}$, $\{\|S_2y_{n_k}-q\|+\|y_{n_k}-q\|\}$ and $\{\|S_3z_{n_k}-q\|+\|z_{n_k}-q\|\}$ are bounded. So we have

$$\begin{aligned}
&\lim_{k\to\infty}(\|S_1x_{n_k}-p\|^2-\|x_{n_k}-p\|^2)\\
&=\lim_{k\to\infty}(\|S_2y_{n_k}-q\|^2-\|y_{n_k}-q\|^2)\\
&=\lim_{k\to\infty}(\|S_3z_{n_k}-r\|^2-\|z_{n_k}-r\|^2)\\
&=0.
\end{aligned}$$

Since $S_i(i=1,2,3)$ are strongly quasi-nonexpansive,

$$\begin{cases}
S_1x_{n_k}-x_{n_k}\to 0\\
S_2y_{n_k}-y_{n_k}\to 0\\
S_3z_{n_k}-z_{n_k}\to 0,
\end{cases}$$

by the iteration scheme (3), we have

$$\begin{cases}
x_{n_k}-x_{n_k+1}\to 0\\
y_{n_k}-y_{n_k+1}\to 0\\
z_{n_k}-z_{n_k+1}\to 0.
\end{cases}$$

It follows from the boundedness of $\{x_{n_k}\}$ that there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_l}}\}\rightharpoonup x$ and

$$\begin{aligned}
&\lim_{l\to\infty}\langle f_1(q)-p,x_{n_{k_l}}-p\rangle\\
&=\limsup_{k\to\infty}\langle f_1(q)-p,x_{n_k}-p\rangle\\
&=\limsup_{k\to\infty}\langle f_1(q)-p,x_{n_k+1}-p\rangle.
\end{aligned}$$

Since $I - S_1$ is demiclosed at zero, it follows that $x \in Fix(S_1)$. It follows from (3), we get

$$\lim_{l\to\infty} \langle f_1(q) - p, x_{n_{k_l}} - p\rangle = \langle f_1(q) - p, x - p\rangle \leq 0.$$

Consequently,

$$\limsup_{k\to\infty} \langle f_1(q) - p, x_{n_k+1} - p\rangle \leq 0.$$

By using the same argument, we have

$$\limsup_{k\to\infty} \langle f_2(r) - q, y_{n_k+1} - q\rangle \leq 0,$$
$$\limsup_{k\to\infty} \langle f_3(p) - r, z_{n_k+1} - r\rangle \leq 0.$$

Therefore, we obtain the desired inequality (6). □

Next, we prove Theorem 3.2. Denote

$$a_n := \|x_n - p\|^2 + \|y_n - q\|^2 + \|z_n - r\|^2$$
$$b_n := 2(\langle f_1(q) - p, x_{n+1} - p\rangle + \langle f_2(r) - q, y_{n+1} - q\rangle + \langle f_3(p) - r, z_{n+1} - r\rangle).$$

Since

$$\begin{aligned}&\|y_n - q\|\|x_{n+1} - p\| + \|z_n - r\|\|y_{n+1} - q\| + \|x_n - p\|\|z_{n+1} - r\|\\ &\leq (\|x_n - p\|^2 + \|y_n - q\|^2 + \|z_n - r\|^2)^{\frac{1}{2}}\\ &\qquad \times(\|x_{n+1} - p\|^2 + \|y_{n+1} - q\|^2 + \|z_{n+1} - r\|^2)^{\frac{1}{2}},\end{aligned}$$

we have the following statements from Lemmas (3.3), (3.4) and (3.5):

- $\{a_n\}$ is a bounded sequence;
- $a_{n+1} \leq (1 - \alpha_n)^2 a_n + 2\alpha_n\lambda\sqrt{a_n}\sqrt{a_{n+1}} + \alpha_n b_n$, for all $n \in N$;
- whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$, it follows that $\limsup_{k\to\infty} b_{n_k} \leq 0$.

Hence, it follows from Lemma 2.6 that $a_n \to 0$, It implies that

$$\lim_{n\to\infty} (\|x_n - p\|^2 + \|y_n - q\|^2\|z_n - r\|^2) = 0.$$

This means that $x_n \to p$, $y_n \to q$ and $z_n \to r$. The proof of Theorem 3.2 is completed. □

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References

[1] E. Buzogány, I. Mezei and V. Varga, Two-variable variational - hemivariational inequalities, *Stud. Univ. Babes-Bolyai Math.*, **47**, (2002), 31-41.

[2] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Stud.*, **63**, (1994), 123-145.

[3] R.U. Verma, Projection methods, algorithms and a new system of nonlinear variational inequalities, *Comput. Math. Appl.* **41**, (2001), 1025-1031.

[4] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

[5] M.A. Noor and K.I. Noor, Sensitivity analysis of quasi variational inclusions, *J. Math. Anal. Appl.*, **236**, (1999), 290-299.

[6] S.S. Chang, Set-valued variational inclusions in Banach spaces, *J. Math. Anal. Appl.*, **248**, (2000), 438-454.

[7] S.S. Chang, Existence and approximation of solutions of set-valued variational inclusions in Banach spaces, *Nonlinear Anal.*, **47**, (2001), 583-494.

[8] S.S. Zhang, Joseph H.W. Lee and C.K. Chan, Algorithms of common solutions for quasi variational inclusion and fixed point problems, *Appl. Math. Mech.*, **29**, (2008), 1-11.

[9] V.F. Demyanov, G. E. Stavroulakis, L.N. Polyakova and P.D. Panagiotopoulos, *Quasidifferentiability and Nonsmooth Modeling in Mechanics*, Engineering and Economics, Kluwer Academic, Dordrecht, 1996.

[10] P.E. Maingé and A. Mouda, Strong convergence of an iterative method for hierarchical fixed point problems, *Pacific J. Optim.*, **3**, (2007), 529-538.

[11] R. Kraikaew and S. Saejung, On Maingé's Approach for Hierarchical Optimization Problems, *J. Optim. Theory Appl.*, **154**, (2012), 71-87.

[12] T.L. Hicks and J.D. Kubicek, On the Mann iteration process in a Hilbert space, *J. Math. Anal. Appl.*, **59**, (1977), 498-504.

[13] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.

[14] A. Genel and J. Lindenstrauss, An example concerning fixed points, *Israel. J. Math.*, **22**, (1975), 81-86.

[15] B. Halpren, Fixed points of nonexpansive maps, *Bull. Amer. Math. Soc.*, **73**, (1967), 957-961.

[16] W.R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4**, (1953), 506-510.

[17] S. Matsushita and W. Takahashi, Weak and strong convergence theorems for relatively nonexpansive mappings in a Banach space, *Fixed Point Theory Appl.*, **2004**, (2004), 37-47.

[18] S. Matsushita and W. Takahashi, An iterative algorithm for relatively nonexpansive mappings by hybrid method and applications, *Proceedings of the Third International Conference on Nonlinear Analysis and Convex Analysis*, (2004), 305-313.

[19] S. Matsushita and W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, *J. Approx. Theory*, **134**, (2005), 257-266.

[20] K. Nakajo and W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.*, **279**, (2003), 372-379.

[21] X.L. Qin, Y.J. Cho, S.M. Kang and H.Y. Zhou, Convergence of a modified Halpern-type iterative algorithm for quasi-ϕ-nonexpansive mappings, *Appl. Math. Lett.*, **22**, (2009), 1051-1055.

[22] Z.M. Wang, Y.F. Su, D.X. Wang and Y.C. Dong, A modified Halpern-type iteration algorithm for a family of hemi-relative nonexpansive mappings and systems of equilibrium problems in Banach spaces, *J. Comput. Appl. Math.*, **235**, (2011), 2364-2371.

[23] B. Nanjaras and B. Panyanak, Demiclosed Principle for Asymptotically Nonexpansive Mappings in $CAT(0)$ Spaces, *Fixed Point Theory and Applications*, **2010**, (2010), Article ID 268780.