Theoretical Mathematics & Applications, vol.3, no.1, 2013, 47-68
ISSN: 1792-9687 (print), 1792-9709 (online)
Scienpress Ltd, 2013

Rotation Surfaces Generated by Planar Profile Curves with Constant Curvature in E_1^3

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Abstract

In this paper, the rotation surfaces whose profile curves are planar and with constant curvature in Minkowski 3-space are investigated according to the causal characters of the profile curves. Some characterizations are given for these rotation surfaces.

Mathematics Subject Classification : 53A05, 53A10, 53B30
Keywords: Rotation Surface; Curvature; Weingarten Surface; Umbilical Surface; Totally Geodesic

1 Introduction

The geometry of rotation surfaces has been studied widely in Euclidean space E^3 as well as Lorentz-Minkowski space E_1^3. It is well known that induced

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Article Info: *Received* : December 11, 2012. *Revised* : January 31, 2013
Published online : April 15, 2013

metric on a surface M in E_1^3 can be non-degenerate or degenerate. If the induced metric is non-degenerate, then M is called a *semi-Riemannian surface.*

The rotation surfaces and ruled surfaces in Euclidean 3-space are studied in [4] and [1], respectively, such that their Gauss maps ξ satisfy the condition $\Delta\xi = A\xi$, where $A \in Mat(3, \mathbb{R})$ and Δ is the Laplacian on M. O.J.Garay [7] investigated the rotation surfaces, in E^3 Euclidean space, whose component functions are eigenfunctions of its Laplacian and he saw that these surfaces must be a Catenoid, a sphere or a right circular cylinder.

Also, the Lorentz version of the non-degenerate surfaces M_s^2, with index $s = 0, 1$ in $\mathbb{R}_1^3$, are classified under the condition $\Delta H = \lambda H$ by Ferrandez and Lucas in [6]. Furthermore, the rotation surfaces are classified under some special conditions in [8] and [3].

Dillen and Kühnel characterized all ruled surfaces in Minkowski 3-space with a relation between the Gauss and mean curvature (*Weingarten surfaces*) [5].

In this paper, we investigate the types of the rotation surfaces whose profile curves are planar and with constant curvature in Minkowski 3-space according to the causal characters of the profile curves. We give some characterizations about these surfaces.

We note that, in this study we shall assume that the constants c_1 and c_2 are non-zero.

2 Preliminaries

In the vector space $\mathbb{R}^n$, if the following Lorentzian inner product

$$\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(x, y) \longrightarrow \langle , \rangle (x, y) = \langle x, y \rangle = -x_1 y_1 + \sum_{i=2}^{n} x_i y_i$$

with 1-index is considered instead of the Euclidean inner product, then it is well-known that the vector space $\mathbb{R}^n$ is a Lorentzian vector space and it is shown by $\mathbb{R}_1^n$. Furthermore, especially if $n = 3$, then this space is called as a Minkowski 3-space and usually shown by E_1^3.

Let M be a 2-dimensional surface of the Minkowski 3-space E_1^3 equipped with induced metric.

A rotation surface in Euclidean space is generated by rotating of an arbitrary curve about an arbitrary axis. In Minkowski space, however, there are different types of curves (spacelike, timelike or lightlike(null)) as well as different types of rotation axes (spacelike, timelike or lightlike(null)), so that there are different types of rotation surfaces in this context [9].

Let $\alpha : I = (a,b) \subset \mathbb{R} \longrightarrow \pi$ be a curve in a plane π of E_1^3 and let ξ be a straight line of π which does not intersect the curve α. A rotation surface M in E_1^3 is defined as a non-degenerate surface, rotating the curve α around the axis ξ.

Now, let us investigate the rotation matrices according to the type of the axis of rotation.

Let A be a 3×3 regular matrix and $0 \neq \xi \in E_1^3$ be a vector. If A satisfies the following conditions, then it is said that A denotes a rotation in positive direction [10]:

i) $A\xi = \xi$,

ii) $AIA^t = I$,

iii) $\det A = 1$,

where I is the 3×3 Lorentzian unit matrix, i. e. ,

$$I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, the rotation matrices which fix the spacelike axes $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ and the timelike axis $e_1 = (1,0,0)$ are the sets of 3×3 matrices defined by

$$A(v) = \begin{bmatrix} \cosh v & 0 & \sinh v \\ 0 & 1 & 0 \\ \sinh v & 0 & \cosh v \end{bmatrix}, \ A(v) = \begin{bmatrix} \cosh v & \sinh v & 0 \\ \sinh v & \cosh v & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ v \in \mathbb{R} \text{ and}$$

$$A(v) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{bmatrix}, \ 0 \leq v \leq 2\pi,$$

respectively.

Suppose that the axis of rotation is a lightlike line of the plane $Sp\{e_1, e_2\}$ spanned by the vector $(1, 1, 0)$. Then, the rotation matrix which fixes the lightlike axis $(1, 1, 0)$ is the set of 3×3 matrices given by

$$A(v) = \begin{bmatrix} 1 + \frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1 - \frac{v^2}{2} & v \\ v & -v & 1 \end{bmatrix}, v \in \mathbb{R}.$$

Let α be a lightlike curve in E_1^3. Then, there exists a re-parametrization of the curve α given by $\beta(s) = \alpha(\theta(s))$ in such way that $\|\beta''(s)\| = 1$. We say that α is length-arc pseudo parametrized [11].

Here, we consider a regular curve α parametrized by the length-arc or by the pseudo-length-arc. The vector $T(s) = \alpha'(s)$ is called the *tangent vector* of the curve α at s. We know that, the curve α is spacelike, timelike or lightlike if the tangent vector is spacelike, timelike or lightlike for any $s \in I$, respectively.

First, we suppose that α is a timelike curve. Then, $T'(s) \neq 0$ is a spacelike vector independent with $T(s)$. The curvature of α at s is defined as $\kappa(s) = \|T'(s)\|$. The *normal vector* $N(s)$ and the *binormal vector* $B(s)$ are

$$N(s) = \frac{T'(s)}{\kappa(s)} = \frac{\alpha''(s)}{\|\alpha''(s)\|} \text{ and } B(s) = T(s) \times N(s).$$

Furthermore $\kappa(s) = \langle T'(s), N(s) \rangle$.

If α is a spacelike curve, then we have three cases with respect to the causal character of $T'(s)$.

Initially, if $T'(s)$ is a spacelike vector, then the curvature, the normal vector and the binormal vector of α are

$$\kappa(s) = \|T'(s)\|, \ N(s) = \frac{T'(s)}{\kappa(s)} \text{ and } B(s) = T(s) \times N(s),$$

respectively.

If $T'(s)$ is a timelike vector, then

$$\kappa(s) = \sqrt{-\langle T'(s), T'(s) \rangle}, N(s) = \frac{T'(s)}{\kappa(s)} \text{ and } B(s) = T(s) \times N(s).$$

And if $T'(s)$ is lightlike, then there isn't a definition of the curvature of α. In this case, the normal vector is $N(s) = T'(s)$ which is linear independent with $T(s)$. The binormal vector B is a unique lightlike vector such that

$\langle N(s), B(s)\rangle = 1$ and $\langle T(s), N(s)\rangle = 0$. Here, the Frenet equations are

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \tau & 0 \\ -1 & 0 & -\tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{1}$$

where the function τ is called the torsion of the curve α.

Finally, if α is a lightlike curve parametrized by pseudo-length-arc, that is, α'' is a unitary spacelike vector, then $T(s) = \alpha'(s)$, $N(s) = T'(s)$ and $B(s)$ is the unique lightlike vector such that $\langle T(s), B(s)\rangle = 1$ and $\langle N(s), B(s)\rangle = 0$. In this case, we don't define the curvature of the curve α [11].

We know that, there are no causal vectors (non-spacelike) which is orthogonal to a timelike vector. So that if α is a planar timelike curve, then the plane which contains the curve α must be timelike.

If the planar curve is spacelike, then it can lie in the spacelike, timelike or lightlike planes. And if the planar curve is lightlike, then it can lie in the timelike or lightlike planes.

3 Rotation Surfaces Generated by Planar Profile Curves with Constant Curvature in E_1^3

In this section, we study the rotation surfaces whose profile curves are planar and with constant curvature in Minkowski 3-space. The planar curves can be spacelike, timelike or lightlike. So, we shall investigate these cases separately.

3.1 Rotation Surfaces Generated by Timelike Planar Profile Curves with Constant Curvature in E_1^3

In this case, since the curve is planar and timelike, the plane which contains the curve must be timelike. Therefore, we suppose that the profile curve lies in the timelike plane $\pi = Sp\{e_1, e_3\}$. Then the curve α can be written as

$$\alpha(u) = f(u)e_1 + g(u)e_3 = (f(u), 0, g(u)), \tag{2}$$

where f and g are smooth functions.

Let us assume that the curve α is parametrized by length-arc, that is,

$$-f'^2(u) + g'^2(u) = -1.$$

So, there exists a smooth function θ such that

$$f'(u) = \cosh\theta(u),\ g'(u) = \sinh\theta(u).$$

Since the curve has constant curvature, we get

$$\kappa = \|\alpha''(u)\| = |\theta'(u)| = c_1,\ c_1 \text{ is a constant.}$$

Therefore, we can take $\theta(u) = c_1 u + c_2$ and so

$$f'(u) = \cosh(c_1 u + c_2),\ g'(u) = \sinh(c_1 u + c_2),\ c_1, c_2 \in \mathbb{R}. \tag{3}$$

From (2) and (3), the timelike planar curve can be written as

$$\alpha(u) = \frac{1}{c_1}(\sinh(c_1 u + c_2), 0, \cosh(c_1 u + c_2)). \tag{4}$$

This is an Euclidean hyperbola in the timelike plane π.

Thus, we can give the following cases:

i) The rotation surface which is obtained by rotating the timelike planar profile curve (4) with constant curvature about the timelike axis e_1 is

$$\Psi(u,v) = \frac{1}{c_1}(\sinh(c_1 u + c_2), -\cosh(c_1 u + c_2)\sin v, \cosh(c_1 u + c_2)\cos v). \tag{5}$$

The Gauss map of this rotation surface is

$$G = (-\sinh(c_1 u + c_2), \cosh(c_1 u + c_2)\sin v, -\cosh(c_1 u + c_2)\cos v).$$

G is a spacelike vector and so, the rotation surface given by (5) is a timelike surface (See Figure 1).

Proposition 3.1. *The mean curvature and Gaussian curvature of the rotation surface (5) which is obtained by rotating the timelike planar profile curve (4) with constant curvature c_1 about the timelike axis e_1 are*

$$H = c_1 \text{ and } K = c_1^2,$$

respectively. Furthermore, the shape operator of the rotation surface (5) is

$$A = \begin{bmatrix} -c_1 & 0 \\ 0 & -c_1 \end{bmatrix}.$$

ii) The rotation surfaces which are obtained by rotating the timelike planar profile curve (4) with constant curvature about the spacelike axes e_2 and e_3 are

$$\Psi(u,v) = \frac{1}{c_1}(\sinh(c_1u + c_2)\cosh v + \cosh(c_1u + c_2)\sinh v, 0, \quad \sinh(c_1u + c_2)\sinh v + \cosh(c_1u + c_2)\cosh v) \tag{6}$$

and

$$\Psi(u,v) = \frac{1}{c_1}(\sinh(c_1u + c_2)\cosh v, \sinh(c_1u + c_2)\sinh v, \cosh(c_1u + c_2)), \tag{7}$$

respectively. The rotation surface given by (6) is a patch of the plane $\pi = Sp\{e_1, e_3\}$ and timelike.

The Gauss map of the rotation surface given by (7) is

$$G = (\sinh(c_1u + c_2)\cosh v, \sinh(c_1u + c_2)\sinh v, \cosh(c_1u + c_2)).$$

G is a spacelike vector. Thus, the rotation surface given by (7) is a timelike surface.

Proposition 3.2. *The mean curvature and Gaussian curvature of the rotation surface (7) which is obtained by rotating the timelike planar profile curve (4) with constant curvature c_1 about the spacelike axis e_3 are*

$$H = -c_1 \text{ and } K = c_1^2,$$

respectively. Furthermore, the shape operator of the rotation surface (7) is

$$A = \begin{bmatrix} c_1 & 0 \\ 0 & c_1 \end{bmatrix}.$$

iii) The rotation surface which is obtained by rotating the timelike planar profile curve (4) with constant curvature about the lightlike axis $(1,1,0)$ is

$$\Psi(u,v) = \frac{1}{c_1}((1+\frac{v^2}{2})\sinh(c_1u+c_2)+v\cosh(c_1u+c_2), \quad (8)$$
$$\frac{v^2}{2}\sinh(c_1u+c_2)+v\cosh(c_1u+c_2), v\sinh(c_1u+c_2)+\cosh(c_1u+c_2)).$$

The Gauss map of this rotation surface is

$$G = ((1+\frac{v^2}{2})\sinh(c_1u+c_2)+v\cosh(c_1u+c_2),$$
$$\frac{v^2}{2}\sinh(c_1u+c_2)+v\cosh(c_1u+c_2), v\sinh(c_1u+c_2)+\cosh(c_1u+c_2)).$$

Because of the Gauss map G is spacelike, the rotation surface given by (8) is a timelike surface, too.

Proposition 3.3. *The mean curvature and Gaussian curvature of the rotation surface (8) which is obtained by rotating the timelike planar profile curve (4) with constant curvature* c_1 *about the lightlike axis* $(1,1,0)$ *are*

$$H = -c_1 \text{ and } K = c_1^2,$$

respectively. Furthermore, the shape operator of the rotation surface (8) is

$$A = \begin{bmatrix} c_1 & 0 \\ 0 & c_1 \end{bmatrix}.$$

Thus, we can give the following table for rotation surfaces with timelike planar profile curves:

Profile Curve	**Plane**	**Axis of Rotation**	**ROTATION SURFACE**
Timelike	Timelike	Timelike (e_1)	Timelike
		Spacelike (e_2) Spacelike (e_3)	Timelike Timelike
		Lightlike ($(1,1,0)$)	Timelike

3.2 Rotation Surfaces Generated by Spacelike Planar Profile Curves with Constant Curvature in E_1^3

Let the profile curve be planar and spacelike. So, it can be lie in spacelike, timelike or lightlike planes.

a) Let us assume that the planar spacelike profile curve lies in the spacelike plane $\pi = Sp\{e_2, e_3\}$.

In this case, the induced metric on π by the Lorentzian metric agrees with the Euclidean metric. With the analogous consideration in subsection 3.1., if we take the curve α which lies in the plane π as $\alpha(u) = (0, f(u), g(u))$, then we obtain the spacelike curve α with constant curvature as

$$\alpha(u) = \frac{1}{c_1}(0, -\cos(c_1 u + c_2), \sin(c_1 u + c_2)). \tag{9}$$

This curve is an Euclidean circle.

Thus, we can give the following cases:

i) The rotation surface which is obtained by rotating the spacelike planar profile curve (9) with constant curvature lying in the spacelike plane $\pi = Sp\{e_2, e_3\}$ about the timelike axis e_1 is

$$\begin{aligned}\Psi(u, v) = \frac{1}{c_1}(0, &-\cos(c_1 u + c_2)\cos v - \sin(c_1 u + c_2)\sin v, \\ &-\cos(c_1 u + c_2)\sin v + \sin(c_1 u + c_2)\cos v).\end{aligned} \tag{10}$$

This is a patch of the plane $\pi = Sp\{e_2, e_3\}$ and spacelike.

ii) The rotation surfaces which are obtained by rotating the spacelike planar profile curve (9) with constant curvature lying in the spacelike plane $\pi = Sp\{e_2, e_3\}$ about the spacelike axes e_2 and e_3 are

$$\Psi(u, v) = \frac{1}{c_1}(\sin(c_1 u + c_2)\sinh v, -\cos(c_1 u + c_2), \sin(c_1 u + c_2)\cosh v) \tag{11}$$

and

$$\Psi(u, v) = \frac{1}{c_1}(-\cos(c_1 u + c_2)\sinh v, -\cos(c_1 u + c_2)\cosh v, \sin(c_1 u + c_2)), \tag{12}$$

respectively. The Gauss map of the rotation surface given by (11) is

$$G = (-\sin(c_1u + c_2)\sinh v, \cos(c_1u + c_2), -\sin(c_1u + c_2)\cosh v).$$

G is a spacelike vector and so, the rotation surface given by (11) is timelike (See Figure 2).

The Gauss map of the rotation surface given by (12) is

$$G = (-\cos(c_1u + c_2)\sinh v, -\cos(c_1u + c_2)\cosh v, \sin(c_1u + c_2)).$$

G is a spacelike vector. Hence, the rotation surface given by (12) is timelike.

Proposition 3.4. *The mean curvatures and Gaussian curvatures of the rotation surfaces (11) and (12) which are obtained by rotating the spacelike planar profile curve (9) with constant curvature c_1 about the spacelike axes e_2 and e_3 are*

$$H = c_1,\ K = c_1^2 \text{ and } H = -c_1,\ K = c_1^2,$$

respectively. Furthermore, the shape operators of the rotation surfaces (11) and (12) are

$$A = \begin{bmatrix} -c_1 & 0 \\ 0 & -c_1 \end{bmatrix} \text{ and } A = \begin{bmatrix} c_1 & 0 \\ 0 & c_1 \end{bmatrix},$$

respectively.

iii) The rotation surface which is obtained by rotating the spacelike planar profile curve (9) with constant curvature lying in the spacelike plane $\pi = Sp\{e_2, e_3\}$ about the lightlike axis $(1, 1, 0)$ is

$$\Psi(u, v) = \frac{1}{c_1}\Big(\frac{v^2}{2}\cos(c_1u + c_2) + v\sin(c_1u + c_2) - (1 - \frac{v^2}{2})\cos(c_1u + c_2) + v\sin(c_1u + c_2), v\cos(c_1u + c_2) + \sin(c_1u + c_2)\Big). \tag{13}$$

The Gauss map of this rotation surface is

$$G = \Big(-\frac{v^2}{2}\cos(c_1u + c_2) - v\sin(c_1u + c_2), (1 - \frac{v^2}{2})\cos(c_1u + c_2) - v\sin(c_1u + c_2), -v\cos(c_1u + c_2) - \sin(c_1u + c_2)\Big).$$

G is a spacelike vector. Thus, the rotation surface given by (13) is timelike (See Figure 3).

Proposition 3.5. *The mean curvature and Gaussian curvature of the rotation surface (13) which is obtained by rotating the spacelike profile curve (9) with constant curvature* c_1 *about the lightlike axis* $(1,1,0)$ *are*

$$H = c_1 \text{ and } K = c_1^2,$$

respectively. Furthermore, the shape operator of the rotation surface (13) is

$$A = \begin{bmatrix} -c_1 & 0 \\ 0 & -c_1 \end{bmatrix}.$$

b) Assume that the planar spacelike profile curve lies in the timelike plane $\pi = Sp\{e_1, e_2\}$**.**

In this case, the spacelike curve α which is parametrized by length-arc can be written by

$$\alpha(u) = (f(u), g(u), 0), \ \ -f'^2(u) + g'^2(u) = 1,$$

where f and g are smooth functions.

Thus, there exists a smooth function θ such that

$$f'(u) = \sinh\theta(u), \ \ g'(u) = \cosh\theta(u).$$

Then, the planar spacelike curve with constant curvature which lies in the timelike plane $\pi = Sp\{e_1, e_2\}$ can be parametrized by

$$\alpha(u) = \frac{1}{c_1}(\cosh(c_1 u + c_2), \sinh(c_1 u + c_2), 0). \tag{14}$$

We have,

i) The rotation surface which is obtained by rotating the spacelike planar profile curve (14) with constant curvature lying in the timelike plane $\pi = Sp\{e_1, e_2\}$ about the timelike axis e_1 is

$$\Psi(u,v) = \frac{1}{c_1}(\cosh(c_1u + c_2), \sinh(c_1u + c_2)\cos v, \sinh(c_1u + c_2)\sin v). \quad (15)$$

The Gauss map of this rotation surface is

$$G = (-\cosh(c_1u + c_2), -\sinh(c_1u + c_2)\cos v, -\sinh(c_1u + c_2)\sin v)$$

and it is timelike. Hence, the rotation surface given by (15) is spacelike.

Proposition 3.6. *The mean curvature and Gaussian curvature of the rotation surface (15) which is obtained by rotating the spacelike planar profile curve (14) with constant curvature c_1 about the timelike axis e_1 are*

$$H = c_1 \text{ and } K = c_1^2,$$

respectively. Furthermore, the shape operator of the rotation surface (15) is

$$A = \begin{bmatrix} -c_1 & 0 \\ 0 & -c_1 \end{bmatrix}.$$

ii) The rotation surfaces which are obtained by rotating the spacelike planar profile curve (14) with constant curvature lying in the timelike plane $\pi = Sp\{e_1, e_2\}$ about the spacelike axes e_2 and e_3 are

$$\Psi(u,v) = \frac{1}{c_1}(\cosh(c_1u + c_2)\cosh v, \sinh(c_1u + c_2), \cosh(c_1u + c_2)\sinh v) \quad (16)$$

and

$$\Psi(u,v) = \frac{1}{c_1}(\cosh(c_1u + c_2)\cosh v + \sinh(c_1u + c_2)\sinh v, \quad (17)$$
$$\cosh(c_1u + c_2)\sinh v + \sinh(c_1u + c_2)\cosh v, 0)$$

respectively. The Gauss map of the rotation surface (16) is

$$G = (-\cosh(c_1u + c_2)\cosh v, -\sinh(c_1u + c_2), -\cosh(c_1u + c_2)\sinh v).$$

G is timelike and so, the rotation surface given by (16) is spacelike.

The the rotation surface given by (17) is a patch of the plane $\pi = Sp\{e_1, e_2\}$ and timelike.

Proposition 3.7. *The mean curvature and Gaussian curvature of the rotation surface (16) which is obtained by rotating the spacelike planar profile curve (14) with constant curvature c_1 about the spacelike axis e_2 are*

$$H = c_1 \text{ and } K = c_1^2,$$

respectively. Furthermore, the shape operator of the rotation surface (16) is

$$A = \begin{bmatrix} -c_1 & 0 \\ 0 & -c_1 \end{bmatrix}.$$

iii) The rotation surface which is obtained by rotating the spacelike planar profile curve (14) with constant curvature lying in the timelike plane $\pi = Sp\{e_1, e_2\}$ about the lightlike axis $(1, 1, 0)$ is

$$\begin{aligned}\Psi(u,v) = \frac{1}{c_1}((1+\frac{v^2}{2})\cosh(c_1u+c_2) - \frac{v^2}{2}\sinh(c_1u+c_2),\\ \frac{v^2}{2}\cosh(c_1u+c_2) + (1-\frac{v^2}{2})\sinh(c_1u+c_2), v\cosh(c_1u+c_2)\\ - v\sinh(c_1u+c_2)).\end{aligned} \tag{18}$$

The Gauss map of this rotation surface is

$$\begin{aligned}G = \frac{1}{\cosh(c_1u+c_2) - \sinh(c_1u+c_2)}(-\frac{v^2}{2}\sinh^2(c_1u+c_2) - \cosh^2(c_1u+c_2)\\ + \sinh(c_1u+c_2)\cosh(c_1u+c_2) - \frac{v^2}{2}\cosh^2(c_1u+c_2)\\ + v^2\sinh(c_1u+c_2)\cosh(c_1u+c_2),\\ - \sinh(c_1u+c_2)\cosh(c_1u+c_2) + \sinh^2(c_1u+c_2) - \frac{v^2}{2}\sinh^2(c_1u+c_2)\\ - \frac{v^2}{2}\cosh^2(c_1u+c_2) + v^2\sinh(c_1u+c_2)\cosh(c_1u+c_2),\\ 2v\sinh(c_1u+c_2)\cosh(c_1u+c_2) - v\sinh^2(c_1u+c_2) - v\cosh^2(c_1u+c_2)).\end{aligned}$$

Because of the Gauss map G is timelike, the rotation surface given by (18) is spacelike.

Proposition 3.8. *The mean curvature and Gaussian curvature of the rotation surface (18) which is obtained by rotating the spacelike planar profile curve (14) with constant curvature* c_1 *about the lightlike axis* $(1,1,0)$ *are*

$$H = c_1 \text{ and } K = c_1^2,$$

respectively. Furthermore, the shape operator of the rotation surface (18) is

$$A = \begin{bmatrix} -c_1 & 0 \\ 0 & -c_1 \end{bmatrix}.$$

c) Suppose that the planar spacelike profile curve lies in the lightlike plane.

There isn't notion of curvature of a curve in this case. After a rigid motion, we suppose that the plane which contains the curve is $\pi = Sp\{e_1 - e_2 = 0\}$. Thus, the curve α can be parametrized as $\alpha(u) = (f(u), f(u), g(u))$.

If the spacelike curve α is parametrized by length-arc, then ${g'}^2(u) = 1$. Let us assume that $g(u) = u$. Hence, $T(u) = (f'(u), f'(u), 1)$, $f'(u) \neq$constant, and so $T'(u)$ is a lightlike vector. We know that, there is a unique lightlike direction in the plane. Therefore, $T'(u)$ is liner dependent with a fixed vector, for example, with $v = (1,1,0)$. There isn't notion of curvature of a curve lying in the lightlike plane but if $\tau = 0$, we say that α has constant curvature [11].

For the spacelike curve α which lies in the lightlike plane, we have (1). If the curve α has constant curvature, then

$$N' = 0 \text{ and } N = v, \text{ thus } T' = v.$$

Therefore, we have

$$\begin{aligned} \alpha(u) &= \frac{u^2}{2} v + uw + p_0 \\ &= \frac{u^2}{2}(e_1 + e_2) + uw + p_0, \end{aligned}$$

where w is a unit spacelike vector.

If we write the unit spacelike vector w by $w = e_3 + c_1(e_1 + e_2)$ and the point p_0 by $p_0 = (c_2, c_2, 0)$, then the curve α can be parametrized by

$$\alpha(u) = (\frac{u^2}{2} + c_1 u + c_2, \frac{u^2}{2} + c_1 u + c_2, u). \tag{19}$$

This is a parabola in the plane $\pi = Sp\{e_1 - e_2 = 0\}$ with axis parallel to the lightlike direction.Thus, we have:

i) The rotation surface which is obtained by rotating the spacelike planar profile curve (19) with constant curvature lying in the lightlike plane about the timelike axis e_1 is

$$\begin{aligned}\Psi(u,v) \;=\; & (\frac{u^2}{2} + c_1 u + c_2, (\frac{u^2}{2} + c_1 u + c_2)\cos v - u\sin v, (\frac{u^2}{2} + c_1 u \\ & + c_2)\sin v + u\cos v).\end{aligned} \tag{20}$$

The normal vector of this rotation surface is

$$N = (-BC - u, -BC\cos v + Bu\sin v, -BC\sin v - Bu\cos v),$$

where $B = u + c_1$, $C = \frac{u^2}{2} + c_1 u + c_2$. If we take $A_1 = -2c_1c_2u - (1 + c_1^2 + 2c_2)u^2 - c_1u^3$, then the rotation surface given by (20) is spacelike, timelike or lightlike if $A_1 < 0$, $A_1 > 0$ or $A_1 = 0$, respectively (See Figure 4).

Proposition 3.9. *The mean curvature and Gaussian curvature of the rotation surface (20) which is obtained by rotating the timelike profile curve (19) with constant curvature c_1 about the timelike axis e_1 are*

$$H = \frac{1}{2\sqrt{|A_1|}}\left[\frac{u[C^2 + u^2] + [BC^2 + Bu^2] - 2D[BC - B^2u]}{C^2 - D^2 + u^2}\right]$$

and

$$K = \frac{1}{A_1}\left[\frac{u[BC^2 + Bu^2] - [BC - B^2u]^2}{C^2 - D^2 + u^2}\right],$$

where $D = -\frac{u^2}{2} + c_2$, respectively.

ii) The rotation surface which is obtained by rotating the spacelike planar profile curve (19) with constant curvature lying in the lightlike plane about the spacelike axes e_2 and e_3 are

$$\Psi(u,v) = ((\frac{u^2}{2}+c_1u+c_2)\cosh v + u\sinh v, \frac{u^2}{2}+c_1u+c_2, (\frac{u^2}{2}+c_1u +c_2)\sinh v + u\cosh v) \quad (21)$$

and

$$\Psi(u,v) = ((\frac{u^2}{2}+c_1u+c_2)(\sinh v+\cosh v), (\frac{u^2}{2}+c_1u+c_2)(\sinh v+\cosh v), u), \quad (22)$$

respectively. The normal vector of the rotation surface (21) is

$$N = (-BC\cosh v - Bu\sinh v, -BC+u, -BC\sinh v - Bu\cosh v).$$

Let $A_2 = -2c_1c_2u-(-1+c_1^2+2c_2)u^2-c_1u^3$. Then the rotation surface given by (21) is spacelike, timelike or lightlike if $A_2 < 0$, $A_2 > 0$ or $A_2 = 0$, respectively.

The normal vector of the rotation surface given by (22) is

$$N = (C(\sinh v+\cosh v), C(\sinh v+\cosh v), 0)$$

and it is lightlike. So the rotation surface (22) is lightlike.

Proposition 3.10. *The mean curvature and Gaussian curvature of the rotation surface (21) which is obtained by rotating the timelike profile curve (19) with constant curvature c_1 about the spacelike axis e_2 are*

$$H = \frac{1}{2\sqrt{|A_2|}}\left[\frac{u[C^2-u^2]+[BC^2-Bu^2]-2D[BC-B^2u]}{C^2-D^2-u^2}\right]$$

and

$$K = \frac{1}{A_2}\left[\frac{u[BC^2-Bu^2]-[BC-B^2u]^2}{C^2-D^2-u^2}\right],$$

respectively.

iii) The rotation surface which is obtained by rotating the spacelike planar profile curve (19) with constant curvature lying in the lightlike plane about the lightlike axis $(1,1,0)$ is

$$\Psi(u,v) = (\frac{u^2}{2}+c_1u+c_2+uv, \frac{u^2}{2}+c_1u+c_2+uv, u). \quad (23)$$

The normal vector of the rotation surface given by (23) is

$$N = (u, u, 0)$$

and it is lightlike. Therefore the rotation surface (23) is lightlike.

Thus, we can give the following table which contains all the cases for rotation surfaces with spacelike planar profile curves:

Profile Curve	Plane	Axis of Rotation	ROTATION SURFACE
		Timelike (e_1)	Spacelike
	Spacelike	Spacelike (e_2) Spacelike (e_3)	Timelike Timelike
		Lightlike ($(1,1,0)$)	Timelike
		Timelike (e_1)	Spacelike
Spacelike	Timelike	Spacelike (e_2) Spacelike (e_3)	Spacelike Timelike
		Lightlike ($(1,1,0)$)	Spacelike
		Timelike (e_1)	Space,Time or Lightlike
	Lightlike	Spacelike (e_2) Spacelike (e_3)	Space,Time or Lightlike Lightlike
		Lightlike ($(1,1,0)$)	Lightlike

3.3 Rotation Surfaces Generated by Lightlike Planar Profile Curves with Constant Curvature in E_1^3

If the curve is lightlike and planar, then the plane which contains the curve can be lightlike or timelike.

a) Let us assume that the planar lightlike curve lies in the lightlike plane $\pi = Sp\{e_1 + e_2, e_3\}$.

Since there is a unique lightlike direction in the plane and $\alpha'(u)$ is lightlike,

$$\alpha'(u) = f(u)(e_1 + e_2) = (f(u), f(u), 0),$$

where f is a smooth function. Because of α is a straight-line, we can't consider the Frenet trihedron.

Therefore, there are no rotation surfaces obtained by rotating the lightlike planar curve with constant curvature in the lightlike plane $\pi = Sp\{e_1 + e_2, e_3\}$.

b) Suppose that the planar lightlike curve lies in the timelike plane $\pi = Sp\{e_1, e_2\}$.

In this case, there are only two lightlike directions $e_1 + e_2$ and $e_1 - e_2$ which are linear independent. Thus,

$$\alpha'(u) = f(u)(e_1 \pm e_2) = (f(u), \pm f(u), 0),$$

where f is a smooth function. Here, α is a straight-line again. Hence, there are no rotation surfaces obtained by rotating the lightlike planar curve with constant curvature in the timelike plane $\pi = Sp\{e_1, e_2\}$.

It is known that [5], if the mean curvature H and the Gaussian curvature K satisfy a nontrivial relation $\Phi(H, K) = 0$, then the surface is called a *Weingarten surface.*

Thus, we can give the following theorem:

Theorem 1. *The rotation surfaces (5), (7), (8), (11), (12), (13), (15), (16) and (18) whose profile curves are planar and with constant curvature in E_1^3 are Weingarten surfaces satisfying the condition*

$$H^2 - K = 0.$$

Let M be a surface in E_1^3 and A_p the shape operator at $s \in M$. s is called an *umbilical point* (respectively, *generalized umbilical point*) of M if A_p can be put into the canonical form A_1 (respectively, A_2), where

$$A_1 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \text{ (respectively } A_2 = \begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix}\text{)}.$$

M is called an *umbilical surface* (respectively, *generalized umbilical surface*) in E_1^3 if there exists a real constant a such that, for any $s \in M$, A_p can be put into the canonical form A_1 (respectively, A_2) [12].

We have:

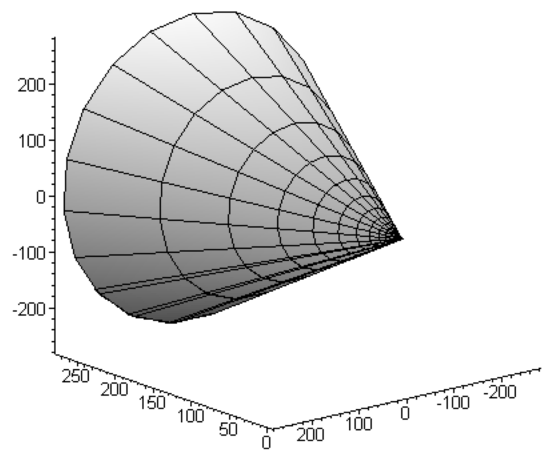

Figure 1: The rotation surface which is obtained by rotating the timelike planar profile curve with constant curvature about the timelike axis e_1

Theorem 2. *The rotation surfaces (5), (7), (8), (11), (12), (13), (15), (16) and (18) whose profile curves are planar and with constant curvature in* E_1^3 *are umbilical surfaces.*

4 Conclusion

The parameter curves of the rotation surfaces (5), (7), (8), (11), (12), (13), (15), (16) and (18) whose profile curves are planar and with constant curvature in E_1^3 are their lines of curvature.

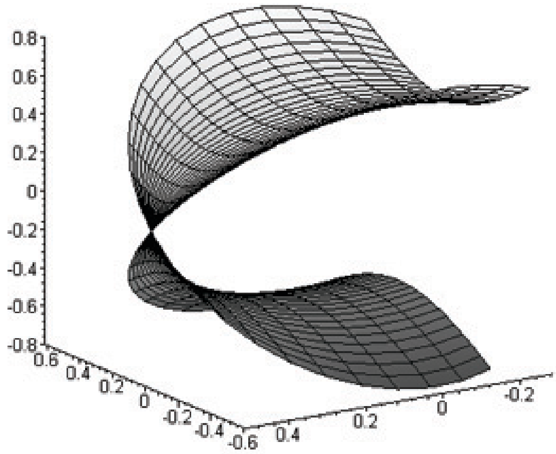

Figure 2: The rotation surface which is obtained by rotating the spacelike planar profile curve with constant curvature lying in the spacelike plane $\pi = Sp\{e_2, e_3\}$ about the spacelike axis e_2

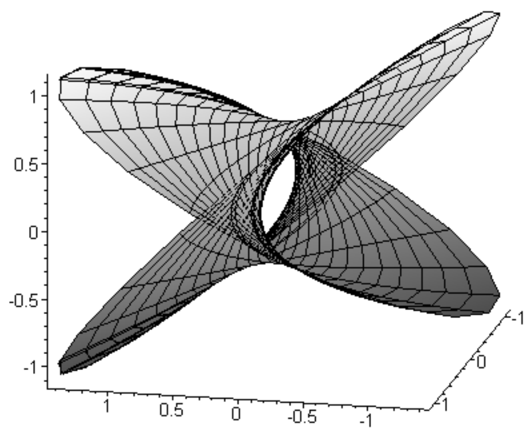

Figure 3: The rotation surface which is obtained by rotating the spacelike planar profile curve with constant curvature lying in the spacelike plane $\pi = Sp\{e_2, e_3\}$ about the lightlike axis $(1, 1, 0)$

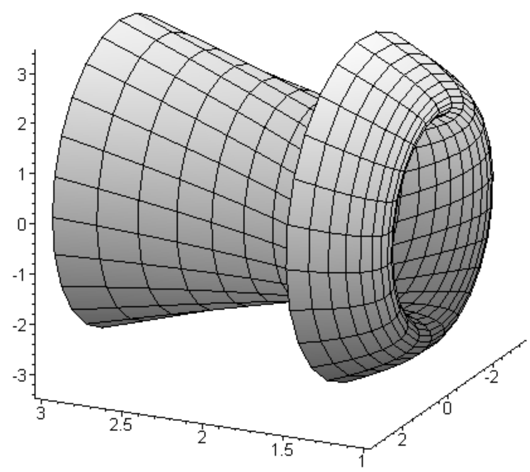

Figure 4: The rotation surface which is obtained by rotating the spacelike planar profile curve with constant curvature lying in the lightlike plane about the timelike axis e_1

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