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Strong convergence to a common fixed point of nonexpansive mappings semigroups

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Abstract

In this paper, we introduce some new iteration methods based on the hybrid method in mathematical programming and the descent-like method for finding a fixed point of a nonexpansive mapping and a common fixed point of a nonexpansive semigroup in Hilbert spaces. The main results in this paper modify and improve some well-known results in the literature.

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Keywords: Metric projection; Fixed point; Nonexpansive Mappings and Semigroups

1 Introduction

Let H be a real Hilbert space with the scalar product and the norm denoted by the symbols $\langle .,. \rangle$ and $\|.\|$, respectively, and let C be a nonempty closed

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and convex subset of H. Denote by $P_C x$ the metric projection of an element $x \in H$ onto C. It is well-known that P_C is a nonexpansive mapping on H for any closed convex subset C in H. Recall that a mapping T is said to be nonexpansive on C, if $T : C \to C$ and $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T, i.e., $F(T) = \{x \in C : x = Tx\}$. We know that $F(T)$ is nonempty, if C is bounded, for more details see [1].

Let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C, that is,
(1) for each $t > 0, T(t)$ is a nonexpansive mapping on C;
(2) $T(0)x = x$ for all $x \in C$;
(3) $T(s + t) = T(s) \circ T(t)$ for all $s, t > 0$; and
(4) for each $x \in C$, the mapping $T(.)x$ from $(0, \infty)$ into C is continuous.

Assume that $\mathcal{F} = \cap_{t>0} F(T(t)) \neq \emptyset$. We know that $\mathcal{F}$ is a closed convex subset [2] and that $\mathcal{F} \neq \emptyset$, if C is bounded [3].

For finding a fixed point of a nonexpansive mapping T on C, Alber [4] proposed the following descent-like method:

$$x_{n+1} = P_C(x_n - \mu_n(I - T)x_n), n \geq 0, x_0 \in C, \tag{1}$$

where I denotes the identity mapping in H, and proved that if the sequence of positive real numbers $\{\mu_n\}$ is chosen such that $\mu_n \to 0$ as $n \to \infty$ and $\{x_n\}$ is bounded, then:

(i) there exists a weak accumulation point $\tilde{x} \in C$ of $\{x_n\}$;
(ii) all weak accumulation points of $\{x_n\}$ belong to $F(T)$;
(iii) if $F(T)$ is a singleton, i.e., $F(T) = \{\tilde{x}\}$, then $\{x_n\}$ converges weakly to $\tilde{x}$.

Motivated by Solodov and Svaiter's algorithm [5] , Nakajo and Takahashi [2] introduced the following strongly convergence iteration procedures:

$$\begin{aligned} x_0 &\in C \quad \text{any element}, \\ y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - x_0, z - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), n \geq 0, \end{aligned} \tag{2}$$

where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$, for finding a fixed point of a nonexpansive mapping T on C, and

$$\begin{aligned} x_0 &\in C \quad \text{any element,} \\ y_n &= \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_n &= \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - x_0, z - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), n \ge 0, \end{aligned} \tag{3}$$

where where T_n is defined by

$$T_n y = \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s) y ds,$$

for each $y \in C$, $\alpha_n \in$ [0,a] for some $a \in$ [0,1) and $\{\lambda_n\}$ is a positive real number divergent sequence, for finding a common fixed point of a nonexpansive semigroup $\{T(t) : t > 0\}$.

Further, in 2008, Takahashi, Takeuchi and Kubota [6] proposed a simple variant of (3) that has the following form:

$$\begin{aligned} x_0 &\in H, C_1 = C, x_1 = P_{C_1} x_0, \\ y_n &= \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}} x_0, n \ge 1. \end{aligned} \tag{4}$$

They showed (Theorem 4.4 in [6]) that if $0 \le \alpha_n \le a < 1, 0 < \lambda_n < \infty$ for all $n \ge 1$ and $\lambda_n \to \infty$, then $\{x_n\}$ converges strongly to $u_0 = P_{\mathcal{F}} x_0$. At the time, Saejung [7] considered the following analogue without Bochner integral:

$$\begin{aligned} x_0 &\in H, C_1 = C, x_1 = P_{C_1} x_0, \\ y_n &= \alpha_n x_n + (1-\alpha_n)T(t_n) x_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}} x_0, n \ge 1, \end{aligned} \tag{5}$$

where $0 \le \alpha_n \le a < 1, \liminf_n t_n = 0, \limsup_n t_n > 0$, and $\lim_n (t_{n+1} - t_n) = 0$. Then $\{x_n\}$ converges strongly to $u_0 = P_{\mathcal{F}} x_0$.

If $C \equiv H$, then C_n and Q_n in (2)-(5) are two halfspaces. So, the projection x_{n+1} onto $C_n \cap Q_n$ or C_{n+1} in these methods can be found by an explicit formula

[5]. Clearly, if C is a proper subset of H, then C_n and Q_n in these algorithms are not two halfspaces. Then, the following problem is posed: how to construct the closed convex subsets C_n and Q_n and if we can express x_{n+1} of the above algoritms in a similar form as in [5]? This problem is solved very recently in [8], [9] and [10]. In the works, C_n and Q_n in (2)-(3) are replaced by two halfspaces and y_n is the right hand side of (1) with a modification. In this paper, using the idea, we present a new variant for (4)-(5) where C_{n+1} becomes a halfspace H_{n+1} defined below. More precisely, we consider the following algorithms:

$$\begin{aligned} x_0 \in H = H_0, y_n &= x_n - \mu_n(I - TP_C)x_n, \\ H_{n+1} &= \{z \in H_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} &= P_{H_{n+1}}x_0, n \ge 0, \end{aligned} \tag{6}$$

for finding an element in $F(T)$;

$$\begin{aligned} x_0 \in H = H_0, y_n &= x_n - \mu_n(I - T_nP_C)x_n), \\ H_{n+1} &= \{z \in H_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} &= P_{H_{n+1}}x_0, n \ge 0; \end{aligned} \tag{7}$$

and

$$\begin{aligned} x_0 \in H = H_0, y_n &= x_n - \mu_n(I - T(t_n)P_C)x_n, \\ H_{n+1} &= \{z \in H_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} &= P_{H_{n+1}}x_0, n \ge 0, \end{aligned} \tag{8}$$

for finding an element in $\mathcal{F}$.

We shall prove that iteration processes (6) and (7), (8) converge strongly to a fixed point of T and a common fixed point of $\{T(t) : t > 0\}$ in sections 2 and 3, respectively.

The symbols $\rightharpoonup$ and $\rightarrow$ denote weak and strong convergences, respectively.

2 Strong convergence to a fixed point of nonexpansive mappings

We formulate the following facts needed in the proof of our results.

Lemma 2.1 [11]. *Let C be a nonempty closed convex subset of a real Hilbert space H. For any $x \in H$, there exists a unique $z \in C$ such that*

$$\|z - x\| \leq \|y - x\|$$

for all $y \in C$, and $z = P_C x$ if and only if $\langle z - x, y - z \rangle \geq 0$ for all $y \in C$.

Theorem 2.2. *Let C be a nonempty closed convex subset in a real Hilbert space H and let T be a nonexpansive mapping on C such that $F(T) \neq \emptyset$. Assume that $\{\mu_n\}$ is a sequence in $(a, 1)$ for some $a \in (0, 1]$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by (6), converge strongly to the same point $u_0 = P_{F(T)} x_0$.*

Proof. First, note that $\|y_n - z\| \leq \|x_n - z\|$ is equivalent to

$$\langle y_n - x_n, x_n - z \rangle \leq -\frac{1}{2}\|y_n - x_n\|^2.$$

Thus, H_n is a halfspace. Next, we show that $F(T) \subset H_n$ for all $n \geq 0$. It is clear that $F(T) = F(TP_C) := \{p \in H : TP_C p = p\}$ for any mapping T from C into C. So, we have for each $p \in F(T)$ that

$$\begin{aligned}
\|y_n - p\| &= \|(1 - \mu_n)x_n + \mu_n TP_C x_n - p\| \\
&= \|(1 - \mu_n)(x_n - p) + \mu_n(TP_C x_n - TP_C p)\| \\
&\leq (1 - \mu_n)\|x_n - p\| + \mu_n\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned}$$

Therefore, $p \in H_n$ for all $n \geq 0$.

Further, since $F(T)$ is a nonempty closed convex subset of H, by Lemma 2.1, there exists a unique element $u_0 \in F(T)$ such that $u_0 = P_{F(T)} x_0$. From $x_{n+1} = P_{H_{n+1}} x_0$, we obtain that

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|$$

for every $z \in H_{n+1}$. As $u_0 \in F(T) \subset H_{n+1}$, we get

$$\|x_{n+1} - x_0\| \leq \|u_0 - x_0\| \quad \forall n \geq 0. \tag{9}$$

Now, we show that

$$\lim_{n \to \infty} \|x_{n+m} - x_n\| = 0, \tag{10}$$

for each fixed integer $m > 0$. Indeed, from the definition of H_{n+1}, it implies that $H_{n+1} \subseteq H_n$ and hence we have that

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\| \quad \forall n \geq 0.$$

Therefore, there exists $\lim_n \|x_n - x_0\| = c$. Next, by Lemma 2.1, $x_{n+m} \in H_n$ and $x_n = P_{H_n} x_0$, we get that

$$\langle x_n - x_0, x_{n+m} - x_n \rangle \geq 0.$$

Thus,

$$\begin{aligned}
\|x_{n+m} - x_n\|^2 &= \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_{n+m} - x_n \rangle \\
&\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2
\end{aligned}$$

from that and $\lim_n \|x_n - x_0\| = c$, (10) is implied. So, $\{x_n\}$ is a Cauchy sequence. We assume that $x_n \to p \in H$. On the other hand, from (10) and the following inequalities

$$\begin{aligned}
\|x_n - TP_C x_n\| &= \frac{1}{\mu_n} \|y_n - x_n\| \\
&\leq \frac{1}{a} (\|y_n - x_{n+m}\| + \|x_{n+m} - x_n\|) \\
&\leq \frac{2}{a} \|x_{n+m} - x_n\|,
\end{aligned}$$

we get

$$\lim_{n \to \infty} \|x_n - TP_C x_n\| = 0.$$

So, $p = TP_C p$. It means that $p \in F(T)$. Now, from (9) and Lemma 2.1, it implies that $p = u_0$. The strong convergence of the sequence $\{y_n\}$ to u_0 is followed from

$$\lim_{n \to \infty} \|y_n - x_n\| = \lim_{n \to \infty} \mu_n \|x_n - TP_C x_n\| = 0$$

and $x_n \to u_0$. This completes the proof. □

3 Strong convergence to a common fixed point of nonexpansive semigroups

Lemma 3.1 [12]. *Let C be a nonempty bounded closed convex subset in a real Hilbert space H and let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C. Then, for any $h > 0$*

$$\limsup_{t\to\infty} \sup_{y\in C} \left\| T(h)\left(\frac{1}{t}\int_0^t T(s)y ds\right) - \frac{1}{t}\int_0^t T(s)y ds \right\| = 0.$$

Theorem 3.2. *Let C be a nonempty closed convex subset in a real Hilbert space H and let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \cap_{t>0} F(T(t)) \neq \emptyset$. Assume that $\{\mu_n\}$ is a sequence in $(a, 1]$ for some $a \in (0, 1]$ and $\{\lambda_n\}$ is a positive real number divergent sequence. Then, the sequences $\{x_n\}$ and $\{y_n\}$ defined by (7), converge strongly to the same point $u_0 = P_{\mathcal{F}} x_0$.*

Proof. For each $p \in \mathcal{F} \subseteq C$, we have from (7) and $p = P_C p$ that

$$\begin{aligned}
\|y_n - p\| &= \left\| (1-\mu_n)(x_n - p) + \mu_n \left(\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_C x_n ds - p \right) \right\| \\
&\le (1-\mu_n)\|x_n - p\| + \mu_n \left\| \frac{1}{\lambda_n}\int_0^{\lambda_n} (T(s)P_C x_n - T(s)P_C p) ds \right\| \\
&\le (1-\mu_n)\|x_n - p\| + \mu_n \frac{1}{\lambda_n}\int_0^{\lambda_n} \|x_n - p\| ds \\
&= \|x_n - p\|.
\end{aligned}$$

Therefore, $p \in H_n$. It means that $\mathcal{F} \subset H_n$ for all $n \ge 0$. As in the proof of Theorem 2.2, we get that $\{x_n\}$ is well defined, it converges strongly to an element $p \in H$, and

$$\|x_{n+1} - x_0\| \le \|u_0 - x_0\|, \quad \lim_{n\to\infty} \left\| x_n - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_C x_n ds \right\| = 0, \tag{11}$$

where $u_0 = P_{\mathcal{F}} x_0$. Since

$$\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_C x_n ds \in C$$

and P_C is a nonexpansive mapping, we have that

$$\begin{aligned}\left\|P_Cx_n - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds\right\| &= \left\|P_Cx_n - P_C\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds\right\| \\ &\leq \left\|x_n - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds\right\|.\end{aligned}$$

So, we obtain from (11) that

$$\lim_{n\to\infty}\left\|P_Cx_n - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds\right\| = 0. \tag{12}$$

This together with (11) and $x_n \to p$ implies that the sequence $\{P_Cx_n\}$ also converges strongly to p. Since C is closed, we get $p \in C$.

On the other hand, we have for each $h > 0$ that

$$\begin{aligned}\|T(h)P_Cx_n - P_Cx_n\| &\leq \left\|T(h)P_Cx_n - T(h)\left(\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds\right)\right\| \\ &\quad + \left\|T(h)\left(\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds\right) - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds\right\| \\ &\quad + \left\|\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds - P_Cx_n\right\| \\ &\leq 2\left\|\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds - P_Cx_n\right\| \\ &\quad + \left\|T(h)\left(\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds\right) - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds\right\|.\end{aligned} \tag{13}$$

Let $C_0 = \{z \in C : \|z - u_0\| \leq 2\|x_0 - u_0\|\}$. Since $u_0 = P_{\mathcal{F}}x_0 \in C$, we have from (11) and

$$\begin{aligned}\|P_Cx_n - u_0\| &= \|P_Cx_n - P_Cu_0\| \\ &\leq \|x_n - u_0\| \\ &\leq \|x_n - x_0\| + \|x_0 - u_0\| \\ &\leq 2\|x_0 - u_0\|.\end{aligned}$$

So, C_0 is a nonempty bounded closed convex subset. It is easy to verify that $\{T(t) : t > 0\}$ also is a nonexpansive semigroup on C_0. By Lemma 3.1, (13) and $P_Cx_n \to p$, we get $p = T(h)p$ for each $h > 0$. So, $p \in \mathcal{F}$. Again, from (11) and $p \in \mathcal{F}$, it implies that $p = u_0$ and $y_n \to u_0$ as $n \to \infty$. This completes the proof. □

Theorem 3.3. *Let C be a nonempty closed convex subset in a real Hilbert space H and let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \cap_{t>0} F(T(t)) \neq \emptyset$. Assume that $\{\mu_n\}$ is a sequence in $(a, 1]$ for some $a \in (0, 1]$ and $\{t_n\}$ is a sequence of positive real numbers satisfying the condition $\liminf_n t_n = 0, \limsup_n t_n > 0$, and $\lim_n(t_{n+1} - t_n) = 0$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ defined by (8), converge strongly to the same point $u_0 = P_{\mathcal{F}} x_0$.*

Proof. As in the proof of Theorems 2.2 and 3.2, we get

$$\|x_{n+1} - x_0\| \leq \|u_0 - x_0\|, \quad \lim_{n\to\infty} \|x_n - T(t_n)P_C x_n\| = 0, \tag{14}$$

$$\lim_{n\to\infty} \|P_C x_n - T(t_n)P_C x_n\| = 0, \tag{15}$$

and the sequence $\{x_n\}$ and $\{P_C x_n\}$ also converge strongly to $p \in C$.

Without loss of generality, as in [7], let

$$\lim_{j\to\infty} t_{n_j} = \lim_{j\to\infty} \frac{\|P_C x_{n_j} - T(t_{n_j})P_C x_{n_j}\|}{t_{n_j}} = 0. \tag{16}$$

Now, we prove that $p = T(t)p$ for a fixed $t > 0$. It is easy to see that

$$\begin{aligned}
\|P_C x_{n_j} - T(t)p\| \leq & \sum_{l=0}^{[t-t_{n_j}]-1} \|T(lt_{n_j})P_C x_{n_j} - T((l+1)t_{n_j})P_C x_{k_j}\| \\
& + \left\| T\left(\left[\frac{t}{t_{n_j}}\right]\right) P_C z_{n_j} - T\left(\left[\frac{t}{t_{n_j}}\right]\right) p \right\| \\
& + \left\| T\left(\left[\frac{t}{t_{k_j}}\right]\right) p - T(t)p \right\| \\
\leq & \frac{t}{t_{n_j}} \|P_C x_{n_j} - T(t_{n_j})P_C x_{n_j}\| + \|P_C x_{n_j} - p\| \\
& + \left\| T\left(t - \left[\frac{t}{t_{n_j}}\right] t_{n_j}\right) p - p \right\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|P_C x_{n_j} - T(t)p\| \leq & \frac{t}{t_{n_j}} \|P_C x_{n_j} - T(t_{n_j})P_C x_{n_j}\| \\
& + \|P_C x_{n_j} - p\| + \sup\{\|T(s)p - p\| : 0 \leq s \leq t_{n_j}\}.
\end{aligned}$$

This fact, together with (16) and property (4) for the semigroup, implies that

$$\lim_{j\to\infty} \|P_C x_{n_j} - T(t)p\| = 0.$$

Therefore, $p \in \mathcal{F}$. So, from (14), we have that the sequence $\{x_n\}$ converges strongly to u_0 as $n \to \infty$. The strong convergence of the sequence $\{y_n\}$ to u_0 is followed from (8), (14), $\mu_n \in (a, 1]$ and $x_n \to u_0$ as $n \to \infty$. The theorem is proved. □

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