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Oscillation and Asymptotic Behavior of Solutions of Certain Third-Order Nonlinear Delay Dynamic Equations

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Abstract

The paper deals with the oscillation and asymptotic behavior of solutions of the third-order nonlinear delay dynamic equation

$$\left\{b(t)\left(\left[a(t)\left(x^{\Delta}(t)\right)^{\alpha}\right]^{\Delta}\right)^{\beta}\right\}^{\Delta}+f(t,x(\tau(t)))=0$$

on a time scale $\mathbb{T}$, where $\alpha,\beta>0$ are quotients of odd positive integers. We obtain some sufficient conditions which ensure that every solution of the equation either oscillates or converges to zero. Our results extend and improve some known results.

Mathematics Subject Classification: 34N05
Keywords: Oscillation; Asymptotic behavior; Third-order nonlinear delay dynamic equation; Time scale

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1 Introduction

In this paper, we are concerned with the oscillation and asymptotic behavior of solutions of the third-order nonlinear delay dynamic equation

$$\left\{b(t)\left(\left[a(t)\left(x^{\Delta}(t)\right)^{\alpha}\right]^{\Delta}\right)^{\beta}\right\}^{\Delta}+f(t,x(\tau(t)))=0 \tag{1}$$

on a time scale $\mathbb{T}$. Throughout this paper we assume that the following conditions hold:

(S_1) $\sup\mathbb{T}=\infty$, and α and β are quotients of odd positive integers;

(S_2) $t_0\in\mathbb{T}$, $\mathbb{I}:=\{t:t\in\mathbb{T},t\geq t_0\}$, $a,b\in \mathrm{C_{rd}}(\mathbb{I},\mathbb{R})$, $a(t),b(t)>0$ for $t\in\mathbb{I}$, $\int_{t_0}^{\infty}a^{-1/\alpha}(t)\Delta t=\infty$, and $\int_{t_0}^{\infty}b^{-1/\beta}(t)\Delta t=\infty$;

(S_3) $\tau\in \mathrm{C_{rd}}(\mathbb{T},\mathbb{T}),\tau(t)\leq t$ for $t\in\mathbb{I}$, and $\lim_{t\to\infty}\tau(t)=\infty$;

(S_4) $f\in \mathrm{C}(\mathbb{I}\times\mathbb{R},\mathbb{R})$, and there exists a positive rd-continuous function q defined on $\mathbb{I}$ such that $f(t,u)/(u^{\gamma})\geq q(t)$ for all $t\in\mathbb{I}$ and for all $u\neq 0$, where $\gamma:=\alpha\beta$;

(S_5) $\tau^{\Delta}(t)>0$ is rd-continuous on $\mathbb{T}$, $\tilde{\mathbb{T}}:=\tau(\mathbb{T})=\{\tau(t):t\in\mathbb{T}\}\subset\mathbb{T}$ is a time scale, and $(\tau^{\sigma})(t)=(\sigma\circ\tau)(t)$ for all $t\in\mathbb{T}$, where σ is the forward jump operator on $\mathbb{T}$ and $(\tau^{\sigma})(t):=(\tau\circ\sigma)(t)$.

Recall that a solution of (1) is a nontrivial real function x such that $x\in \mathrm{C^1_{rd}}[t_x,\infty)$, $a(x^{\Delta})^{\alpha}\in \mathrm{C^1_{rd}}[t_x,\infty)$, $b([a(x^{\Delta})^{\alpha}]^{\Delta})^{\beta}\in \mathrm{C^1_{rd}}[t_x,\infty)$ for a certain $t_x\geq t_0$, and x satisfies (1) for $t\geq t_x$. Our attention is restricted to those solutions of (1) which exist on the half-line $[t_x,\infty)$ and satisfy $\sup\{|x(t)|:t>t_*\}>0$ for any $t_*\geq t_x$. A solution x of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

In this work a knowledge and understanding of time scales and of time scale notations is assumed; for an excellent introduction to the calculus on time scales, see Bohner and Peterson [15, 16]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential equations and of difference equations. Many interesting time scales exist, and they give rise to many applications (see [15]). The new theory of the so-called "dynamic equations" not only can unify the theories of differential equations and of difference equations, but also is able to extend these classical cases to cases "in between," e.g., to the so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k = 0, 1, 2, \cdots, q > 1\}$ (which has important applications in quantum theory) and can be applied to different types of time scales like $\mathbb{T} = h\mathbb{Z} := \{hk : k \in \mathbb{Z}, h > 0\}, \mathbb{T} = \mathbb{N}_0^2 := \{k^2 : k = 0, 1, 2, \cdots\}$ and $\mathbb{T} = \mathbb{H}_n$ the space of harmonic numbers. In the last few years, there has been an increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation and asymptotic behavior of solutions of different classes of dynamic equations, and we refer the reader to the papers [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 17, 18, 19, 20, 21, 22, 23, 24] and the references cited therein.

Recently, Erbe et al. [11] considered the case when $\alpha = 1$, $\beta \geq 1$ is a quotient of odd positive integers and $\tau(t) = t$ in (1) and established some sufficient conditions which ensure that every solution of (1) either oscillates or has a finite limit at ∞. Besides, Erbe et al. [14] studied the case when $\alpha = 1$ and β is a quotient of odd positive integers in (1) and improved and extended the results in [11]. Hassan [22] investigated the case when $\alpha = 1$ and $\beta \geq 1$ is a quotient of odd positive integers in (1) and gave several oscillation criteria for (1). Yu and Wang [24] were concerned with the case when α and β are quotients of odd positive integers, $\alpha\beta = 1$ and $\tau(t) = t$ in (1) and obtained two sufficient conditions for the asymptotic and oscillatory behavior of solutions of (1).

It is clear that what the papers [11, 14, 22, 24] considered are some special cases of (1) and that the results in [11, 14, 22, 24] cannot be applied to the general cases of (1). For instance, all the results in [11, 14, 22] cannot be applied to (1) when $\alpha \neq 1$, the results in [11, 24] are invalid when $\tau(t) \neq t$, and the results in [24] fail to be applied to (1) when $\alpha\beta \neq 1$. Therefore, it is of great interest to investigate the oscillation and asymptotic behavior of solutions of (1) in the general cases. In this paper, for the case when α and β are quotients of odd positive integers and $\tau(t) \leq t$, we establish several sufficient conditions which ensure that every solution of (1) either oscillates or tends to zero. Our results extend and improve some of the results presented in [11, 14, 22, 24].

In what follows, for convenience, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large t.

2 Lemmas

Lemma 2.1 (Chen [7], Lemma 2.3). *Suppose that* (S_5) *holds. Let* $x : \mathbb{T} \to \mathbb{R}$. *If* $x^{\Delta}(t)$ *exists for all sufficiently large* $t \in \mathbb{T}$, *then* $(x \circ \tau)^{\Delta}(t) = (x^{\Delta} \circ \tau)(t)\tau^{\Delta}(t)$ *for all sufficiently large* $t \in \mathbb{T}$.

Lemma 2.2 (Chen [7], Lemma 2.4). *Let* $\psi : \mathbb{T} \to \mathbb{R}$ *and* $\lambda > 0$ *be a constant. Furthermore, assume* $\psi^{\Delta}(t) > 0$ *and* $\psi(t) > 0$ *for all sufficiently large* $t \in \mathbb{T}$. *Then we have the following:*

(i) If $0 < \lambda < 1$, *then* $(\psi^{\lambda})^{\Delta}(t) \geq \lambda(\psi^{\sigma})^{\lambda-1}(t)\psi^{\Delta}(t)$ *for all sufficiently large* $t \in \mathbb{T}$, *where* $\psi^{\sigma} := \psi \circ \sigma$;

(ii) If $\lambda \geq 1$, *then* $(\psi^{\lambda})^{\Delta}(t) \geq \lambda\psi^{\lambda-1}(t)\psi^{\Delta}(t)$ *for all sufficiently large* $t \in \mathbb{T}$.

Lemma 2.3 (Hardy et al. [8]). *If* A *and* B *are nonnegative, then*

$$\lambda AB^{\lambda-1} - A^{\lambda} \leq (\lambda - 1)B^{\lambda} \quad \text{when } \lambda > 1,$$

where the equality holds if and only if A=B.

Lemma 2.4. *Suppose that* (S_1)–(S_4) *and the following condition hold:*

$$\int_{t_0}^{\infty} \left\{ a^{-1}(s) \int_{s}^{\infty} \left[b^{-1}(u) \int_{u}^{\infty} q(v)\Delta v \right]^{1/\beta} \Delta u \right\}^{1/\alpha} \Delta s = \infty. \tag{2}$$

Furthermore, suppose that (1) has an eventually positive solution x. *Then*

$$\left[a(t)\big(x^{\Delta}(t)\big)^{\alpha} \right]^{\Delta} > 0, \tag{3}$$

and either $x^{\Delta}(t) > 0$ *or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Since x is an eventually positive solution of (1), form (S_3) there exists $t_1 \geq t_0$ such that

$$x(t) > 0 \quad \text{and} \quad x(\tau(t)) > 0 \quad \text{for} \quad t \in [t_1, \infty). \tag{4}$$

Therefore, from (1) and (S_4) we have for $t \in [t_1, \infty)$,

$$\left\{ b(t) \left(\left[a(t) \big(x^{\Delta}(t) \big)^{\alpha} \right]^{\Delta} \right)^{\beta} \right\}^{\Delta} = -f(t, x(\tau(t))) \leq -q(t) x^{\gamma}(\tau(t)) < 0, \tag{5}$$

which implies that $b(t)\Big(\Big[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\Big]^{\Delta}\Big)^{\beta}$ is strictly decreasing on $[t_1, \infty)$. Thus, $\Big[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\Big]^{\Delta}$ is eventually of one sign, i.e., $\Big[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\Big]^{\Delta}$ is eventually positive or eventually negative.

We now claim

$$\left[a(t) \big(x^{\Delta}(t) \big)^{\alpha} \right]^{\Delta} > 0 \quad \text{for} \quad t \in [t_1, \infty). \tag{6}$$

If not, then there exists $t_2 \geq t_1$ such that

$$\left[a(t) \big(x^{\Delta}(t) \big)^{\alpha} \right]^{\Delta} < 0 \quad \text{for} \quad t \in [t_2, \infty). \tag{7}$$

Since $b(t)\Big(\Big[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\Big]^{\Delta}\Big)^{\beta}$ is strictly decreasing on $[t_1, \infty)$, we get

$$b(t) \left(\left[a(t) \big(x^{\Delta}(t) \big)^{\alpha} \right]^{\Delta} \right)^{\beta} \leq b(t_2) \left(\left[a(t_2) \big(x^{\Delta}(t_2) \big)^{\alpha} \right]^{\Delta} \right)^{\beta} := c_1 < 0$$

for $t \in [t_2, \infty)$. Therefore, we conclude $\Big[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\Big]^{\Delta} \leq c_1^{1/\beta} b^{-1/\beta}(t)$ for $t \in [t_2, \infty)$. Integrating both sides of the last inequality from t_2 to t, we obtain

$$a(t) \big(x^{\Delta}(t) \big)^{\alpha} \leq a(t_2) \big(x^{\Delta}(t_2) \big)^{\alpha} + c_1^{1/\beta} \int_{t_2}^{t} b^{-1/\beta}(s) \Delta s \quad \text{for} \quad t \in [t_2, \infty).$$

Letting $t \to \infty$ and using (S_2), we get $\lim_{t\to\infty} a(t)\big(x^{\Delta}(t)\big)^{\alpha} = -\infty$. Thus, there exists $t_3 \geq t_2$ such that $a(t_3)\big(x^{\Delta}(t_3)\big)^{\alpha} < 0$. It follows from (7) that $a(t)\big(x^{\Delta}(t)\big)^{\alpha}$ is strictly decreasing on $[t_2, \infty)$. Hence, we obtain $a(t)\big(x^{\Delta}(t)\big)^{\alpha} \leq a(t_3)\big(x^{\Delta}(t_3)\big)^{\alpha} := c_2 < 0$ and $x^{\Delta}(t) \leq c_2^{1/\alpha} a^{-1/\alpha}(t)$ for $t \in [t_3, \infty)$. Integrating both sides of the last inequality from t_3 to t, we find

$$x(t) \leq x(t_3) + c_2^{1/\alpha} \int_{t_3}^{t} a^{-1/\alpha}(s) \Delta s \quad \text{for} \quad t \in [t_3, \infty).$$

Letting $t \to \infty$ and using (S_2), we get $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that x is an eventually positive solution of (1). Therefore, (6) holds.

From (6) we conclude that $a(t)\big(x^{\Delta}(t)\big)^{\alpha}$ is strictly increasing on $[t_1,\infty)$. Thus, $a(t)\big(x^{\Delta}(t)\big)^{\alpha}$ as well as $x^{\Delta}(t)$ is eventually of one sign, i.e., $a(t)\big(x^{\Delta}(t)\big)^{\alpha}$ as well as $x^{\Delta}(t)$ is eventually positive or eventually negative. If $x^{\Delta}(t)$ is eventually negative, then we obtain $\lim_{t\to\infty} x(t) := l_1 \geq 0$ and

$$\lim_{t\to\infty} a(t)\big(x^{\Delta}(t)\big)^{\alpha} := l_2 \leq 0, \tag{8}$$

and we conclude that there exists $t_4 \geq t_1$ such that $x(t) \geq l_1$ for $t \in [t_4,\infty)$. Thus, from (S_3) there exists $t_5 \in [t_4,\infty)$ such that

$$x(\tau(t)) \geq l_1 \quad \text{for} \quad t \in [t_5,\infty). \tag{9}$$

Next, we prove $l_1 = 0$. From (5) and (9) we obtain

$$\left\{b(t)\left(\left[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\right]^{\Delta}\right)^{\beta}\right\}^{\Delta} \leq -q(t)x^{\gamma}(\tau(t)) \leq -l_1^{\gamma}q(t) \quad \text{for} \quad t \in [t_5,\infty),$$

Integrating both sides of the last inequality from t to u, we get

$$\begin{aligned} &- b(t)\left(\left[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\right]^{\Delta}\right)^{\beta} \\ &< b(u)\left(\left[a(u)\big(x^{\Delta}(u)\big)^{\alpha}\right]^{\Delta}\right)^{\beta} - b(t)\left(\left[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\right]^{\Delta}\right)^{\beta} \\ &\leq -l_1^{\gamma}\int_t^u q(v)\Delta v \quad \text{for} \quad u \geq t \geq t_5. \end{aligned}$$

Letting $u \to \infty$, we have $-b(t)\left(\left[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\right]^{\Delta}\right)^{\beta} \leq -l_1^{\gamma}\int_t^{\infty} q(v)\Delta v$ and $-\left[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\right]^{\Delta} \leq -l_1^{\gamma/\beta}\left[b^{-1}(t)\int_t^{\infty} q(v)\Delta v\right]^{1/\beta}$ for $t \in [t_5,\infty)$. Integrating both sides of the last inequality from t to ∞, we conclude for $t \in [t_5,\infty)$

$$a(t)\big(x^{\Delta}(t)\big)^{\alpha} \leq -l_2 + a(t)\big(x^{\Delta}(t)\big)^{\alpha} \leq -l_1^{\gamma/\beta}\int_t^{\infty}\left[b^{-1}(u)\int_u^{\infty} q(v)\Delta v\right]^{1/\beta}\Delta u,$$

where l_2 is defined as in (8). Hence, we have

$$x^{\Delta}(t) \leq -l_1\left\{a^{-1}(t)\int_t^{\infty}\left[b^{-1}(u)\int_u^{\infty} q(v)\Delta v\right]^{1/\beta}\Delta u\right\}^{1/\alpha} \quad \text{for} \quad t \in [t_5,\infty).$$

Integrating both sides of the last inequality from t_5 to t, we obtain for $t \geq t_5$,

$$x(t) \leq x(t_5) - l_1 \int_{t_5}^{t} \left\{ a^{-1}(s) \int_{s}^{\infty} \left[b^{-1}(u) \int_{u}^{\infty} q(v)\Delta v \right]^{1/\beta} \Delta u \right\}^{1/\alpha} \Delta s.$$

Assume $l_1 > 0$. Letting $t \to \infty$ and using (2), we see $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that x is an eventually positive solution of (1). Therefore, we have $l_1 = 0$, which implies $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Lemma 2.5. *Suppose that (S_1)–(S_4) hold and that x is an eventually positive solution of (1). Furthermore, assume that there exists $T_* \in [t_0, \infty)$ such that $\left[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\right]^{\Delta} > 0$ and $x^{\Delta}(t) > 0$ for $t \in [T_*, \infty)$. Then there exists $T \geq T_*$ such that*

$$x^{\Delta}(t) > g_1(t,T) b^{1/\gamma}(t) \Big(\left[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\right]^{\Delta} \Big)^{1/\alpha} \quad \text{for} \quad t \in [T, \infty), \tag{10}$$

where $g_1(t,T) := a^{-1/\alpha}(t) \Big(\int_T^t b^{-1/\beta}(s)\Delta s \Big)^{1/\alpha}$, here γ is defined as in (S_4).

Proof. Proceeding as in the proof of Lemma 2.4, we obtain (4) and (5). Let $T := \max\{t_1, T_*\}$. Since $x^{\Delta}(t) > 0$ for $t \in [T, \infty)$, we have for $t \in [T, \infty)$,

$$\begin{aligned} a(t)\big(x^{\Delta}(t)\big)^{\alpha} &> a(t)\big(x^{\Delta}(t)\big)^{\alpha} - a(T)\big(x^{\Delta}(T)\big)^{\alpha} \\ &= \int_T^t \left\{ b^{1/\beta}(s) \left[a(s)\big(x^{\Delta}(s)\big)^{\alpha}\right]^{\Delta} \right\} b^{-1/\beta}(s)\Delta s. \end{aligned} \tag{11}$$

From (5) we obtain that $b(t)\Big(\left[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\right]^{\Delta}\Big)^{\beta}$ is strictly decreasing on $[T, \infty)$. Thus, we get $b(s)\Big(\left[a(s)\big(x^{\Delta}(s)\big)^{\alpha}\right]^{\Delta}\Big)^{\beta} \geq b(t)\Big(\left[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\right]^{\Delta}\Big)^{\beta}$ for $t \geq s \geq T$ and

$$b^{1/\beta}(s)\left[a(s)\big(x^{\Delta}(s)\big)^{\alpha}\right]^{\Delta} \geq b^{1/\beta}(t)\left[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\right]^{\Delta} \quad \text{for} \quad t \geq s \geq T. \tag{12}$$

It follows from (11) and (12) that

$$a(t)\big(x^{\Delta}(t)\big)^{\alpha} > b^{1/\beta}(t)\left[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\right]^{\Delta} \int_T^t b^{-1/\beta}(s)\Delta s$$

and

$$\begin{aligned} x^{\Delta}(t) &> a^{-1/\alpha}(t) b^{1/\gamma}(t) \Big(\left[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\right]^{\Delta} \Big)^{1/\alpha} \Big(\int_T^t b^{-1/\beta}(s)\Delta s \Big)^{1/\alpha} \\ &= g_1(t,T) b^{1/\gamma}(t) \Big(\left[a(t)\big(x^{\Delta}(t)\big)^{\alpha}\right]^{\Delta} \Big)^{1/\alpha} \quad \text{for} \quad t \in [T, \infty), \end{aligned} \tag{13}$$

where $g_1(t,T)$ is defined as in Lemma 2.5. The proof is complete. □

3 Main Results

Theorem 3.1. *Assume that* (S_1)–(S_5) *and (2) hold. Furthermore, suppose that, for all sufficiently large* $T \in [t_0, \infty)$*, there exist* $T_1 > T$ *and a positive function* $\varphi \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I}, \mathbb{R})$ *such that* $\tau(T_1) > T$ *and*

$$\limsup_{t\to\infty} \int_{T_1}^{t} \left\{ \varphi(s)q(s) - \frac{(\varphi_+^\Delta(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\tau^\Delta(s)\varphi(s)g_1(\tau(s),T)]^\gamma} \right\} \Delta s = \infty, \quad (14)$$

where $\varphi_+^\Delta(s) := \max\{\varphi^\Delta(s), 0\}$ *and the function* g_1 *is defined as in Lemma 2.5. Then every solution of (1) either oscillates or tends to zero.*

Proof. Let x be a nonoscillatory solution of (1). Without loss of generality, we may assume that x is an eventually positive solution of (1). Proceeding as in the proof of Lemma 2.4, we see that there exists $t_1 \in [t_0, \infty)$ such that (4) and (5) hold. By Lemma 2.4, there exists $t_2 \in [t_1, \infty)$ such that (3) holds for $t \in [t_2, \infty)$ and either $x^\Delta(t) > 0$ for $t \in [t_2, \infty)$ or $\lim_{t\to\infty} x(t) = 0$. Assume $x^\Delta(t) > 0$ for $t \in [t_2, \infty)$. Consider the generalized Riccati substitution

$$w(t) = Q(t)\frac{\varphi(t)}{x^\gamma(\tau(t))} \quad \text{for} \quad t \in [t_2, \infty), \quad (15)$$

where $Q(t) := b(t)\Big(\Big[a(t)\big(x^\Delta(t)\big)^\alpha\Big]^\Delta\Big)^\beta$ and γ is defined as in (S_4). It is easy to see that $w(t) > 0$ for $t \in [t_2, \infty)$. By the following product and quotient rules for the delta derivatives of the product FG and the quotient F/G of two delta differentiable functions F and G:

$$(FG)^\Delta = F^\Delta G + F^\sigma G^\Delta \quad \text{and} \quad \left(\frac{F}{G}\right)^\Delta = \frac{F^\Delta G - FG^\Delta}{GG^\sigma} = \frac{F^\Delta}{G^\sigma} - \frac{FG^\Delta}{GG^\sigma},$$

where $F^\sigma := F \circ \sigma, G^\sigma := G \circ \sigma$ and $GG^\sigma \neq 0$, from (15) we get

$$\begin{aligned} w^\Delta &= Q^\Delta \frac{\varphi}{(x\circ\tau)^\gamma} + Q^\sigma\left[\frac{\varphi}{(x\circ\tau)^\gamma}\right]^\Delta \\ &= Q^\Delta \frac{\varphi}{(x\circ\tau)^\gamma} + Q^\sigma\left[\frac{\varphi^\Delta}{(x\circ\tau^\sigma)^\gamma} - \varphi\frac{[(x\circ\tau)^\gamma]^\Delta}{(x\circ\tau)^\gamma(x\circ\tau^\sigma)^\gamma}\right] \quad \text{on} \quad [t_2,\infty). \quad (17) \end{aligned}$$

Hence, from (5), (15) and (17) we have

$$\begin{aligned} w^\Delta &\le -\varphi q + \frac{\varphi^\Delta}{\varphi^\sigma}w^\sigma - \varphi\frac{Q^\sigma[(x\circ\tau)^\gamma]^\Delta}{(x\circ\tau)^\gamma(x\circ\tau^\sigma)^\gamma} \\ &\le -\varphi q + \frac{\varphi_+^\Delta}{\varphi^\sigma}w^\sigma - \varphi\frac{Q^\sigma[(x\circ\tau)^\gamma]^\Delta}{(x\circ\tau)^\gamma(x\circ\tau^\sigma)^\gamma} \quad \text{on} \quad [t_2,\infty), \quad (18) \end{aligned}$$

where φ_+^Δ is defined as in Theorem 3.1. From (S_5) and Lemma 2.1, there exists $t_3 \in [t_2, \infty)$ such that

$$(x \circ \tau)^\Delta = (x^\Delta \circ \tau)\tau^\Delta > 0 \quad \text{on} \quad [t_3, \infty). \tag{19}$$

If $0 < \gamma < 1$, then by taking $\psi = x \circ \tau$ and by Lemma 2.2 (i) and (19) there exists $t_4 \in [t_3, \infty)$ such that

$$[(x \circ \tau)^\gamma]^\Delta \geq \gamma(x \circ \tau^\sigma)^{\gamma-1}(x \circ \tau)^\Delta = \gamma(x \circ \tau^\sigma)^{\gamma-1}(x^\Delta \circ \tau)\tau^\Delta \quad \text{on} \quad [t_4, \infty). \tag{20}$$

It follows from (18) and (20) that

$$\begin{aligned} w^\Delta &\leq -\varphi q + \frac{\varphi_+^\Delta}{\varphi^\sigma} w^\sigma - \varphi \frac{Q^\sigma \cdot \gamma(x \circ \tau^\sigma)^{\gamma-1}(x^\Delta \circ \tau)\tau^\Delta}{(x \circ \tau)^\gamma (x \circ \tau^\sigma)^\gamma} \\ &= -\varphi q + \frac{\varphi_+^\Delta}{\varphi^\sigma} w^\sigma - \gamma \tau^\Delta \varphi \frac{Q^\sigma}{(x \circ \tau^\sigma)^{\gamma+1}} \cdot \frac{(x \circ \tau^\sigma)^\gamma}{(x \circ \tau)^\gamma}(x^\Delta \circ \tau) \end{aligned} \tag{21}$$

on $[t_4, \infty)$. If $\gamma \geq 1$, then by taking $\psi = x \circ \tau$ and by Lemma 2.2 (ii) and (19) there exists $t_5 \geq t_4$ such that

$$[(x \circ \tau)^\gamma]^\Delta \geq \gamma(x \circ \tau)^{\gamma-1}(x \circ \tau)^\Delta = \gamma(x \circ \tau)^{\gamma-1}(x^\Delta \circ \tau)\tau^\Delta \quad \text{on} \quad [t_5, \infty). \tag{22}$$

It follows from (18) and (22) that

$$\begin{aligned} w^\Delta &\leq -\varphi q + \frac{\varphi_+^\Delta}{\varphi^\sigma} w^\sigma - \varphi \frac{Q^\sigma \cdot \gamma(x \circ \tau)^{\gamma-1}(x^\Delta \circ \tau)\tau^\Delta}{(x \circ \tau)^\gamma (x \circ \tau^\sigma)^\gamma} \\ &= -\varphi q + \frac{\varphi_+^\Delta}{\varphi^\sigma} w^\sigma - \gamma \tau^\Delta \varphi \frac{Q^\sigma}{(x \circ \tau^\sigma)^{\gamma+1}} \cdot \frac{(x \circ \tau^\sigma)}{(x \circ \tau)}(x^\Delta \circ \tau) \end{aligned} \tag{23}$$

on $[t_5, \infty)$. From (S_5) we see $\tau(t)$ is increasing on $\mathbb{T}$. Since $t \leq \sigma(t)$ for $t \in \mathbb{T}$, we have $\tau(t) \leq \tau^\sigma(t)$ for $t \in \mathbb{T}$. In view of $x^\Delta(t) > 0$ for $t \in [t_2, \infty)$, we have $(x \circ \tau)(t) \leq (x \circ \tau^\sigma)(t)$ for $t \in [t_2, \infty)$. Therefore, for all $\gamma > 0$, from (21) and (23) we get

$$w^\Delta \leq -\varphi q + \frac{\varphi_+^\Delta}{\varphi^\sigma} w^\sigma - \gamma \tau^\Delta \varphi \frac{Q^\sigma}{(x \circ \tau^\sigma)^{\gamma+1}}(x^\Delta \circ \tau) \quad \text{on} \quad [t_5, \infty). \tag{24}$$

From (10) and the definition of the function Q, there exists $T \in [t_5, \infty)$ such that $x^\Delta(t) > g_1(t, T)Q^{1/\gamma}(t)$ for $t \in [T, \infty)$. Take $t_6 \in (T, \infty)$ such that $\tau(t) > T$ for $t \in [t_6, \infty)$. Then we get $(x^\Delta \circ \tau)(t) > g_1(\tau(t), T)Q^{1/\gamma}(\tau(t))$ for $t \in [t_6, \infty)$. Since $\tau(t) \leq t \leq \sigma(t)$ for $t \in \mathbb{T}$ and (5) implies that $Q(t)$ is decreasing on $[t_1, \infty)$, we have $Q(\tau(t)) \geq Q^\sigma(t)$ for $t \in [t_1, \infty)$. Therefore, we get $(x^\Delta \circ \tau)(t) > g_1(\tau(t), T)(Q^\sigma)^{1/\gamma}(t)$ for $t \in [t_6, \infty)$. Hence, from (24) we obtain for $t \in [t_6, \infty)$,

$$w^\Delta(t) < -\varphi(t)q(t) + \frac{\varphi_+^\Delta(t)}{\varphi^\sigma(t)} w^\sigma(t) - \gamma\tau^\Delta(t)\varphi(t)\frac{Q^\sigma(t)g_1(\tau(t), T)(Q^\sigma)^{1/\gamma}(t)}{(x \circ \tau^\sigma)^{\gamma+1}(t)}. \tag{25}$$

From (15) and (25) we have for $t \in [t_6, \infty)$,

$$w^\Delta(t) < -\varphi(t)q(t) + \frac{\varphi_+^\Delta(t)}{\varphi^\sigma(t)} w^\sigma(t) - \gamma\tau^\Delta(t)\varphi(t)g_1(\tau(t), T)\left(\frac{w^\sigma(t)}{\varphi^\sigma(t)}\right)^\lambda, \tag{26}$$

where $\lambda = 1 + 1/\gamma$. For $t \in [t_6, \infty)$, taking $A = \left[\gamma\tau^\Delta(t)\varphi(t)g_1(\tau(t), T)\right]^{1/\lambda} \frac{w^\sigma(t)}{\varphi^\sigma(t)}$ and $B = \left\{\frac{\varphi_+^\Delta(t)}{\lambda\left[\gamma\tau^\Delta(t)\varphi(t)g_1(\tau(t),T)\right]^{1/\lambda}}\right\}^\gamma$, by Lemma 2.3 and (26) we obtain

$$w^\Delta(t) < -\varphi(t)q(t) + \frac{(\varphi_+^\Delta(t))^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\tau^\Delta(t)\varphi(t)g_1(\tau(t), T)]^\gamma} \tag{27}$$

for $t \in [t_6, \infty)$. Integrating both sides of the last inequality from t_6 to t, we obtain for $t \in [t_6, \infty)$,

$$w(t) - w(t_6) \leq -\int_{t_6}^t \left\{\varphi(s)q(s) - \frac{(\varphi_+^\Delta(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\tau^\Delta(s)\varphi(s)g_1(\tau(s), T)]^\gamma}\right\}\Delta s.$$

Since $w(t) > 0$ for $t \in [t_2, \infty)$, we have for $t \in [t_6, \infty)$,

$$\int_{t_6}^t \left\{\varphi(s)q(s) - \frac{(\varphi_+^\Delta(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\tau^\Delta(s)\varphi(s)g_1(\tau(s), T)]^\gamma}\right\}\Delta s < w(t_6).$$

Thus, we get $\limsup_{t\to\infty} \int_{t_6}^t \left\{\varphi(s)q(s) - \frac{(\varphi_+^\Delta(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\tau^\Delta(s)\varphi(s)g_1(\tau(s),T)]^\gamma}\right\}\Delta s \leq w(t_6)$ $< \infty$, which contradicts (14). Hence, the proof is complete. □

We now introduce a function class $\mathfrak{R}$ to present our next theorem. Let $\mathbb{D} := \{(t, s) \in \mathbb{T} \times \mathbb{T} : t \geq s \geq t_0\}$ and $\mathbb{D}_0 := \{(t, s) \in \mathbb{T} \times \mathbb{T} : t > s \geq t_0$. A function $H \in \mathrm{C_{rd}}(\mathbb{D}, \mathbb{R})$ is said to belong to the class $\mathfrak{R}$ if $H(t, t) = 0$ for $t \geq t_0, H(t, s) > 0$ for $(t, s) \in \mathbb{D}_0$, and H has a rd-continuous delta partial derivative $H^{\Delta_s}(t, s)$ on $\mathbb{D}_0$ with respect to the second variable.

Theorem 3.2. *Assume that* (S_1)*–*(S_5)*and (2) hold. Furthermore, suppose that, for all sufficiently large* $T \in [t_0, \infty)$*, there exist* $T_1 > T$*, a positive function* $\varphi \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I}, \mathbb{R})$*, a function* $H \in \mathfrak{R}$ *and a function* $h \in \mathrm{C}_{\mathrm{rd}}(\mathbb{D}, \mathbb{R})$ *such that* $\tau(T_1) > T$*,*

$$H^{\Delta_s}(t,s) + H(t,s)\frac{\varphi_+^{\Delta}(s)}{\varphi^{\sigma}(s)} = \frac{h(t,s)}{\varphi^{\sigma}(s)} H^{\frac{\gamma}{\gamma+1}}(t,s) \quad \textit{for} \quad (t.s) \in \mathbb{D} \tag{28}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T_1)} \int_{T_1}^{t} \left[H(t,s)\varphi(s)q(s) - \frac{h_+^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}\Psi^{\gamma}(s,T)} \right] \Delta s = \infty, \tag{29}$$

where $\varphi_+^{\Delta}(s)$ *is defined as in Theorem 3.1,* $\Psi(s,T) := \tau^{\Delta}(s)\varphi(s)g_1(\tau(s),T)$ *and* $h_+(t,s) := \max\{0,\ h(t,s)\}$*, here* g_1 *is defined as in Lemma 2.5. Then every solution of (1) either oscillates or tends to zero.*

Proof. Assume that x is a nonoscillatory solution of (1). Without loss of generality, assume that x is an eventually positive solution of (1). Proceeding as in the proof of Theorem 3.1, we see that (26) holds. Multiplying (26) by $H(t,s)$ and then integrating from t_6 to t, we find for $t \in [t_6, \infty)$,

$$\begin{aligned}\int_{t_6}^{t} H(t,s)\varphi(s)q(s)\Delta s \le -\int_{t_6}^{t} H(t,s)w^{\Delta}(s)\Delta s + \int_{t_6}^{t} H(t,s)\frac{\varphi_+^{\Delta}(s)}{\varphi^{\sigma}(s)} w^{\sigma}(s)\Delta s \\ - \int_{t_6}^{t} H(t,s)\gamma\Psi(s,T)\left(\frac{w^{\sigma}(s)}{\varphi^{\sigma}(s)}\right)^{1+1/\gamma} \Delta s,\end{aligned} \tag{30}$$

where $\Psi(s,T)$ is defined as in Theorem 3.2. Applying the integration by parts formula

$$\int_c^d F(s)G^{\Delta}(s)\Delta s = \Big[F(s)G(s)\Big]_c^d - \int_c^d F^{\Delta}(s)G(\sigma(s))\Delta s,$$

we get for $t \in [t_6, \infty)$,

$$\begin{aligned}-\int_{t_6}^{t} H(t,s)w^{\Delta}(s)\Delta s &= \Big[-H(t,s)w(s)\Big]_{s=t_6}^{s=t} + \int_{t_6}^{t} H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s \\ &= H(t,t_6)w(t_6) + \int_{t_6}^{t} H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s.\end{aligned} \tag{31}$$

Substituting (31) in (30) and then using (28), we obtain for $t \in [t_6, \infty)$

$$\begin{aligned}
&\int_{t_6}^{t} H(t,s)\varphi(s)q(s)\Delta s \\
&\le H(t,t_6)w(t_6) + \int_{t_6}^{t} \Bigg\{ \Big[H^{\Delta_s}(t,s) + H(t,s)\frac{\varphi_+^{\Delta}(s)}{\varphi^{\sigma}(s)} \Big] w^{\sigma}(s) \\
&\qquad\qquad - H(t,s)\gamma\Psi(s,T)\left(\frac{w^{\sigma}(s)}{\varphi^{\sigma}(s)}\right)^{1+1/\gamma} \Bigg\} \Delta s
\end{aligned}$$

$$\begin{aligned}
&= H(t,t_6)w(t_6) + \int_{t_6}^{t} \Big[\frac{h(t,s)}{\varphi^{\sigma}(s)} H^{\frac{\gamma}{\gamma+1}}(t,s) w^{\sigma}(s) \\
&\qquad\qquad - H(t,s)\gamma\Psi(s,T)\left(\frac{w^{\sigma}(s)}{\varphi^{\sigma}(s)}\right)^{1+1/\gamma} \Big] \Delta s \\
&\le H(t,t_6)w(t_6) + \int_{t_6}^{t} \Big[\frac{h_+(t,s)}{\varphi^{\sigma}(s)} H^{\frac{\gamma}{\gamma+1}}(t,s) w^{\sigma}(s) \\
&\qquad\qquad - H(t,s)\gamma\Psi(s,T)\left(\frac{w^{\sigma}(s)}{\varphi^{\sigma}(s)}\right)^{1+1/\gamma} \Big] \Delta s, \qquad (32)
\end{aligned}$$

where $h_+(t,s)$ is defined as in Theorem 3.2. Taking $\lambda = 1 + 1/\gamma$,

$$A = \big[H(t,s)\gamma\Psi(s,T)\big]^{1/\lambda} \frac{w^{\sigma}(t)}{\varphi^{\sigma}(t)}$$

and $B = \left\{ \frac{h_+(t,s)}{\lambda\left[\gamma\Psi(s,T)\right]^{1/\lambda}} \right\}^{\gamma}$ for $t \ge s \ge t_6$, by Lemma 2.3 and (32) we have

$$\int_{t_6}^{t} H(t,s)\varphi(s)q(s)\Delta s \le H(t,t_6)w(t_6) + \int_{t_6}^{t} \frac{h_+^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}\Psi^{\gamma}(s,T)} \Delta s$$

for $t \ge t_6$ and $\frac{1}{H(t,t_6)} \int_{t_6}^{t} \left[H(t,s)\varphi(s)q(s) - \frac{h_+^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}\Psi^{\gamma}(s,T)} \right] \Delta s \le w(t_6)$ for $t \in (t_6, \infty)$. Hence, we get

$$\limsup_{t\to\infty} \frac{1}{H(t,t_6)} \int_{t_6}^{t} \left[H(t,s)\varphi(s)q(s) - \frac{h_+^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}\Psi^{\gamma}(s,T)} \right] \Delta s \le w(t_6) < \infty,$$

which implies a contradiction to (29). Thus, this completes the proof. □

Remark 3.1. *The results obtained in this paper are very general. From Theorems 3.1 and 3.2, we can get many different sufficient conditions for the oscillation and asymptotic behavior of solutions of (1) with different choices of the functions φ and H.*

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