Theoretical Mathematics & Applications, vol.3, no.1, 2013, 221-246
ISSN: 1792- 9687 (print), 1792-9709 (online)
Scienpress Ltd, 2013

On Vector valued Variational Problems

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Abstract

A Mond-Weir type second-order dual associated with a class of vector-valued variational problem in the presence of equality and inequality constraints is formulated. Under the generalized second-order invexity, various second-order duality results are derived for this pair of multiobjective variational problems. A mixed type second-order duality for this class of multiobjective variational problems is also presented. Finally, the relationship between our second-order duality results and those of multiobjective nonlinear programming problems is incorporated through the pair of second-order dual multiobjective variational problems with natural boundary values.

Mathematics Subject Classification: 90C30, 90C11, 90C20, 90C26

Keywords: Vector-valued variational problem, Second-order duality, Mixed type second-order duality, Second-order generalized invexity, Multiobjective nonlinear programming problem.

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Article Info: *Received* : December 13, 2012. *Revised* : February 21, 2013
Published online : April 15, 2013

1 Introduction

Second-order duality in mathematical programming has been extensively studied. This type of duality enjoys computational advantage over the first order duality because it offers tighter bounds. Following the approach of Mangasarian [1], Chen [2] formulated a Wolfe type second-order dual associated with a class of constrained variational problems. Later Husain et al [3] formulated a Mond-Weir type dual to the variational problem considered in [2] to derive various duality results under generalized invexity.

In the modeling of real-life problems, there can be more than one objective in mathematical programming problems representing them. So the subject of multiobjective programming has an important place in optimization theory. Motivated with this faint glimpse of vast applications of multicriteria optimization problems, Husain and Jain [4] have recently presented Wolfe type second-order duality for multiobjective variational problems with equality and inequality constraints under second-order pseudoinvexity.

In this research, we present Mond-Weir type duality for the class of variational problems considered in [4] in order to relax the second-order invexity requirements on the functions that constitute this pair of second-order dual multiobjective variational problems. Further, in order to combine the Wolfe type second-order duality results of [4] and Mond-Weir type second-order duality results for multiobjective variational problems derived in this research, mixed type second-order duality is presented and various second-order duality results are obtained under second-order generalized invexity. Finally, a relationship between our duality results and those of nonlinear multiobjective programming problems is briefly outlined through the pair of second-order multiobjective variational problems with natural boundary values.

2 Pre-Requisites and statement of the problem

Let $I=[a,b]$ be a real interval, $\phi : I\times R^n\times R^n\to R$ *and* $\psi : I\times R^n\times R^n\to R^m$

be twice continuously differentiable functions. In order to consider $\phi\left(t,x(t),\dot{x}(t)\right)$, where $x:I\to R^n$ is differentiable with derivative $\dot{x}$, denoted by ϕ_x *and* $\phi_{\dot{x}}$, the first order derivatives of φ with respect to $x(t)$ *and* $\dot{x}(t)$, respectively, that is,

$$\phi_x=\left(\frac{\partial\phi}{\partial x^1},\frac{\partial\phi}{\partial x^2},\cdots,\frac{\partial\phi}{\partial x^n}\right)^T,\ \phi_{\dot{x}}=\left(\frac{\partial\phi}{\partial \dot{x}^1},\frac{\partial\phi}{\partial \dot{x}^2},\cdots,\frac{\partial\phi}{\partial \dot{x}^n}\right)^T$$

Denote by ϕ_{xx} the Hessian matrix of ϕ, and ψ_x the $m\times n$ Jacobian matrix respectively, that is, with respect to $x(t)$, that is, $\phi_{xx}=\left(\frac{\partial^2\phi}{\partial x^i\partial x^j}\right)$, $i,j=1,2,\ldots n$, ψ_x the $m\times n$ Jacobian matrix

$$\psi_x=\begin{pmatrix}\frac{\partial\psi^1}{\partial x^1} & \frac{\partial\psi^1}{\partial x^2} \cdots & \frac{\partial\psi^1}{\partial x^n}\\ \frac{\partial\psi^2}{\partial x^1} & \frac{\partial\psi^2}{\partial x^2} \cdots & \frac{\partial\psi^2}{\partial x^n}\\ \vdots & \vdots & \vdots\\ \frac{\partial\psi^m}{\partial x^1} & \frac{\partial\psi^m}{\partial x^2} \cdots & \frac{\partial\psi^m}{\partial x^n}\end{pmatrix}_{m\times n}.$$

The symbols $\phi_{\dot{x}}$, $\phi_{\dot{x}x}$, $\phi_{x\dot{x}}$ *and* $\psi_{\dot{x}}$ have analogous representations.

Designate by X, the space of piecewise smooth functions $x:I\to R^n$, with the norm $\|x\|=\|x\|_\infty+\|Dx\|_\infty$, where the differentiation operator D is given by

$$u=Dx\Leftrightarrow x(t)=\int_a^t u(s)\,ds,$$

Thus $\frac{d}{dt}=D$ except at discontinuities.

Consider the following constrained multiobjective variational problem:

(VEP): Minimize $\left(\int_I f^1(t,x,\dot{x})dt,\ldots,\int_I f^p(t,x,\dot{x})dt\right)$

subject to $x(a)=\alpha\ ,x(b)=\beta$

$$g(t,x,\dot{x}) \leqq 0, t\in I \tag{1}$$

$$h(t,x,\dot{x})=0, t\in I \tag{2}$$

where $f^i : I\times R^n\times R^n \to R,\ i\in K=\{1,2,\ldots,p\},\ \ g: I\times R^n\times R^n \to R^m$,

$h: I\times R^n\times R^n \to R^l$ are continuously differentiable functions.

The following convention for equality and inequality will be used. If $\alpha,\beta\in R^n$, then

$$\alpha=\beta \Leftrightarrow \alpha^i=\beta^i \qquad i=1,2,\ldots,n$$
$$\alpha \geqq \beta \Leftrightarrow \alpha^i \geqq \beta^i \qquad i=1,2,\ldots,n$$
$$\alpha \geq \beta \Leftrightarrow \alpha \geqq \beta \quad and \quad \alpha\neq\beta$$
$$\alpha > \beta \Leftrightarrow \alpha^i > \beta^i \qquad i=1,2,\ldots,n$$

Definition 2.1 A feasible solution $\bar{x}$ is efficient for (VEP) if there is no feasible $\hat{x}$ for (VEP) such that

$$\int_I f^i(t,\hat{x},\dot{\hat{x}})dt < \int_I f^i(t,\bar{x},\dot{\bar{x}})dt\ , \quad \text{for some} \quad i\in\{1,2,\ldots,p\}$$

$$\int_I f^j(t,\hat{x},\dot{\hat{x}})dt \leqq \int_I f^j(t,\bar{x},\dot{\bar{x}})dt\ , \quad \text{for all} \quad j\in\{1,2,\ldots,p\}$$

In the case of maximization, the signs of above inequalities are reversed.

The optimality conditions for the problems (VEP), derived by Husain and Jain [4] are reproduced in the following theorems (Theorem 2.1 and Theorem 2.2).

Theorem 2.1 (Karush-Kuhn-Tucker type necessary optimality conditions): If $\bar{x}$ is an normal and optimal solution of (VEP) and $h_x(.,\bar{x}(.),\dot{\bar{x}}(.))$ maps onto a closed subspace of $C(I, R^l)$, then there exist piecewise smooth $\bar{y} : I \to R^m$, $\bar{z} : I \to R^l$ such that

$$f_x(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{y}(t)^T g_x(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{z}(t)^T h_x(t,\bar{x}(t),\dot{\bar{x}}(t))$$
$$= D\Big(f_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{y}(t)^T g_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{z}(t)^T h_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t))\Big),\ t \in I$$

$$\bar{y}(t)^T g(t,\bar{x}(t),\dot{\bar{x}}(t)) = 0, \quad t \in I$$
$$\bar{y}(t) \geqq 0, \quad t \in I$$

Theorem 2.2 (Fritz John type necessary optimality conditions): Let $\bar{x}$ be an efficient solution of (VEP).Then there exist $\bar{\lambda}^i \in R, i \in K$ and piecewise smooth functions $\bar{y} : I \to R^m$, $\bar{z} : I \to R^l$ such that

$$\sum_{i=1}^{p} \bar{\lambda}^i \left(f_x^i - Df_{\dot{x}}^i \right) + \left(\bar{y}(t)^T g_x + \bar{z}(t)^T h_x \right) - D\left(\bar{y}(t)^T g_{\dot{x}} + \bar{z}(t)^T h_{\dot{x}} \right) = 0,\ t \in I$$

$$\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}}\right) = 0,\ t \in I$$
$$\left(\bar{\lambda}, \bar{y}(t)\right) \geqq 0,\ t \in I$$
$$\left(\bar{\lambda}, \bar{y}(t), \bar{z}(t)\right) \neq 0,\ t \in I$$

We shall make use of the following definitions in the subsequent analysis:

Definition 2.2 The function $\int_I \phi(t,.,.)dt$ is second-order pseudoinvex if for all $\beta(t) \in R^n$, there exist an $\eta = \eta(t,x,u)$ such that

$$\int_I \left\{ \eta^T \phi_u(t,u,\dot{u}) + \left(\frac{d\eta}{dt}\right)^T \phi_{\dot{u}}(t,u,\dot{u}) + \eta^T A\beta(t) \right\} dt \geqq 0$$
$$\Rightarrow \int_I \phi(t,x,\dot{x})dt \geqq \int_I \left(\phi(t,u,\dot{u}) - \frac{1}{2}\beta^T(t)A\beta(t) \right) dt$$

or equivalently,

$$\int_I \phi(t,x,\dot{x})dt < \int_I \left(\phi(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T A\beta(t) \right) dt$$

$$\Rightarrow \int_I \left\{ \eta^T \phi_u(t,u,\dot{u}) + \left(\frac{d\eta}{dt}\right)^T \phi_{\dot{u}}(t,u,\dot{u}) + \eta^T A\beta(t) \right\} dt < 0$$

where $\quad A = f^i_{xx} - 2Df^i_{x\dot{x}} + D^2 f^i_{\dot{x}\dot{x}} - D^3 f^i_{\ddot{x}\ddot{x}}$

Definition 2.3 The function $\int_I \phi(t,.,.)dt$ is said to second-order quasi-invex if for all $\beta(t) \in R^n$, there exist an $\eta = \eta(t,x,u)$ such that

$$\int_I \phi(t,x,\dot{x})dt \leqq \int_I \left(\phi(t,u,\dot{u}) - \frac{1}{2}\beta^T(t) A\beta(t) \right) dt$$

$$\Rightarrow \int_I \left\{ \eta^T \phi_u(t,u,\dot{u}) + \left(\frac{d\eta}{dt}\right)^T \phi_{\dot{u}}(t,u,\dot{u}) + \eta^T A\beta(t) \right\} dt \leqq 0$$

The function $\int_I \phi(t,.,.)dt$ is said to strictly pseudoinvex if for all $\beta(t) \in R^n$ *and* $x(t) \neq u(t), t \in I$, there exist an $\eta = \eta(t,x,u)$ such that

$$\int_I \left\{ \eta^T \phi_u(t,u,\dot{u}) + \left(\frac{d\eta}{dt}\right)^T \phi_{\dot{u}}(t,u,\dot{u}) + \eta^T A\beta(t) \right\} dt \geqq 0$$

$$\Rightarrow \int_I \phi(t,x,\dot{x})dt > \int_I \phi(t,u,\dot{u})dt$$

or equivalently

$$\int_I \phi(t,x,\dot{x})dt \leqq \int_I \phi(t,u,\dot{u})dt$$

$$\int_I \left\{ \eta^T \phi_u(t,u,\dot{u}) + \left(\frac{d\eta}{dt}\right)^T \phi_{\dot{u}}(t,u,\dot{u}) + \eta^T A\beta(t) \right\} dt < 0$$

Remark 2.1 If ϕ does not depend explicitly on t then the above definitions reduced to those given in [5].

3 Mond-Weir type second-order Duality

In this section, we propose the following Mond-Weir type second-order to the problem(VEP):

(M-WED): Maximize

$$\left(\int_I \left(f^1(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T F^1 \beta(t) \right) dt, \ldots, \int_I \left(f^p(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T F^p \beta(t) \right) dt \right)$$

subject to

$$u(a) = \alpha \ , u(b) = \beta \tag{3}$$

$$\lambda^T f_u + y(t)^T g_u + z(t)^T h_u - D\left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}} + z(t)^T h_{\dot{u}}\right) + B\beta(t) = 0 \,, t \in I \tag{4}$$

$$\int_I \left(y(t)^T g(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T G \beta(t) \right) dt \geqq 0 \tag{5}$$

$$\int_I \left(z(t)^T h(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T H \beta(t) \right) dt \geqq 0 \tag{6}$$

$$y(t) \geqq 0 \,, t \in I \tag{7}$$

$$\lambda > 0 \tag{8}$$

where $B = \lambda^T F + G + H$ with

$$F^i = f^i_{uu} - 2Df^i_{u\dot{u}} + D^2 f^i_{\dot{u}\dot{u}} - D^3 f^i_{\dot{u}\ddot{u}} \ , \ t \in I, i \in K$$

$$G = \left(y(t)^T g_u\right)_u - 2D\left(y(t)^T g_u\right)_{\dot{u}} + D^2\left(y(t)^T g_{\dot{u}}\right)_{\dot{u}} - D^3\left(y(t)^T g_{\dot{u}}\right)_{\ddot{u}} \,, t \in I$$

and

$$H = \left(z(t)^T h_u\right)_u - 2D\left(z(t)^T h_u\right)_{\dot{u}} + D^2\left(z(t)^T h_{\dot{u}}\right)_{\dot{u}} - D^3\left(z(t)^T h_{\dot{u}}\right)_{\ddot{u}} \,, t \in I$$

are $n \times n$ symmetric matrices.

In the following analysis, we shall designate the sets of feasible solutions of the problems (VEP) and (M-WED) by X and Y respectively.

Theorem 3.1(Weak duality): Let $x(t) \in X$ and $\left(u(t), \lambda, y(t), z(t), \beta(t)\right) \in Y$ such that with respect to the same η

(A$_1$): $\int_I \left(\lambda^T f\left(t,.,.\right)\right)dt$ is second-order pseudoinvex

(A$_2$): $\int_I \left(y(t)^T g\left(t,.,.\right)\right)dt$ is second-order quasi-invex and

(A$_3$): $\int_I \left(z(t)^T h\left(t,.,.\right)\right)dt$ is second-order quasi-invex,

then

$$\int_I f^r(t,x,\dot{x})dt < \int_I \left\{ f^r(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T F^r \beta(t) \right\} dt \text{, for some } r \in K \tag{9}$$

and

$$\int_I f^i(t,x,\dot{x})dt \leqq \int_I \left\{ f^i(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T F^i \beta(t) \right\} dt \text{ , } i \in K_r \tag{10}$$

cannot hold.

Proof: Suppose, to the contrary, that (9) and (10) holds. Since $\lambda > 0$, the above inequalities give

$$\int_I \lambda^T f(t,x,\dot{x})dt < \int_I \left(\lambda^T f(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T \lambda^T F \beta(t) \right) dt$$

which because of second-order pseudoinvexity of $\int_I \lambda^T f(t,.,.)dt$, this implies

$$\int_I \left\{ \eta^T \left(\lambda^T f_u(t,u,\dot{u})\right) + \left(\frac{d\eta}{dt}\right)^T \left(\lambda^T f_{\dot{u}}(t,u,\dot{u})\right) + \eta^T (\lambda^T F)\beta(t) \right\} dt < 0 \tag{11}$$

Using $x(t) \in X$ and $\left(u(t), \lambda, y(t), z(t), x(t)\right) \in Y$, we have

$$\int_I y(t)^T g(t,x,\dot{x})dt \leqq \int_I \left(y(t)^T g(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T G \beta(t) \right) dt$$

By the hypothesis (A$_2$), this implies

$$\int_I \left\{ \eta^T \left(y(t)^T g_u \right) + \left(\frac{d\eta}{dt} \right)^T \left(y(t)^T g_{\dot{u}} \right) + \eta^T G\beta(t) \right\} dt \leqq 0 \tag{12}$$

Also $\int_I z(t)^T h(t,x,\dot{x})dt \leqq \int_I z(t)^T h(t,u,\dot{u})dt$, because ($A_3$) implies

$$\int_I \left\{ \eta^T \left(z(t)^T h_u \right) + \left(\frac{d\eta}{dt} \right)^T \left(z(t)^T h_{\dot{u}} \right) + \eta^T H\beta(t) \right\} dt \leqq 0 \tag{13}$$

Combining (11), (12) and (13), we have

$$0 > \int_I \left\{ \eta^T \left(\lambda^T f_u + y(t)^T g_u + z(t)^T h_u \right) + \left(\frac{d\eta}{dt} \right)^T \left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}} + z(t)^T h_{\dot{u}} \right) + \eta^T (\lambda^T F + G + H)\beta(t) \right\} dt$$

$$= \int_I \eta^T \left[\left(\lambda^T f_u + y(t)^T g_u + z(t)^T h_u \right) - D\left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}} + z(t)^T h_{\dot{u}} \right) + (\lambda^T F + G + H)\beta(t) \right] dt$$

$$+ \eta^T \left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}} + z(t)^T h_{\dot{u}} \right) \Big|_{t=a}^{t=b}$$

which, because of $\eta = 0,$ at $t = a$ *and* $t = b,$ yields

$$\int_I \eta^T [\left(\lambda^T f_u + y(t)^T g_u + z(t)^T h_u \right) - D\left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}} + z(t)^T h_{\dot{u}} \right)$$

$$+ (\lambda^T F + G + H)\beta(t)] dt < 0$$

i.e.

$$\int_I \eta^T \left[\left(\lambda^T f_u + y(t)^T g_u + z(t)^T h_u \right) - D\left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}} + z(t)^T h_{\dot{u}} \right) + B\beta(t) \right] dt < 0$$

This is contrary to the equality constraint of the dual problem (M-WED). Hence the theorem fully follows.

Theorem 3.2 (Strong duality): Let $\bar{x}(t)$ be normal and efficient solution of (VEP). Then there exist $\bar{\lambda} \in R^p$, piecewise smooth functions $\bar{y} : I \to R^m$ and $\bar{z} : I \to R^l$ such that $\left(\bar{x}(t), \bar{\lambda}, \bar{y}(t), \bar{z}(t), \bar{\beta}(t) = 0 \right)$ is feasible for (M-WED) and the two objective functional are equal. Furthermore, if the hypotheses of Theorem 2.1

hold for all feasible solutions of (VEP) and (M-WED),the $\left(\bar{x}(t), \bar{\lambda}, \bar{y}(t), \bar{z}(t), \bar{\beta}(t)=0\right)$ is an efficient for (M-WED).

Proof: Since $\bar{x}(t)$ is normal and an efficient solution of the problem (VEP), therefore, by Theorem 2.1, there exist $\bar{\lambda} \in R^p$, piecewise smooth functions $\bar{y}: I \to R^m$ and $\bar{z}: I \to R^l$ such that

$$\begin{aligned}&\bar{\lambda}^T f_x + \bar{y}(t)^T g_x(t,\bar{x},\dot{\bar{x}}) + \bar{z}(t)^T h_x(t,\bar{x},\dot{\bar{x}}) \\ &-D\left(\bar{\lambda}^T f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{z}(t)^T h_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\right) = 0, t \in I\end{aligned} \tag{14}$$

$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0,\ t \in I \tag{15}$$

$$\bar{\lambda} > 0, \sum_{i=1}^{p} \bar{\lambda}^i = 1 \tag{16}$$

$$\bar{y}(t)^T \geqq 0, t \in I \tag{17}$$

The relation (15) implies

$$\int_I \left(\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\bar{\beta}(t)^T G \bar{\beta}(t) \right) dt = 0$$

Since $\bar{x}(t)$ is feasible for (VEP), we have $h\left(t,\bar{x},\dot{\bar{x}}\right) = 0, t \in I$ and

$$\int_I \left(\bar{z}(t)^T h(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\bar{\beta}(t)^T H \bar{\beta}(t) \right) dt = 0 .$$

Consequently, $\left(\bar{x}, \bar{\lambda}, \bar{y}(t), \bar{z}(t), \bar{\beta}(t)=0\right)$ is feasible for (M-WED).

Consider

$$\begin{aligned}&\left(\int_I f^1(t,\bar{x},\dot{\bar{x}})dt, \ldots, \int_I f^p(t,\bar{x},\dot{\bar{x}})dt \right) \\ &= \left(\int_I \left(f^1(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\bar{\beta}(t)^T F^1 \bar{\beta}(t) \right) dt, \ldots, \int_I \left(f^p(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\bar{\beta}(t)^T F^p \bar{\beta}(t) \right) dt \right)\end{aligned}$$

yielding the equality of the objective functionals.

If $\left(\bar{x}, \lambda, \bar{y}(t), \bar{z}(t), \bar{\beta}(t)=0\right)$ is not efficient, there exists $\left(\hat{u}, \hat{\lambda}, \hat{y}, \hat{z}, \hat{\beta}\right)$ feasible for

(M-WED) such that

$$\int_I \left\{ f^k(t,\hat{u},\dot{\hat{u}}) - \frac{1}{2}\hat{\beta}(t)^T \hat{F}^k \hat{\beta}(t) \right\} dt > \int_I f^k(t,\bar{x},\dot{\bar{x}})dt, \ for \ some \ k \in K$$

and

$$\int_I \left\{ f^r(t,\hat{u},\dot{\hat{u}}) - \frac{1}{2}\hat{\beta}(t)^T \hat{F}^r \hat{\beta}(t) \right\} dt \geqq \int_I f^r(t,\bar{x},\dot{\bar{x}})dt, r \neq k.$$

These inequality with $\hat{\lambda} > 0$ implies

$$\int_I \left\{ \hat{\lambda}^T f(t,\hat{u},\dot{\hat{u}}) - \frac{1}{2}\hat{\beta}(t)^T (\hat{\lambda}^T \hat{F}) \hat{\beta}(t) \right\} dt > \int_I \hat{\lambda}^T f(t,\bar{x},\dot{\bar{x}})dt$$

which because of pseudoinvexity of $\int_I \hat{\lambda}^T f(t,.,.)dt$ yield for $\eta(t,\bar{x},\hat{u}) = \eta$

$$\int_I \left\{ \eta^T \left(\hat{\lambda}^T f_u(t,\hat{u},\dot{\hat{u}}) \right) + \left(\frac{d\eta}{dt} \right)^T \left(\hat{\lambda}^T f_{\dot{u}}(t,\hat{u},\dot{\hat{u}}) \right) + \eta^T (\hat{\lambda}^T \hat{F}) \hat{\beta}(t) \right\} dt < 0 \tag{18}$$

where $\hat{F} = F(t,\hat{u},\dot{\hat{u}},\ddot{\hat{u}},\dddot{\hat{u}})$

From the feasibility, we have

$$\int_I \hat{y}(t)^T g(t,\bar{x},\dot{\bar{x}})dt \leqq \int_I \left(\hat{y}(t)^T g(t,\hat{u},\dot{\hat{u}}) - \frac{1}{2}\hat{\beta}(t)^T \hat{G} \hat{\beta}(t) \right) dt \tag{19}$$

$$\int_I \hat{z}(t)^T h(t,\bar{x},\dot{\bar{x}})dt \leqq \int_I \left(\hat{z}(t)^T h(t,\hat{u},\dot{\hat{u}}) - \frac{1}{2}\hat{\beta}(t)^T \hat{H} \hat{\beta}(t) \right) dt \tag{20}$$

where $\hat{G} = G(t,\hat{u},\dot{\hat{u}},\ddot{\hat{u}},\dddot{\hat{u}}),\ \hat{H} = H(t,\hat{u},\dot{\hat{u}},\ddot{\hat{u}},\dddot{\hat{u}})$

These, because second-order quasi-invexity of $\int_I \hat{y}(t)^T g(t,.,.)dt$ and $\int_I \hat{z}(t)^T h(t,.,.)dt$ with respect to the same η respectively, yield

$$\int_I \left\{ \eta^T \left(\hat{y}(t)^T g_u(t,\hat{u},\dot{\hat{u}}) \right) + \left(\frac{d\eta}{dt} \right)^T \left(\hat{y}(t)^T g_{\dot{u}}(t,\hat{u},\dot{\hat{u}}) \right) + \eta^T \hat{G} \hat{\beta}(t) \right\} dt \leqq 0$$

and

$$\int_I \left\{ \eta^T \left(\hat{z}(t)^T h_u(t,\hat{u},\dot{\hat{u}}) \right) + \left(\frac{d\eta}{dt} \right)^T \left(\hat{z}(t)^T h_{\dot{u}}(t,\hat{u},\dot{\hat{u}}) \right) + \eta^T \hat{H} \hat{\beta}(t) \right\} dt \leqq 0$$

Combining (18), (19) and (20), we have

$$0 > \int_I \eta^T \left[\left(\hat{\lambda}^T f_u + \hat{y}(t)^T g_u + \hat{z}(t)^T h_u \right) - D\left(\hat{\lambda}^T f_{\dot{u}} + \hat{y}(t)^T g_{\dot{u}} + \hat{z}(t)^T h_{\dot{u}} \right) + (\hat{\lambda}^T \hat{F} + \hat{G} + \hat{H})\hat{\beta}(t) \right] dt$$

As earlier, this will yield

$$\int_I \eta^T \left[\left(\hat{\lambda}^T f_u + \hat{y}(t)^T g_u + \hat{z}(t)^T h_u \right) - D\left(\hat{\lambda}^T f_{\dot{u}} + \hat{y}(t)^T g_{\dot{u}} + \hat{z}(t)^T h_{\dot{u}} \right) + B\hat{\beta}(t) \right] dt < 0,$$

contradicting the feasibility of $\left(\hat{u}, \hat{\lambda}, \hat{y}, \hat{z}, \hat{\beta}\right)$ for (M-WED).

We now give a Mangasarian type [5] strict-converse duality theorem for the dual (M-WED) to (VEP).

Theorem 3.3 (Strict-converse duality): Let $\bar{x}(t)$ and $\left(\bar{u}(t), \bar{\lambda}, \bar{y}(t), \bar{z}(t), \bar{\beta}(t)\right) \in Y$ be efficient solutions of (VEP) and (M-WED) respectively, such that

$$\int_I \bar{\lambda}^T f(t, \bar{x}, \dot{\bar{x}}) dt = \int_I \left\{ \bar{\lambda}^T f(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2} \bar{\beta}(t)^T (\bar{\lambda}^T F) \bar{\beta}(t) \right\} dt \tag{21}$$

If with respect to the same η

(B_1): $\int_I \left(\lambda^T f(t,.,.) \right) dt$ is second-order strictly pseudoinvex

(B_2): $\int_I \left(y(t)^T g(t,.,.) \right) dt$ is second-order quasi-invex,

and

(B_3): $\int_I \left(z(t)^T h(t,.,.) \right) dt$ is second-order quasi-invex,

then

$$\bar{x}(t) = \bar{u}(t), t \in I.$$

Proof: Suppose $\bar{x}(t) \neq u(t), t \in I$

By hypothesis (B_1), (21) implies for $\eta = \eta(t, \bar{x}, \bar{u})$

$$\int_I \left\{ \eta^T \left(\bar{\lambda}^T f_u \right) + \left(\frac{d\eta}{dt} \right)^T \left(\bar{\lambda}^T f_{\dot{u}} \right) + \eta^T (\bar{\lambda}^T \bar{F}) \bar{\beta}(t) \right\} dt < 0 \tag{22}$$

From feasibility of (VEP) and (M-WED), we have

$$\int_I \bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}})dt \leqq \int_I \left(\bar{y}(t)^T g(t,\bar{u},\dot{\bar{u}}) - \frac{1}{2}\bar{\beta}(t)^T \bar{G}\bar{\beta}(t) \right) dt$$

where $\bar{G} = G(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}},\dddot{\bar{u}},\dot{\bar{y}},\ddot{\bar{y}},\dddot{\bar{y}})$

This by the hypothesis (B_2) gives

$$\int_I \left\{ \eta^T \left(\bar{y}(t)^T g_u(t,\bar{u},\dot{\bar{u}}) \right) + \left(\frac{d\eta}{dt} \right)^T \left(\bar{y}(t)^T g_{\dot{u}}(t,\bar{u},\dot{\bar{u}}) \right) + \eta^T \bar{G}\bar{\beta}(t) \right\} dt \leqq 0 \tag{23}$$

Also we have

$$\int_I \bar{z}(t)^T h(t,\bar{x},\dot{\bar{x}})dt \leqq \int_I \bar{z}(t)^T h(t,\bar{u},\dot{\bar{u}})dt$$

by the hypothesis (B_3), this implies

$$\int_I \left\{ \eta^T \left(\bar{z}(t)^T h_u(t,\bar{u},\dot{\bar{u}}) \right) + \left(\frac{d\eta}{dt} \right)^T \left(\bar{z}(t)^T h_{\dot{u}}(t,\bar{u},\dot{\bar{u}}) \right) + \eta^T \bar{H}\beta(t) \right\} dt \leqq 0 \tag{24}$$

Combining (22), (23) and (24), we have

$$\int_I \{ \eta^T \left(\bar{\lambda}^T f_u(t,\bar{u},\dot{\bar{u}}) + \bar{y}(t)^T g_u(t,\bar{u},\dot{\bar{u}}) + \bar{z}(t)^T h_u(t,\bar{u},\dot{\bar{u}}) \right)$$
$$+ \left(\frac{d\eta}{dt} \right)^T \left(\bar{\lambda}^T f_{\dot{u}}(t,\bar{u},\dot{\bar{u}}) + \bar{y}(t)^T g_{\dot{u}}(t,\bar{u},\dot{\bar{u}}) + \bar{z}(t)^T h_{\dot{u}}(t,\bar{u},\dot{\bar{u}}) \right) + \eta^T (\bar{\lambda}^T \bar{F} + \bar{G} + \bar{H})\bar{\beta}(t) \} dt < 0$$

This, as earlier analysis, implies

$$\int_I \{ \eta^T \left(\bar{\lambda}^T f_u(t,\bar{u},\dot{\bar{u}}) + \bar{y}(t)^T g_u(t,\bar{u},\dot{\bar{u}}) + \bar{z}(t)^T h_u(t,\bar{u},\dot{\bar{u}}) \right)$$
$$-D\left(\bar{\lambda}^T f_{\dot{u}}(t,\bar{u},\dot{\bar{u}}) + \bar{y}(t)^T g_{\dot{u}}(t,\bar{u},\dot{\bar{u}}) + \bar{z}(t)^T h_{\dot{u}}(t,\bar{u},\dot{\bar{u}}) \right) + \bar{B}\bar{\beta}(t) \} dt < 0$$

This contradicts the feasibility of $\left(\bar{u},\bar{\lambda},\bar{y},\bar{z},\bar{\beta}\right)$ for (M-WED). Hence

$$\bar{x}(t) = \bar{u}(t), t \in I.$$

4 Mixed type second-order duality

Let $M = \{1,2,...,m\}, L = \{1,2,...,l), I_\alpha \subseteq M, \alpha = 0,1,2,...,r$ with $I_\alpha \cap I_\beta = \phi$, $\alpha \neq \beta, \bigcup_{\alpha=1}^{r} I_\alpha = M$ and $J_\alpha \subseteq L, \alpha = 0,1,2,...,r$ with $J_\alpha \cap J_\beta = \phi, \alpha \neq \beta, \bigcup_{\alpha=1}^{r} J_\alpha = L.$

In relation to the problem (VEP), consider the following multiobjective variational problem:

(MVED): Maximize

$$(\int_I (f^1(t,u,\dot{u}) + \sum_{j\in I_0} y^j(t)g^j(t,u,\dot{u}) + \sum_{k\in J_0} z^k(t)h^k(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T A^1\beta(t))dt,$$

$$..., \int_I (f^p(t,u,\dot{u}) + \sum_{j\in I_0} y^j(t)g^j(t,u,\dot{u}) + \sum_{k\in J_0} z^k(t)h^k(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T A^p\beta(t))dt)$$

subject to

$$u(a) = \alpha,\ u(b) = \beta \tag{25}$$

$$\begin{aligned}&\lambda^T f_u(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) + z(t)^T h_u(t,u,\dot{u})\\ &-D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u})\right) + BA\beta(t) = 0,\ t \in I\end{aligned} \tag{26}$$

$$\sum_{j\in I_\alpha}\int_I \left(y^j(t)g^j(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T G^j\beta(t)\right)dt \geqq 0\ ,\alpha = 1,2,...,r \tag{27}$$

$$\sum_{k\in J_\alpha}\int_I \left(z^k(t)h^k(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T H^k\beta(t)\right)dt \geqq 0,\ ,\alpha = 1,2,...,r \tag{28}$$

$$y(t) \geqq 0,\ t \in I \tag{29}$$

$$\lambda > 0, \sum_{i=1}^{p}\lambda^i = 1,, \alpha = 1,2,...,r \tag{30}$$

where

$$A^i = f^i_{uu} - 2Df_{\dot{u}u} + D^2 f_{\dot{u}\dot{u}} - D^3 f_{\ddot{u}\dot{u}} + \sum_{j\in I_\alpha}\left(y^j(t)g^j_u(t,u,\dot{u})\right)_u - 2D\sum_{j\in I_\alpha}\left(y^j(t)g^j_u(t,u,\dot{u})\right)_{\dot{u}}$$
$$+D^2\sum_{j\in I_\alpha}\left(y^j(t)g^j_{\dot{u}}(t,u,\dot{u})\right)_{\dot{u}} - D^3\sum_{j\in I_\alpha}\left(y^j(t)g^j_{\dot{u}}(t,u,\dot{u})\right)_{\ddot{u}} + \sum_{k\in J_\alpha}\left(z^k(t)h^k_u(t,u,\dot{u})\right)_u$$
$$-2D\sum_{k\in J_\alpha}\left(z^k(t)h^k_u(t,u,\dot{u})\right)_{\dot{u}} + D^2\sum_{k\in J_\alpha}\left(z^k(t)h^k_{\dot{u}}(t,u,\dot{u})\right)_{\dot{u}} - D^3\sum_{k\in J_\alpha}\left(z^k(t)h^k_{\dot{u}}(t,u,\dot{u})\right)_{\ddot{u}},$$

the matrix B is the same as defined in the previous section.

$$G^j = \left(y^j(t)g^j_u(t,u,\dot{u})\right)_u - 2D\left(y^j(t)g^j_u(t,u,\dot{u})\right)_{\dot{u}}$$
$$+ D^2\left(y^j(t)g^j_{\dot{u}}(t,u,\dot{u})\right)_{\dot{u}} - D^3\left(y^j(t)g^j_{\dot{u}}(t,u,\dot{u})\right)_{\ddot{u}},$$

$$H^k = \left(z^k(t)h^k_u(t,u,\dot{u})\right)_u - 2D\left(z^k(t)h^k_u(t,u,\dot{u})\right)_{\dot{u}}$$
$$+ D^2\left(z^k(t)h^k_{\dot{u}}(t,u,\dot{u})\right)_{\dot{u}} - D^3\left(z^k(t)h^k_{\dot{u}}(t,u,\dot{u})\right)_{\ddot{u}},$$

We denote by Ω the set of feasible solutions of (MVED).

Theorem 4.1(Weak duality): Let $\bar{x} \in X$ and $\left(u,\lambda,y,z,\beta(t)\right)\in\Omega$ be efficient solutions of (VEP) and (MVED) respectively, such that with respect to the same η

(C$_1$): $\int_I\left(\lambda^T f(t,.,.) + \sum_{j\in I_0} y^j(t)g^j(t,.,.) + \sum_{k\in J_0} z^k(t)h^k(t,.,.)\right)dt$ is second-order pseudoinvex

(C$_2$): $\int_I\left(y(t)^T g(t,.,.)\right)dt$ is second-order quasi-invex, and

(C$_3$): $\int_I\left(z(t)^T h(t,.,.)\right)dt$ is second-order quasi-invex.

Then

$$\int_I f^r(t,x,\dot{x})dt$$
$$< \int_I\left\{f^r(t,u,\dot{u}) + \sum_{j\in I_0} y^j(t)g^j(t,u,\dot{u}) + \sum_{k\in J_0} z^k(t)h^k(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T A^r\beta(t)\right\}dt$$

for some $r \in K$, and

$$\int_I f^q(t,x,\dot{x})dt$$

$$\leqq \int_I \left\{ f^q(t,u,\dot{u}) + \sum_{j\in I_0} y^j(t) g^j(t,u,\dot{u}) + \sum_{k\in J_0} z^k(t) h^k(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T A^q \beta(t) \right\} dt$$

cannot hold.

Proof: Suppose to the contrary that there is x feasible for (VEP) and $\left(u,\lambda,y,z,\beta(t)\right)$ feasible for (MVED) such that

$$\int_I f^r(t,x,\dot{x})dt$$

$$< \int_I \left\{ f^r(t,u,\dot{u}) + \sum_{j\in I_0} y^j(t) g^j(t,u,\dot{u}) + \sum_{k\in J_0} z^k(t) h^k(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T A^r \beta(t) \right\} dt$$

for some $r \in \{1,2,...,,p\}$

$$\int_I f^q(t,x,\dot{x})dt$$

$$\leqq \int_I \left\{ f^q(t,u,\dot{u}) + \sum_{j\in I_0} y^j(t) g^j(t,u,\dot{u}) + \sum_{k\in J_0} z^k(t) h^k(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T A^q \beta(t) \right\} dt$$

$, q \neq r$

Multiplying these by $\lambda > 0$, we have

$$\int_I \lambda^T f(t,x,\dot{x})dt$$

$$< \int_I \left\{ \lambda^T f(t,u,\dot{u}) + \sum_{j\in I_0} y^j(t) g^j(t,u,\dot{u}) + \sum_{k\in J_0} z^k(t) h^k(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T (\lambda^T A) \beta(t) \right\} dt$$

which by the hypothesis (C_1) yields

$$\int_I \{\eta^T (\lambda^T f_u + \sum_{j\in I_0} y^j(t) g_u^j + \sum_{k\in J_0} z^k(t) h_u^k)$$

$$+\left(\frac{d\eta}{dt}\right)^T (\lambda^T f_{\dot{u}} + \sum_{j\in I_0} y^j(t) g_{\dot{u}}^j + \sum_{k\in J_0} z^k(t) h_{\dot{u}}^k) + \eta^T (\lambda^T A)\beta(t)\} dt < 0 \tag{31}$$

for $\alpha = 1,2,...,r$

$$\sum_{j\in I_\alpha} \int_I \left(y^j(t)^T g^j(t,x,\dot{x}) \right) dt \leqq \sum_{j\in I_\alpha} \int_I \left(y^j(t)^T g^j(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T G^j \beta(t) \right) dt,$$

which by the hypothesis (C_2) gives

$$\sum_{j\in I_\alpha}\int_I \left[\eta^T\left(y^j(t)^T g_u^j(t,u,\dot{u})\right)+\left(\frac{d\eta}{dt}\right)^T\left(y^j(t)^T g_{\dot{u}}^j(t,u,\dot{u})\right)+\eta^T G^j\beta(t)\right]dt \leqq 0,$$

$\alpha = 1,2,\ldots,r$

and so

$$\sum_{j\in\bigcup_{\alpha=1}^{r} I_\alpha}\int_I \left[\eta^T\left(y^j(t)^T g_u^j(t,u,\dot{u})\right)+\left(\frac{d\eta}{dt}\right)^T\left(y^j(t)^T g_{\dot{u}}^j(t,u,\dot{u})\right)+\eta^T G^j\beta(t)\right]dt \leqq 0$$

implies

$$\int_I \left[\eta^T\left(y(t)^T g_u(t,u,\dot{u})\right)+\left(\frac{d\eta}{dt}\right)^T\left(y(t)^T g_{\dot{u}}(t,u,\dot{u})\right)+\eta^T G\beta(t)\right]dt \leqq 0 \tag{32}$$

Also for $\alpha = 1,2,\ldots,r$,

$$\sum_{k\in J_\alpha}\int_I \left(z^k(t)^T h^k(t,x,\dot{x})\right)dt \leqq \sum_{k\in J_\alpha}\int_I \left(z^k(t)^T h^k(t,u,\dot{u})-\frac{1}{2}\beta(t)^T H^k\beta(t)\right)dt,$$

By the hypothesis (C_3), this yields

$$\sum_{k\in J_\alpha}\int_I \left[\eta^T\left(z^k(t)^T h_u^k(t,u,\dot{u})\right)+\left(\frac{d\eta}{dt}\right)^T\left(z^k(t)^T h_{\dot{u}}^k(t,u,\dot{u})\right)+\eta^T H^k\beta(t)\right]dt \leqq 0,$$

$\alpha = 1,2,\ldots,r$

and so

$$\sum_{k\in\bigcup_{\alpha=1}^{r} J_\alpha}\int_I \left[\eta^T\left(z^k(t)^T h_u^k(t,u,\dot{u})\right)+\left(\frac{d\eta}{dt}\right)^T\left(z^k(t)^T h_{\dot{u}}^k(t,u,\dot{u})\right)+\eta^T H^k\beta(t)\right]dt \leqq 0$$

implies

$$\int_I \left[\eta^T\left(z(t)^T h_u(t,u,\dot{u})\right)+\left(\frac{d\eta}{dt}\right)^T\left(z(t)^T h_{\dot{u}}(t,u,\dot{u})\right)+\eta^T H\beta(t)\right]dt \leqq 0 \tag{33}$$

Combining (31), (32) and (33), we have

$$\int_I [\eta^T \left(\lambda^T f_u + y(t)^T g_u + z(t)^T h_u\right) + \left(\frac{d\eta}{dt}\right)^T \left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}} + z(t)^T h_{\dot{u}}\right)$$
$$+\eta^T (\lambda^T A + G + H)\beta(t)]dt < 0$$

$$0 > \int_I \eta^T \left[\left(\lambda^T f_u + y(t)^T g_u + z(t)^T h_u\right) - D\left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}} + z(t)^T h_{\dot{u}}\right) + B\beta(t)\right] dt$$
$$+\eta^T \left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}} + z(t)^T h_{\dot{u}}\right)\Big|_{t=a}^{t=b}$$

which, by using $\eta = \eta(t,x,u)\Big|_{t=a}^{t=b} = 0$ implies

$$\int_I \eta^T \left[\left(\lambda^T f_u + y(t)^T g_u + z(t)^T h_u\right) - D\left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}} + z(t)^T h_{\dot{u}}\right) + B\beta(t)\right] dt < 0,$$

contradicting to the feasibility of $\left(u, \lambda, y, z, \beta(t)\right)$ for (MVED).

Hence the conclusion of the theorem is true under stated hypotheses.

Theorem 4.2 (Strong Duality): Let $\bar{x}$ be normal and efficient. Then there exist $\bar{\lambda} \in R^p$ and piecewise smooth functions $\bar{y} : I \to R^m$ and $\bar{z} : I \to R^l$ such that $\left(\bar{x}(t), \bar{\lambda}, \bar{y}(t), \bar{z}(t), \bar{\beta}(t) = 0\right)$ is feasible for (MVED) and the objective values of (VEP) and (MVED) are equal. If also hypothesis of Theorem 4.1 hold, then $\left(\bar{x}(t), \bar{\lambda}, \bar{y}(t), \bar{z}(t), \bar{\beta}(t) = 0\right)$ is efficient for (MVED).

Proof: By Theorem 2.1, there exist $\bar{\lambda} \in R^p$, piecewise smooth functions $\bar{y} : I \to R^m$ and $\bar{z} : I \to R^l$ such that

$$\bar{\lambda}^T f_x + \bar{y}(t)^T g_x + \bar{z}(t)^T h_x - D\left(\bar{\lambda}^T f_{\dot{x}} + \bar{y}(t)^T g_{\dot{x}} + \bar{z}(t)^T h_{\dot{x}}\right) = 0, t \in I$$

$$\bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}) = 0$$

$$\bar{z}(t)^T h(t, \bar{x}, \dot{\bar{x}}) = 0$$

$$\bar{\lambda} > 0, \sum_{i=1}^{p} \bar{\lambda}^i = 1$$

$$\bar{y}(t)^T \geqq 0, t \in I$$

From the above relation, we have

$$\int_I \left(\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\bar{\beta}(t)^T G\bar{\beta}(t) \right) dt = 0$$

and

$$\int_I \left(\bar{z}(t)^T h(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\bar{\beta}(t)^T G\bar{\beta}(t) \right) dt = 0$$

consequently $(\bar{x},\bar{\lambda},\bar{y},\bar{z},\bar{\beta}=0)$ for (MVED).

Consider

$$\int_I f^i(t,\bar{x},\dot{\bar{x}})dt$$

$$= \int_I \left\{ f^i(t,\bar{x},\dot{\bar{x}}) + \sum_{j\in I_0} y^j(t) g^j(t,\bar{x},\dot{\bar{x}}) + \sum_{k\in J_0} z^k(t) h^k(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\beta(t)^T A^i \beta(t) \right\} dt$$

for $i \in K$.

This implies the objective values of (VEP) and (MVED) are equal.

If $\left(\bar{x}(t),\bar{\lambda},\bar{y}(t),\bar{z}(t),\bar{\beta}(t)=0\right)$ is not efficient solution for (MVED), then there exist feasible $\left(\bar{u},\bar{\lambda},\bar{y},\bar{z},\bar{\beta}\right)$ for (MVED) such that

$$\int_I \left\{ f^r(t,\bar{u},\dot{\bar{u}}) + \sum_{j\in I_0} \bar{y}^j(t) g^j(t,\bar{u},\dot{\bar{u}}) + \sum_{k\in J_0} \bar{z}^k(t) h^k(t,\bar{u},\dot{\bar{u}}) - \frac{1}{2}\bar{\beta}(t)^T A^r \bar{\beta}(t) \right\} dt$$

$$> \int_I \left\{ f^r(t,\bar{x},\dot{\bar{x}}) + \sum_{j\in I_0} \bar{y}^j(t) g^j(t,\bar{x},\dot{\bar{x}}) + \sum_{k\in J_0} \bar{z}^k(t) h^k(t,\bar{x},\dot{\bar{x}}) \right\} dt,$$

for some $r \in K$

$$\int_I \left\{ f^q(t,\bar{u},\dot{\bar{u}}) + \sum_{j\in I_0} \bar{y}^j(t) g^j(t,\bar{u},\dot{\bar{u}}) + \sum_{k\in J_0} \bar{z}^k(t) h^k(t,\bar{u},\dot{\bar{u}}) - \frac{1}{2}\bar{\beta}(t)^T A^q \bar{\beta}(t) \right\} dt$$

$$\geqq \int_I \left\{ f^q(t,\bar{x},\dot{\bar{x}}) + \sum_{j\in I_0} \bar{y}^j(t) g^j(t,\bar{x},\dot{\bar{x}}) + \sum_{k\in J_0} \bar{z}^k(t) h^k(t,\bar{x},\dot{\bar{x}}) \right\} dt, q \neq r.$$

Multiplying by $\lambda(>0)$, these imply

$$\int_I \left\{ \bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}}) + \sum_{j\in I_0} \bar{y}^j(t) g^j(t,\bar{u},\dot{\bar{u}}) + \sum_{k\in J_0} \bar{z}^k(t) h^k(t,\bar{u},\dot{\bar{u}}) - \frac{1}{2}\bar{\beta}(t)^T (\bar{\lambda}^T A)\bar{\beta}(t) \right\} dt$$
$$> \int_I \left\{ \bar{\lambda}^T f(t,\bar{x},\dot{\bar{x}}) + \sum_{j\in I_0} \bar{y}^j(t) g^j(t,\bar{x},\dot{\bar{x}}) + \sum_{k\in J_0} \bar{z}^k(t) h^k(t,\bar{x},\dot{\bar{x}}) \right\} dt$$

which because of pseudoinvexity of

$$\int_I \left(\bar{\lambda}^T f(t,.,.) + \sum_{j\in I_0} \bar{y}^j(t) g^j(t,.,.) + \sum_{k\in J_0} \bar{z}^k(t) h^k(t,.,.) \right) dt \quad \text{implies}$$

$$\int_I [\eta^T (\bar{\lambda}^T f_u + \sum_{j\in I_0} \bar{y}^j(t) g_u^j + \sum_{k\in J_0} \bar{z}^k(t) h_u^k) + \left(\frac{d\eta}{dt}\right)^T (\bar{\lambda}^T f_{\dot{u}} + \sum_{j\in I_0} \bar{y}^j(t) g_{\dot{u}}^j + \sum_{k\in J_0} \bar{z}^k(t) h_{\dot{u}}^k)$$
$$-\frac{1}{2}\bar{\beta}(t)^T (\bar{\lambda}^T A)\bar{\beta}(t)]dt < 0$$

As earlier, it implies

$$\int_I \eta^T [\left(\bar{\lambda}^T f_u + \sum_{j\in I_0} \bar{y}^j(t) g_u^j + \sum_{k\in J_0} \bar{z}^k(t) h_u^k \right) - D\left(\bar{\lambda}^T f_{\dot{u}} + \sum_{j\in I_0} \bar{y}^j(t) g_{\dot{u}}^j + \sum_{k\in J_0} \bar{z}^k(t) h_{\dot{u}}^k \right)$$
$$+ (\bar{\lambda}^T A)\bar{\beta}(t)]dt < 0 \tag{34}$$

Also for $\alpha = 1,2,...,r$, we have

$$\sum_{j\in I_\alpha} \int_I \left(\bar{y}^j(t)^T g^j(t,\bar{x},\dot{\bar{x}}) \right) dt \leqq \sum_{j\in I_\alpha} \int_I \left(\bar{y}^j(t)^T g^j(t,\bar{u},\dot{\bar{u}}) - \frac{1}{2}\bar{\beta}(t)^T G^j \bar{\beta}(t) \right) dt$$

$$\sum_{k\in J_\alpha} \int_I \left(\bar{z}^k(t)^T h^k(t,\bar{x},\dot{\bar{x}}) \right) dt \leqq \sum_{k\in J_\alpha} \int_I \left(\bar{z}^k(t)^T h^k(t,\bar{u},\dot{\bar{u}}) - \frac{1}{2}\bar{\beta}(t)^T H^k \bar{\beta}(t) \right) dt$$

This, because of the second-order quasi-invexity of $\sum_{j\in I_\alpha} \int_I \left(\bar{y}^j(t)^T g^j(t,.,.) \right) dt$

and $\sum_{k\in J_\alpha} \int_I \left(\bar{z}^k(t)^T h^k(t,.,.) \right) dt$, for $\alpha = 1,2,...,r$ gives

$$\sum_{j\in I_\alpha} \int_I \left[\eta^T \left(\bar{y}^j(t)^T g_u^j(t,\bar{u},\dot{\bar{u}}) \right) + \left(\frac{d\eta}{dt}\right)^T \left(\bar{y}^j(t)^T g_{\dot{u}}^j(t,\bar{u},\dot{\bar{u}}) \right) + G^j \bar{\beta}(t) \right] dt \leqq 0,$$

$\alpha = 1,2,...,r$ and

$$\sum_{k\in J_\alpha}\int_I\left[\eta^T\left(\bar{z}^k(t)^T h_u^k(t,\bar{u},\dot{\bar{u}})\right)+\left(\frac{d\eta}{dt}\right)^T\left(\bar{z}^k(t)^T h_{\dot{u}}^k(t,\bar{u},\dot{\bar{u}})\right)+H^k\bar{\beta}(t)\right]dt\leqq 0,$$

$\alpha=1,2,\ldots,r$.

These imply

$$\int_I\eta^T\left[\sum_{j\in\bigcup_{\alpha=1}^r I_\alpha}\left(\bar{y}^j(t)g_u^j\right)-D\sum_{j\in\bigcup_{\alpha=1}^r I_\alpha}\left(\bar{y}^j(t)g_{\dot{u}}^j\right)+\left(\sum_{j\in\bigcup_{\alpha=1}^r I_\alpha}G^j\right)\bar{\beta}(t)\right]dt\leqq 0 \tag{35}$$

and

$$\int_I\eta^T\left[\sum_{k\in\bigcup_{\alpha=1}^r J_\alpha}\left(\bar{z}^k(t)h_u^k\right)-D\sum_{k\in\bigcup_{\alpha=1}^r J_\alpha}\left(\bar{z}^k(t)h_{\dot{u}}^k\right)+\left(\sum_{k\in\bigcup_{\alpha=1}^r J_\alpha}H^k\right)\bar{\beta}(t)\right]dt\leqq 0 \tag{36}$$

Combining (34), (35) and (36), we have

$$\int_I\eta^T\left[\left(\bar{\lambda}^T f_u+\bar{y}(t)^T g_u+\bar{z}(t)^T h_u\right)-D\left(\bar{\lambda}^T f_{\dot{u}}+\bar{y}(t)^T g_{\dot{u}}+\bar{z}(t)^T h_{\dot{u}}\right)+B\bar{\beta}(t)\right]dt<0,$$

contradicting the feasibility of $\left(\bar{u},\bar{\lambda},\bar{y},\bar{z},\bar{\beta}\right)$ for (MVED).Hence the efficiency of $\left(\bar{x},\bar{\lambda},\bar{y},\bar{z},\bar{\beta}=0\right)$ for (MVED) follows.

The following theorem gives Mangasarian type [5] strict-converse duality for (VEP) and (MVED):

Theorem 4.3(Strict converse duality): Let $\bar{x}$ and $\left(\bar{u},\bar{\lambda},\bar{y},\bar{z},\bar{\beta}(t)\right)\in\Omega$ be efficient solutions of (VEP) and (MVED) respectively, such that

$$\int_I\bar{\lambda}^T f(t,\bar{x},\dot{\bar{x}})dt$$

$$=\int_I\left\{\bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}})+\sum_{j\in I_0}\bar{y}^j(t)g^j(t,\bar{u},\dot{\bar{u}})+\sum_{k\in J_0}\bar{z}^k(t)h^k(t,\bar{u},\dot{\bar{u}})-\frac{1}{2}\bar{\beta}(t)^T(\bar{\lambda}^T A)\bar{\beta}(t)\right\}dt \tag{37}$$

If with respect to the same η

(D₁): $\int_I \left(\lambda^T f\left(t,.,.\right) + \sum_{j\in I_0} y^j(t) g^j\left(t,.,.\right) + \sum_{k\in J_0} z^k(t) h^k\left(t,.,.\right) \right) dt$ is second-order strictly pseudoinvex

(D₂): $\sum_{j\in I_\alpha} \int_I \left(y^j(t)^T g^j\left(t,.,.\right)\right) dt,\ \alpha = 1,2,\dots,r$ is second-order quasi-invex, and

(D₃): $\sum_{k\in J_\alpha} \int_I \left(z^k(t)^T h^k\left(t,.,.\right)\right) dt,\ \alpha = 1,2,\dots,r$ is second-order quasi-invex,

then

$$\bar{x}(t) = \bar{u}(t), \quad t \in I$$

Proof: Suppose $\bar{x}(t) \neq \bar{u}(t),\ \ t \in I$. By the hypothesis ($D_1$), (37) implies for $\eta = \eta(t, \bar{x}, \bar{u})$

$$\int_I [\eta^T (\bar{\lambda}^T f_u + \sum_{j\in I_0} \bar{y}^j(t) g_u^j + \sum_{k\in J_0} \bar{z}^k(t) h_u^k) + \left(\frac{d\eta}{dt}\right)^T (\bar{\lambda}^T f_{\dot{u}} + \sum_{j\in I_0} \bar{y}^j(t) g_{\dot{u}}^j + \sum_{k\in J_0} \bar{z}^k(t) h_{\dot{u}}^k)$$
$$+ \eta^T (\bar{\lambda}^T A) \bar{\beta}(t)] dt < 0 \tag{38}$$

As earlier, we have

$$\int_I [\bigcup_{j\in \bigcup_{\alpha=1}^r I_\alpha} (\eta^T \left(\bar{y}^j(t) g_u^j\right) + \left(\frac{d\eta}{dt}\right)^T \left(\bar{y}^j(t) g_{\dot{u}}^j\right) + \eta^T G^j \bar{\beta}(t))$$
$$+ \bigcup_{k\in \bigcup_{\alpha=1}^r J_\alpha} (\eta^T \left(\bar{z}^k(t) h_u^k\right) + \left(\frac{d\eta}{dt}\right)^T \left(\bar{z}^k(t) h_{\dot{u}}^k\right) + \eta^T H^k \bar{\beta}(t))] dt \leqq 0 \tag{39}$$

implying

$$\int_I \left[\eta^T \left(\bar{y}(t)^T g_u + \bar{z}(t)^T h_u \right) + \left(\frac{d\eta}{dt}\right)^T \left(\bar{y}(t)^T g_{\dot{u}} + \bar{z}(t)^T h_{\dot{u}} \right) + \eta^T (G+H) \bar{\beta}(t) \right] dt \leqq 0$$

Combining (38) and (39) and then integrating by parts with $\eta = 0,$ at $t = a$ and $t = b$, we have

$$\int_I \eta^T \left[\left(\bar{\lambda}^T f_u + \bar{y}(t)^T g_u + \bar{z}(t)^T h_u \right) - D\left(\bar{\lambda}^T f_{\dot{u}} + \bar{y}(t)^T g_{\dot{u}} + \bar{z}(t)^T h_{\dot{u}} \right) + B\bar{\beta}(t) \right] dt < 0,$$

contradicting the feasibility of $\left(\bar{u}, \bar{\lambda}, \bar{y}, \bar{z}, \bar{\beta}\right)$ for (MVED). Hence

$$\bar{x}(t) = \bar{u}(t),\ t \in I.$$

5 Natural boundary values

It is possible to extend the duality theorems validated in the previous two sections to the corresponding multiobjective variational problem with natural boundary values rather than fixed end points. The proofs of the duality theorems for these pairs of dual problems can be proved analogously to the proofs of the theorems of the preceding sections except that some slight modifications are needed.

(VEP)N: Minimize $\left(\int_I f^1(t,x,\dot{x})dt,\dots,\int_I f^p(t,x,\dot{x})dt\right)$

subject to

$$g\left(t,x,\dot{x}\right) \leqq 0,\ t\in I$$

$$h\left(t,x,\dot{x}\right) = 0,\ t\in I$$

(M-WED)N: Maximize $\left(\int_I \left(f^1(t,u,\dot{u})\right)dt,\dots,\int_I \left(f^p(t,u,\dot{u})\right)dt\right)$

subject to

$$\lambda^T f_u + y(t)^T g_u + z(t)^T h_u - D\left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}} + z(t)^T h_{\dot{u}}\right) + B\beta(t) = 0$$

$$\int_I \left(y(t)^T g\left(t,u,\dot{u}\right) - \frac{1}{2}\beta(t)^T G\beta(t)\right)dt \geqq 0$$

$$\int_I \left(z(t)^T h\left(t,u,\dot{u}\right) - \frac{1}{2}\beta(t)^T H\beta(t)\right)dt \geqq 0$$

$$\lambda > 0,$$

$$y(t) \geqq 0,\ t\in I$$

$$\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}} + z(t)^T h_{\dot{u}} = 0, \text{at} \quad t=a \quad \text{and} \quad t=b.$$

(MVED)N: Maximize

$$\left(\int_I \left(f^1(t,u,\dot{u}) + \sum_{j\in I_0} y^j(t) g^j(t,u,\dot{u}) + \sum_{k\in J_0} z^k(t) h^k(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T A^1 \beta(t)\right)dt,\right.$$

$$\left. \ldots, \int_I \left(f^p(t,u,\dot{u}) + \sum_{j\in I_0} y^j(t) g^j(t,u,\dot{u}) + \sum_{k\in J_0} z^k(t) h^k(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T A^p \beta(t)\right)dt\right)$$

subject to

$$\left(\lambda^T f_u + y(t)^T g_u + z(t)^T h_u\right) - D\left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}} + z(t)^T h_{\dot{u}}\right) + B\beta(t) = 0,\ t\in I$$

$$\sum_{j\in I_\alpha}\int_I \left(y^j(t) g^j(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T G^j \beta(t)\right)dt \geqq 0,\ \alpha = 1,2,\ldots,r$$

$$\sum_{k\in J_\alpha}\int_I \left(z^k(t) h^k(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T H^k \beta(t)\right)dt \geqq 0,\ \alpha = 1,2,\ldots,r$$

$$\lambda > 0, \sum_{i=1}^{p} \lambda^i = 1$$

$$y(t) \geqq 0,\ t\in I$$

$$\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}} + z(t)^T h_{\dot{u}} = 0, \text{ at } t = a \text{ and } t = b.$$

6 Multiobjective non-linear programming problems

If all the function in the problems are independent of t, then we have

(VEP)$_0$: Minimize $\left(f^1(x),\ldots, f^p(x)\right)$

subject to

$$g(x) \leqq 0$$

$$h(x) = 0$$

(M-WED)$_0$: Maximize $\left(f^1(u) - \frac{1}{2}\beta(t)^T \nabla^2 f^1(u), \ldots, f^p(u) - \frac{1}{2}\beta(t)^T \nabla^2 f^p(u)\right)$

subject to

$$\lambda^T f_u + y^T g_u + z^T h_u = 0$$

$$y^T g(u) - \frac{1}{2}\beta^T \nabla^2 y^T g(u)\beta \geqq 0$$

$$z^T h(u) - \frac{1}{2}\beta^T \nabla^2 z^T h(u)\beta \geqq 0$$

$$\lambda > 0, \quad y \geqq 0 .$$

(MVED)$_0$: Maximize

$$(f^1(u) + \sum_{j\in I_0} y^j g^j(u) + \sum_{k\in J_0} z^k h^k(u) - \frac{1}{2}\beta^T A^1 \beta, \ldots,$$
$$f^p(u) + \sum_{j\in I_0} y^j g^j(u) + \sum_{k\in J_0} z^k h^k(u) - \frac{1}{2}\beta^T A^p \beta)$$

subject to

$$\lambda^T f_u + y^T g_u + z^T h_u = 0$$

$$\sum_{j\in I_\alpha}\left(y^j g^j(u) - \frac{1}{2}\beta^T \nabla^2 y^j g^j(u)\beta \right) \geqq 0, \ \alpha = 1, 2, \ldots, r$$

$$\sum_{k\in J_\alpha}\left(z^k h^k(u) - \frac{1}{2}\beta^T \nabla^2 z^k h^k(u)\beta \right) \geqq 0, \ \alpha = 1, 2, \ldots, r$$

$$\lambda > 0, \sum_{i=1}^{p} \lambda^i = 1$$

$$y(t) \geqq 0 .$$

where

$$A^i = f^i_{uu} + \sum_{j\in I_\alpha} y^j g^j_{uu}(u) + \sum_{k\in J_\alpha} z^k h^k_{uu}(u), \ i \in K$$

These problems (M-WED)$_0$ and (MVED)$_0$ are not explicitly reported in the literature. However, if $\beta = 0$, these reduce to the problems treated by Weir [6].

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