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Around Prime Numbers And Twin Primes

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Abstract

In this paper, we characterize primes; we give two characterizations of twin primes and we state the general conjecture. We recall (see [1] or [2] or [3] or [4] or [5] or [6] or [7] or [8]) that an integer t is a *twin prime*, if t is a prime ≥ 3 and if $t-2$ or $t+2$ is also a prime ≥ 3; for example, it is easy to check that $(881, 883)$ is a couple of twin primes.

Mathematics Subject Classification : $05xx$ and $11xx$.
Keywords: twin primes.

Introduction

This paper is divided into three sections. In section 1, we state and prove a Theorem which implies characterizations of primes and twin primes. In section 2, we use the Theorems of section 1 to characterize primes and to give two characterizations of twin primes. In section 3, using the Theorem of section 1, we state the general conjecture from which all the results mentioned follow.

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1 Statement and proof of Theorem which implies characterizations of primes and twin primes

We recall that for every integer $n \geq 1$, $n!$ is defined as follow:

$$n! = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n = 2, \\ 1 \times 2 \times .... \times n & \text{if } n \geq 3. \end{cases}$$

We also recall that if x and z are integers such that $x \geq 1$ and $z \geq 1$, then x divides z if there exists an integer $y \geq 1$, such that $z = xy$.

Theorem 1.1. *Let n be an integer ≥ 4. Then, $n+1$ is prime or $n+1$ divides $n!$.*

Before proving Theorem 1.1, let us remark the following.

Remark 1.2. *Let n be an integer ≥ 4. If $n+1 = p^2$ (where p is prime), then $n+1$ divides $n!$.*

Proof. Otherwise [we reason by reduction to absurd], clearly

$$p^2 \textit{ does not divide } n! \tag{1}$$

and we observe the following.

Observation 1.2.1. *p is a prime ≥ 3.*

Otherwise, clearly $p = 2$, and noticing (via the hypotheses) that $n + 1 = p^2$, then using the previous two equalities, it becomes trivial to deduce that $n = 3$; a contradiction, since $n \geq 4$ (via the hypotheses).

Observation 1.2.2. *$p \leq n$.*

Otherwise, $p > n$ and the previous inequality clearly says that

$$p \geq n + 1. \tag{2}$$

Now noticing (via the hypotheses) that $n + 1 = p^2$, then, using the previous equality and using (2), we trivially deduce that

$$p \geq n + 1 \textit{ and } n + 1 = p^2.$$

From these relations we clearly have that $p \geq p^2$; a contradiction, since $p \geq 3$ (by using Observation 1.2.1).

Observation 1.2.3. *$2p \leq n$.*
Otherwise, $2p > n$ and the previous inequality clearly says that

$$2p \geq n+1. \tag{3}$$

Now noticing (via the hypotheses) that $n+1 = p^2$, then, using the previous equality and using (3), we trivially deduce that

$$2p \geq n+1 \textit{ and } n+1 = p^2.$$

which clearly says that $2p \geq p^2$; a contradiction, since $p \geq 3$ (by using Observation 1.2.1). Observation 1.2.3 follows.
Observation 1.2.4. *$2p \neq p$.*
Indeed, it is immediate that $2p \neq p$, since $p \geq 3$ (by using Observation 1.2.1). Observation 1.2.4 follows.
The previous trivial observations made, look at p (recall that p is prime); observing (by Observations 1.2.2 and 1.2.3 and 1.2.4) that $p \leq n$ and $2p \leq n$ and $p \neq 2p$, then, it becomes trivial to deduce that

$$\{p, 2p\} \subseteq \{1, 2, 3, ..., n-1, n\}.$$

which immediately implies that

$$p \times 2p \quad divides \quad 1 \times 2 \times 3 \times ... \times n-1 \times n.$$

This clearly says that $2p^2$ divides $n!$; in particular p^2 clearly divides $n!$ and this contradicts (1). Remark 1.2 follows. □

The previous remark made, now we prove Theorem 1.1.
Proof of Theorem 1.1. **If** $n+1$ is prime, then the proof is ended. **If** $n+1$ is not prime, then $n+1$ divides $n!$. Otherwise (we reason by reduction absurd)

$$n+1 \, is \, not \ prime \ and \ n+1 \ does \ not \ divide \ n!, \tag{4}$$

and we observe the following.
Observation 1.1.1. *Let p be a prime such that $n+1$ is divisible by p (such a p clearly exists). Then $\frac{n+1}{p}$ is an integer* **and** $\frac{n+1}{p} \leq n$ **and** $p \leq n$ **and** $\frac{n+1}{p} = p$.
Indeed, it is immediate that $\frac{n+1}{p}$ is an integer [since p divides $n+1$], **and** it is also immediate that $\frac{n+1}{p} \leq n$ [otherwise, $n+1 > np$; now, remarking that $p \geq 2$ (since p is prime), then the previous two inequalities imply that

$n+1 > 2n$; so $1 > n$ and we have a contradiction, since $n \geq 4$, by the hypotheses]. **Clearly** $p \leq n$ [otherwise, $p > n$; now, recalling that $n+1$ is divisible by p, then the previous inequality implies that $n+1 = p$. Recalling that p is prime, then the previous equality clearly says that $n+1$ is prime and this contradicts (4). That being so, to prove Observation 1.1.1, it suffices to prove that $\frac{n+1}{p} = p$. **Fact:** $\frac{n+1}{p} = p$ [otherwise, clearly $\frac{n+1}{p} \neq p$; now, remarking (by using the previous) that $\frac{n+1}{p}$ is an integer **and** $\frac{n+1}{p} \leq n$ **and** $p \leq n$; then it becomes trivial to deduce that $\frac{n+1}{p}$ **and** p are two different integers such that $\{p, \frac{n+1}{p}\} \subseteq \{1, 2, 3, ..., n-1, n\}$. The previous inclusion immediately implies that $p \times \frac{n+1}{p}$ divides $1 \times 2 \times 3 \times ... \times n-1 \times n$; therefore $n+1$ divides $n!$, and this contradicts (4). So $\frac{n+1}{p} = p$]. Observation 1.1.1 follows.

The previous trivial observation made, look at $n+1$; observing (by using Observation 1.1.1) that p is prime such that $\frac{n+1}{p} = p$, clearly

$$n+1 = p^2, \textit{ where } p \textit{ is prime.}$$

Using now this and Remark 1.2, then it becomes trivial to deduce that $n+1$ divides $n!$, and this contradicts (4). Theorem 1.1 follows. □

Theorem 1.1 immediately implies the characterizations of primes and twin primes.

2 Characterizations of primes and twin primes

In this section, using Theorem 1.1, we characterize prime numbers and we also give two characterizations of twin primes.

Theorem 2.1. *(Characterization of primes). Let n be an integer ≥ 4 and look at $n+1$. Then the following are equivalent.*

(1). $n+1$ *is prime.*

(2). $n+1$ *does not divide* $n!$.

To prove Theorem 2.1, we need a Theorem of Euclid.

Theorem 2.2 **(Euclid)**. *Let a, b and c, be integers such that $a \geq 1$, $b \geq 1$ and $c \geq 1$. If a divides bc and if the greatest common divisor of a and b is 1, then a divides c.*□

Corollary 2.3. *Let n be an integer ≥ 1 and look at $n!$. Now let p be a prime $\geq n+1$; then the greatest common divisor of $n!$ and p is 1 (in particular, p*

does not divide $n!$).

Proof. Immediate, and follows immediately by using Theorem 2.2 and the definition of $n!$, and by observing that p is a prime $\geq n+1$.$\square$

Now, we prove Theorem 2.1.

Proof of Theorem 2.1. (1) $\Rightarrow$ (2)]. Immediate, by remarking that $n+1$ is prime and by using Corollary 2.3.

(2) $\Rightarrow$ (1)]. Immediate, by remarking that $n+1$ does not divide $n!$ and by using Theorem 1.1. $\square$

Using Theorem 2.1, then the following corollary becomes immediate.

Corollary 2.4.*Let n be an integer ≥ 4 and look at $n+3$. Then the following are equivalent.*

(i). $n+3$ *is prime.*

(ii). $n+3$ *does not divide* $(n+2)!$.

Proof. Immediate, and follows from Theorem 2.1, where we replace n by $n+2$.$\square$

Using Theorem 2.1 and Corollary 2.4, then we have the following weak characterization of twin primes.

Theorem 2.5. *(A weak characterization of twin primes). Let n be an integer ≥ 4 and look at the couple $(n+1, n+3)$. Then the following are equivalent*

(a). $(n+1, n+3)$ *is a couple of twin primes.*

(b). $n+1$ *does not divide* $n!$ *and* $n+3$ *does not divide* $(n+2)!$.

Proof. $(a) \Rightarrow (b)$]. Immediate. Indeed, if $(n+1, n+3)$ is a couple of twin primes, then, $n+1$ does not divide $n!$ (by using Theorem 2.1) and $n+3$ does not divide $(n+2)!$ (by using Corollary 2.4).

$(b) \Rightarrow (a)$]. Immediate. Indeed, if $n+1$ does not divide $n!$ and if $n+3$ does not divide $(n+2)!$, then $n+1$ is prime (by using Theorem 2.1) and $n+3$ is also prime (by using Corollary 2.4); consequently $(n+1, n+3)$ is a couple of twin primes. $\square$

Theorem 2.6. *(A non-weak characterization of twin primes). Let n be an integer ≥ 4 and look at the couple $(n+1, n+3)$. Then the following are equivalent*

(c). $(n+1, n+3)$ *is a couple of twin primes.*

(d). $n+1$ *does not divide* $n!$ *and* $n+3$ *does not divide* $n!$.

To prove simply Theorem 2.6, we need the following two lemmas.

Lemma 2.7. *Let n be an integer ≥ 4. If $n+1$ is prime and if $n+3$ does not*

divide $n!$*, then* $n+3$ *is prime.*
Proof. Otherwise [we reason by reduction to absurd], clearly

$$n+1\,\textit{is prime and}\ n+3\ \textit{does not divide}\ n!\ \textit{and}\, n+3\ \textit{is not prime}, \tag{5}$$

and we observe the following.
Observation 2.7.1. *Let* p *be a prime such that* $n+3$ *is divisible by* p *(such a* p *clearly exists). Then* $p \neq n+3$ **and** $p \neq n+2$ **and** $p \neq n+1$ **and** $p \leq n$.
Indeed, it is immediate that $p \neq n+3$ [since $n+3$ is not prime (use (5), and since p is prime]. It is also immediate that $p \neq n+2$ **and** $p \neq n+1$ [[otherwise

$$p = n+2 \textit{ or } p = n+1.$$

From this and recalling that p divides $n+3$, it becomes trivial to deduce that $n+2$ divides $n+3$ or $n+1$ divides $n+3$. We have a contradiction [since $n \geq 4$ (use the hypotheses), therefore $n+2$ does not divide $n+3$ and $n+1$ does not divide $n+3$]. So $p \neq n+2$ and $p \neq n+1$]]. That being so, to prove Observation 2.7.1, it suffices to prove that $p \leq n$. **Fact:** $p \leq n$ [indeed, recalling that p divides $n+3$, and since we have proved that $p \neq n+3$ and $p \neq n+2$ and $p \neq n+1$, then it becomes trivial to deduce that $p \leq n$]. Observation 2.7.1 follows.
Observation 2.7.2. *Let* p *be a prime such that* $n+3$ *is divisible by* p *(such a* p *clearly exists). Then* $\frac{n+3}{p}$ *is an integer* **and** $\frac{n+3}{p} \leq n$ **and** $\frac{n+3}{p} = p$.
Indeed, it is immediate that $\frac{n+3}{p}$ is an integer [since p divides $n+3$], **and** it is also immediate that $\frac{n+3}{p} \leq n$ [otherwise, $n+3 > np$; now, remarking that $p \geq 2$ (since p is prime), then the previous two inequalities imply that $n+3 > 2n$; so $3 > n$ and we have a contradiction, since $n \geq 4$, by the hypotheses]. That being so, to prove Observation 2.7.2, it suffices to prove that $\frac{n+3}{p} = p$. **Fact:** $\frac{n+3}{p} = p$ [otherwise, clearly $\frac{n+3}{p} \neq p$; now, remarking (by using the previous) that $\frac{n+3}{p}$ is an integer **and** $\frac{n+3}{p} \leq n$, **and** observing (by using Observation 2.7.1) that $p \leq n$; then it becomes trivial to deduce that $\frac{n+3}{p}$ **and** p are two different integers such that $\{p, \frac{n+3}{p}\} \subseteq \{1, 2, 3, ..., n-1, n\}$. The previous inclusion immediately implies that $p \times \frac{n+3}{p}$ divides $1 \times 2 \times 3 \times ... \times n-1 \times n$; therefore $n+3$ divides $n!$, and this contradicts (5). So $\frac{n+3}{p} = p$]. Observation 2.7.2 follows.
Observation 2.7.3. *Let* p *be a prime such that* $n+3$ *is divisible by* p *(such a* p *clearly exists). Then* $p \neq 2$. Otherwise, $p = 2$; now, observing (by using

Observation 2.7.2) that $\frac{n+3}{p} = p$, then the previous two equalities immediately imply that $\frac{n+3}{2} = 2$; clearly $n + 3 = 4$ and so $n = 1$. We have a contradiction, since $n \geq 4$, by the hypotheses. Observation 2.7.3 follows.

Observation 2.7.4. *Let p be a prime such that $n + 3$ is divisible by p (such a p clearly exists). Then $p \neq 3$.* Otherwise, $p = 3$; now, observing (by using Observation 2.7.2) that $\frac{n+3}{p} = p$, then the previous two equalities immediately imply that $\frac{n+3}{3} = 3$; clearly $n + 3 = 9$ and so $n = 6$. Clearly $n + 3$ divides $n!$, and this contradicts (5) (note that $n+3$ divides $n!$, since $n+3 = 9$, $n = 6$ and 9 divides 6!). Observation 2.7.4 follows.

Observation 2.7.5. *Let p be a prime such that $n + 3$ is divisible by p (such a p clearly exists). Then $p \geq 5$* **and** *$\frac{2(n+3)}{p}$ is an integer* **and** *$\frac{2(n+3)}{p} \leq n$* **and** *$\frac{2(n+3)}{p} \neq p$.*

Clearly $p \geq 5$ [indeed, observe that $p \neq 2$ (by Observation 2.7.3) and $p \neq 3$ (by Observation 2.7.4); now using the previous and the fact that p is prime, then it becomes trivial to deduce that $p \geq 5$]. Indeed, it is immediate that $\frac{2(n+3)}{p}$ is an integer [since p divides $n + 3$], **and** it is also immediate that $\frac{2(n+3)}{p} \leq n$ [otherwise, $2(n+3) > np$; now, remarking (by using the previous) that $p \geq 5$, then the previous two inequalities imply that $2(n + 3) > 5n$; so $6 > 3n$ and therefore $2 > n$. We have a contradiction, since $n \geq 4$, by the hypotheses]. That being so, to prove Observation 2.7.5, it suffices to prove that $\frac{2(n+3)}{p} \neq p$.

Fact: $\frac{2(n+3)}{p} \neq p$ [otherwise, clearly $\frac{2(n+3)}{p} = p$; now, remarking (by using Observation 2.7.2) that $\frac{n+3}{p} = p$, then the previous two equalities immediacy imply that $\frac{2(n+3)}{p} = \frac{n+3}{p}$; so $2(n + 3) = n + 3$ and this last equality is clearly impossible. So $\frac{2(n+3)}{p} \neq p$]. Observation 2.7.5 follows.

The previous trivial observations made, look at $n + 3$; observing (by using Observation 2.7.5) that $\frac{2(n+3)}{p}$ is an integer **and** $\frac{2(n+3)}{p} \leq n$ **and** $\frac{2(n+3)}{p} \neq p$, and remarking (by using Observation 2.7.1) that $p \leq n$, then it becomes trivial to deduce that $\frac{2(n+3)}{p}$ and p are two different integers such that $\{p, \frac{2(n+3)}{p}\} \subseteq \{1, 2, 3, ..., n - 1, n\}$. The previous inclusion immediately implies that $p \times \frac{2(n+3)}{p}$ divides $1 \times 2 \times 3 \times ... \times n - 1 \times n$; therefore $2(n + 3)$ divides $n!$; clearly $n + 3$ divides $n!$ and this contradicts (5). Lemma 2.7 follows. □

Lemma 2.8. *Let n be an integer ≥ 4. If $n + 1$ is prime and if $n + 3$ does not divide $n!$, then $(n + 1, n + 3)$ is a couple of twin primes.*

*Proof.*Indeed, since $n + 1$ is prime and $n + 3$ does not divide $n!$, then Lemma 2.7 implies that $n + 3$ is prime; therefore $(n + 1, n + 3)$ is a couple of twin primes. $\square$

The previous simple two lemmas made, we now prove Theorem 2.6.

Proof of Theorem 2.6. $(c) \Rightarrow (d)$]. Immediate. Indeed, since $(n+1, n+3)$ is a couple of twin primes, clearly $n+1$ does not divide $n!$ (by observing that $n+1$ is prime and by using Theorem 2.1) and clearly $n+3$ does not divide $n!$ [otherwise $n+3$ divides $n!$ and we trivially deduce that

$$n + 3 \text{ divides } (n+2)!.$$

Using this and Corollary 2.4, then it becomes immediate to deduce that $n+3$ is not prime. A contradiction, since $(n+1, n+3)$ is a couple of twin primes, and in particular $n+3$ is prime].

$(d) \Rightarrow (c)$]. Immediate. Indeed, if $n+1$ does not divide $n!$ and if $n+3$ does not divide $n!$, clearly $n+1$ is prime (by observing that $n+1$ does not divide $n!$ and by using Theorem 2.1) and clearly $(n+1,\ n+3)$ is a couple of twin primes [by remarking that $n+1$ is prime (use the previous) and by observing that $n+3$ does not divide $n!$ and by using Lemma $2, 8$]. $\square$

3 Statement of the general conjecture from which stated results follow

Indeed, using Theorem 1.1, then it becomes a little natural to us to conjecture the following:

Conjecture 3.1. *For every integer $j \geq 0$ and for every integer $n \geq 4 + j$; then, $n + 1 + j$ is prime or $n + 1 + j$ divides $n!$.*

It is immediate that the previous conjecture implies Theorem 1.1, and therefore, all the results proved in section 1 and section 2, are only an immediate consequence of Conjecture 3.1. Moreover, if Conjecture 3.1 is true, then, putting $j = n - 4$ (n can be very large), it immediately follows that $2n - 3$ is prime or $2n - 3$ divides $n!$.$\square$.

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