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Solving initial value problems by residual power series method

Mohammed H. Al-Smadi[1]

Abstract

In this article, the residual power series method for solving first-order initial value problems is introduced. The new method provides the solution in the form of a power series with easily computable components using Maple13 software package. The proposed method obtains Maclaurin expansion of the solution and reproduces the exact solution when the solution is polynomial. The proposed technique is applied to several test examples to illustrate the accuracy, efficiency, and applicability of the method.

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Keywords: Initial value problems; Residual power series method; Maclaurin expansion

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[1] Department of Mathematics and Computer Science, Tafila Technical University, Tafila 66110, P.O. Box 179, Tafila - Jordan. e-mail: mhm.smadi@yahoo.com

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1 Introduction

Initial value problems (IVPs) of ordinary differential equations arise in a number of important applications in many fields. Various applications of IVPs to physical, biological, and chemical processes are well documented in the literature, for more about this area one can see [1-5] and the references therein. In general, IVPs do not always have solutions which we can obtain using analytical methods. In fact, many of real physical phenomena encountered, are almost impossible to solve by this technique. Due to this, some authors have proposed numerical and approximate methods to approximate the solutions of IVPs [7-10].

However, none of previous studies propose a methodical way to solve IVPs. Moreover, previous studies require more effort to achieve the results and usually they are suited for a linear form. But on the other aspects as well, the applications of other versions of series solutions to linear and nonlinear problems can be found in [11-16] and for numerical solvability of different categories of differential equations one can consult the references [17, 18].

In the present paper, we apply the residual power series (RPS) method [19] to find series solutions to strongly linear and nonlinear IVPs. The RPS method is effective and easy to use for solving linear and nonlinear IVPs without linearization, perturbation, or discretization [19]. This method constructs an analytical approximate solution in the form of a polynomial. The RPS method is different from the traditional higher order Taylor series method. The Taylor series method is computationally expensive for large orders. The RPS method is an alternative procedure for obtaining analytic Maclaurin series solution of IVPs. By using residual error concept, we get a series solution, in practice a truncated series solution.

The RPS method has the following characteristics [19]; first, the method obtains Maclaurin expansion of the solution; as a result, the exact solution is available when the solution is polynomial. Moreover the solutions and all its derivatives are applicable for each arbitrary point in the given interval. Second, the

RPS method needs small computational requirements with high precision and less time.

The purpose of this paper is to obtain approximate power series solutions for IVPs of the following form:

$$x'(t) = f\big(t, x(t)\big), t \in [0, a], \tag{1}$$

subject to the initial conditions

$$x(0) = \mathrm{x}_0, \tag{2}$$

where, $f: [0, a] \times \mathbb{R} \to \mathbb{R}$ are nonlinear continuous function, $x(t)$ are unknown functions of independent variable t to be determined, and $a > 0$. Throughout this paper, we assume that f, x are analytic functions on the given interval. Also, we assume that f satisfies all the necessary requirements for the existence of a unique solution.

The reminder of the paper is as follows: in the next section, we present the basic idea of the RPS method. In section 3, numerical examples are given to illustrate the capability of proposed method. This article ends in section 4 with some concluding remarks.

2 Solution of system of IVPs by RPS method

In this section, we employ our technique of the RPS method to find out series solution for IVPs subject to given initial conditions.

The RPS method consists in expressing the solutions of IVPs (1) and (2) as a power series expansion about the initial point $t = t_0$ [19]. To achieve our goal, we suppose that this solution takes the form

$$x(t) = \sum_{m=0}^{\infty} c_m t^m ,$$

where $x_m(t) = c_m t^m$ are terms of approximations.

Obviously, when $m = 0$, since $x_0(t)$ satisfy the initial conditions (2), where

$x_0(t)$ is the initial guess approximation of $x(t)$, we have $c_0 = x_0(0) = x(0)$.

If we choose $x_0(t) = x(0)$ as initial guess approximation of $x(t)$, then we can calculate $x_m(t)$ for $m = 1, 2, \ldots$ and approximate the solution $x(t)$ of IVP (1) and (2) by the k^{th}-truncated series

$$x^k(t) = \sum_{m=0}^{k} c_m t^m \tag{3}$$

Prior to applying the RPS method, we rewrite IVP (1) and (2) in the form of the following:

$$x'(t) - f(t, x(t)) = 0 \tag{4}$$

The subsisting of k^{th}-truncated series $x^k(t)$ into Eq. (4) leads to the following definition for the k^{th} residual function:

$$\mathrm{Res}^k(t) = \sum_{m=1}^{k} m c_m t^{m-1} - f(t, \sum_{m=0}^{k} c_m t^m) \tag{5}$$

and the following ∞th residual function:

$$\mathrm{Res}^\infty(t) = \lim_{k\to\infty} \mathrm{Res}^k(t).$$

It easy to see that, $\mathrm{Res}^\infty(t) = 0$ for each $t \in [0, a]$. This show that $\mathrm{Res}^\infty(t)$ is infinitely many times differentiable at $t = 0$. On the other hand, $\frac{d^s}{dt^s}\mathrm{Res}^\infty(0) = \frac{d^s}{dt^s}\mathrm{Res}^k(0) = 0$, for each $s = 1, 2, \ldots, k$. In fact, this relation is a fundamental rule in RPS method and its applications [19].

Now, in order to obtain the 1^{st}-approximate solutions, we put $k = 1$, substitute $t = 0$ into Eq. (5), and using the fact that $\mathrm{Res}^\infty(0) = \mathrm{Res}^1(0) = 0$, to conclude $c_1 = f(0, c_0) = f(0, x(0))$.

Thus, using 1^{st}-truncated series the first approximation for IVP (1) and (2) can be written as

$$x^1(t) = x(0) + f(0, x(0))t.$$

Similarly, to find the 2nd approximation, we put $k=2$ and $x^2(t)=\sum_{m=0}^{2} c_m t^m$. On the other hand, we differentiate both sides of Eq. (5) with respect to t and substitute $t=0$, to get

$$\frac{d}{dt}\operatorname{Re}s^2(0)=2c_2-\frac{\partial}{\partial t}f(0,c_0)-c_1\frac{\partial}{\partial x^2}f(0,c_0).$$

In fact $\frac{d}{dt}\operatorname{Re}s^2(0)=\frac{d}{dt}\operatorname{Re}s^\infty(0)=0$. Thus, we can write

$$c_2=\frac{1}{2}\left[\frac{\partial}{\partial t}f(0,x(0))+c_1\frac{\partial}{\partial x^2}f(0,x(0))\right].$$

Hence, using 2nd-truncated series the second approximation for IVP (1) and (2) can be written as

$$x^2(t)=x(0)+f(0,x_1(0))t+\frac{1}{2}\left[\frac{\partial}{\partial t}f(0,x(0))+f(0,x(0))\frac{\partial}{\partial x^2}f(0,x(0))\right]t^2.$$

This procedure can be repeated till the arbitrary order coefficients of RPS solution for IVP (1) and (2) are obtained. Moreover, higher accuracy can be achieved by evaluating more components of the solution. In other words, choose large k in the truncation series (3). The next theorem shows convergence of the RPS method.

Theorem 2.1 [19] Suppose that $x(t)$ is the exact solution for IVP (1) and (2). Then, the approximate solution obtained by the RPS method is just the Maclaurin expansion of $x(t)$.

Corollary 2.1 [19] If $x(t)$ or some components of $x(t)$ is a polynomial, then the RPS method will be obtained the exact solution.

It will be convenient to have a notation for the error in the approximation $x(t)\approx x^k(t)$. Accordingly, we will let $\operatorname{Re}m^k(t)$ denote the difference between

$x(t)$ and its k th Maclaurin polynomial; that is,

$$\mathrm{Re}\,m^k(t) = x(t) - x^k(t) = \sum_{m=k+1}^{\infty} x^{(m)}(0)\,t^m$$

The functions $\mathrm{Re}\,m^k(t)$ are called the k^{th} remainder for the Maclaurin series of $x(t)$. In fact, it often happens that the remainders $\mathrm{Re}\,m^k(t)$ become smaller and smaller, approaching zero, as k gets large.

3 Numerical results and discussion

In this section, the validity and efficiency of the proposed method is illustrated by three examples. The examples reflect the behavior of the solution with different nonhomogeneous terms and type of nonlinearity. Throughout this paper, all the symbolic and numerical computations performed by using Maple 13 software package.

To show the accuracy of the present method for our problems, we report three types of error. The first one is the exact error, $\mathrm{Ext}^k(t)$, and defined as $\mathrm{Ext}^k(t) := \left| x(t) - x^k(t) \right|$, while the residual, Res, and relative, Rel, errors are defined, respectively, by $\mathrm{Res}^k(t) := \left| \frac{d}{dt} x^k(t) - f(t, x^k(t)) \right|$ and $\mathrm{Rel}^k(t) := \frac{|x(t) - x^k(t)|}{|x(t)|}$, where $t \in [0, a]$, x^k is the kth-order approximation of $x(t)$ obtained by the RPS method, and $x(t)$ is the exact solution.

Example 3.1 Consider the following linear stiff IVP:

$$\begin{aligned} &x'(t) = 10^{10}[(x(t) - (x(t)-2)^3)] + f(t), \quad t \geq 0, \\ &f(t) = (10^{20}+3)t^2 - 10^{10}(10^{10}t^2 - t^3 - 2t + 1)^3 - 4(10^{10})t - 10^{10}t^3 - (10^{10} - 2), \end{aligned} \tag{6}$$

subject to the initial conditions

$$x(0)=1 \tag{7}$$

As we mentioned earlier, if we select the initial guess approximation as $x^0(t) = 1$, then the power series expansion of the solution takes the form

$$x(t)=1+c_1t+c_2t^2+c_3t^3+\cdots \tag{8}$$

Consequently, the 3ed-order approximations of the RPS solution for Eqs. (6) and (7) according to this initial guess is as follows:

$$x^3(t)=1+2t-10^{10}t^2+t^3,$$

with full agreement with Corollary 2.1. It easy to discover that the each of the coefficients c_m for $m>4$ in the expansion (8) is vanished. In other words, we have

$$\sum_{m=0}^{\infty}c_mt^m=\sum_{m=0}^{3}c_mt^m .$$

Thus, the analytic approximate solution of Eqs. (6) and (7) agree well with the exact solution $x(t)=-8+12t-t^2+t^3$.

Example 3.2 Consider the following nonlinear IVP:

$$\begin{aligned}&x'(t)=t^2x(t)+\cos^{-1}(x(t))-\frac{\sin(t^2)}{\sqrt{x^2(t)+\sin^2(t^2)}}+f(t), \quad t\ge 0,\\&f(t)=\sin(t^2)(1-2t)-t^2\cos(t^2)-t^2,\end{aligned} \tag{9}$$

subject to the initial conditions

$$x(0)=1 \tag{10}$$

As in the previous example, if we select the initial guess approximation as $x^0(t) = 1$, then the first few terms approximations of the RPS solution for Eqs. (9) and (10) are

$$x_1(t)=0, \quad x_2(t)=0, \quad x_3(t)=0, \quad x_4(t)=-\frac{1}{2}t^4,\ldots,$$

If we collect the above results, then the 20th-truncated series of the RPS solution for $x(t)$ is as follows:

$$x^{20}(t) = 1 - \frac{1}{2}t^4 + \frac{1}{24}t^8 - \frac{1}{720}t^{12} + \frac{1}{40320}t^{16} - \frac{1}{3628800}t^{20} = \sum_{j=0}^{5}(-1)^j \frac{(t^2)^{2j}}{(2j)!}$$

Thus, the exact solution of Eqs. (9) and (10) has the general form which are coinciding with the exact solution

$$x(t) = \sum_{j=0}^{\infty}(-1)^j \frac{(t^2)^{2j}}{(2j)!} = \cos t^2 \quad .$$

Let us now carry out the error analysis of the RPS method for this example. Figure 1 shows the exact solution $x(t)$ and the four iterates approximations $x^k(t)$ for $k = 5,10,15,20$. These graphs exhibit the convergence of the approximate solutions to the exact solution with respect to the order of the solution.

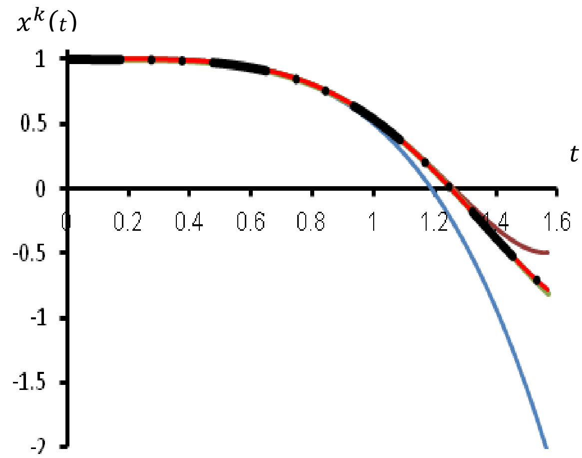

Figure 1: Plots of RPS solution for Eqs. (9) and (10) blue, brown, green, and red solid lines, denote four iterates approximations when $k = 5,10,15,20$, respectively, and black dashed-dot-dotted line, denote exact solution.

In Figure 2, we plot the residual error function $\mathrm{Res}^k(t)$ for $k = 5,10,15,20$ which are approaching the axis $y = 0$ as the number of iterations increase. These graphs show that the exact error is getting smaller as the order of the solution is increasing.

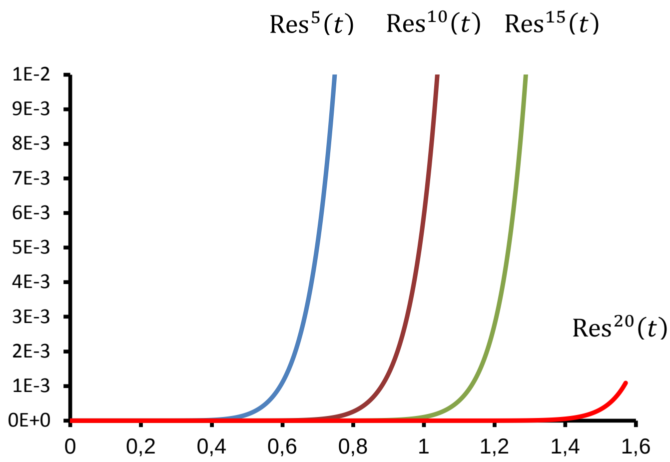

Figure 2: Plots of residual error function for Eqs. (9) and (10), when $k = 5,10,15,20$.

Example 3.3 Consider the following nonlinear IVP:

$$\begin{aligned} x'(t) &= \sin x(t) + \cos x(t) + f(t), t \geq 0, \\ f(t) &= -\sin t - \cos(\cos t) - \sin(\cos t), \end{aligned} \tag{11}$$

subject to the initial conditions

$$x(0) = 1. \tag{12}$$

Assuming that the initial guess approximation has the form $x^0(t) = 1 + t$. Then, the 10th-truncated series of the RPS solution of $x(t)$ for Eqs. (11) and (12) is as follows:

$$x^{10}(t) = 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720} + \frac{t^8}{40320} - \frac{t^{10}}{3628800} = \sum_{j=0}^{5} (-1)^j \frac{(t^2)^{2j}}{(2j)!}.$$

It easy to see that, the 10th-truncated series of the RPS solution for $x(t)$ above agree well with the general form

$$x(t) = \sum_{j=0}^{\infty} (-1)^j \frac{(\mathrm{t})^{2j}}{(2j)!} = \cos(t).$$

So, the exact solution of Eqs. (11) and (12) will be $x(t) = \cos(t)$.

Our next goal is to show how the value of k in the truncation series (3) affects the RPS approximate solutions. To determine this effect an error analysis is performed. We calculate the approximations $x^k(t)$ for various k and obtain the error functions. The maximum and average errors when $k = 5,10,20$ for Eqs. (11) and (12) have been listed in Table 1 for $t_i = \frac{1}{10} i,\ i = 0,1,2,\dots,10,\ t \in [0,1]$.

Table 1: The maximum error functions of $x(t)$ when $k = 5,10,15,20$.

Description	$k = 5$	$k = 10$	$k = 15$	$k = 20$
$\max\{\text{Ext}^k(t_i)\}$	1.36436×10^{-3}	2.07625×10^{-9}	4.77396×10^{-14}	1.11022×10^{-16}
$\max\{\text{Res}^k(t_i)\}$	4.03023×10^{-2}	2.73497×10^{-7}	1.12955×10^{-11}	7.99893×10^{-12}
$\max\{\text{Rel}^k(t_i)\}$	2.52518×10^{-3}	3.84276×10^{-9}	8.83572×10^{-14}	2.05483×10^{-16}

4 Conclusion

The fundamental objective of this work is to introduce in an algorithmic form and implement a new symbolic treatment for the linear and nonlinear IVPs. Our treatment in principal is the use of the new analytic method for IVPs introduced by the author in [19] with some slight modifications considered by the nature of the initial condition. There is an important point to make here, the results obtained by the RPS method are very effective and convenient in linear and nonlinear cases with less computational work and time. This confirms our belief that the efficiency of our technique gives it much wider applicability for general classes of linear and nonlinear problems.

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