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# Compact and Fredholm composition operators defined by modulus functions

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**Abstract**

In this paper we characterize Compact and Fredholm Composition Operators on the spaces $W_\infty(A, f)$.

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**Keywords:** Modulus function, Composition Operator, Compact Operator, Fredholm Operator, Bounded Operator

## 1 Introduction

Let X and Y be two non-empty sets. Let $F(X, C)$ and $F(Y, C)$ be two topological vector spaces of complex valued functions on X and Y respectively. Suppose $T : Y \to X$ is a mapping such that $foT \in F(Y, C)$, whenever $f \in F(X, C)$. Then we can define a composition transformation.

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$$C_T : F(X,C) \to F(Y,C)$$

by

$$C_T f = foT \text{ for every } f \in F(X,C)$$

If $C_T$ is continuous, we call it a Composition Operator induced by T.
A convex function $f : [0,\infty) \to [o,\infty)$ such that
(i) f(x) = 0 if and only if x = 0
(ii) $f(x+y) \leq f(x) + f(y)$ for all $x \geq 0, y \geq 0$
(iii) f is increasing
(iv) f is continuous from right at 0.

Then f is called a *modulus function.* Let $(a_{nk})$ be an infinite matrix of non-negative real numbers such that

$$\sup_n \sum_{k=1}^{\infty} a_{nk} < \infty$$

and let $W_\infty(A,f)$ be defined as

$$W_\infty(A,f) = \{\{x_k\} \in C : \sup_n \sum_{k=1}^{\infty} a_{nk} f(|x_k|) < \infty\}$$

Set

$$||x||_{A,f} = \sup_n \sum_{k=1}^{\infty} a_{nk} f(|x_k|)$$

It is well known that $W_\infty(A,f)$ is a complete topological vector space under the topology induced by the paranorm $||x||_{A,f}$.

A bounded linear operator A from a Hilbert space H into itself is called
(i) *Compact*, if the closure of the image of unit ball in H is compact i.e. $\overline{A(B_1)}$ is a compact set, where $B_1$ is the closed unit ball of H.
(ii) *Fredholm operator*, if the range of A is closed and the dimensions of kernel of A and co-kernel of A are finite.

A study of composition operators on several function spaces like $L^p(\lambda), C(X)$, $H^p(D), 1 \leq p < \infty$ has been the subject matter of intensive study over the past several decades. It is known that no composition operators on $L^p(\lambda)$ is compact see Singh and Kumar [7]. Compact and Fredholm composition operators on $C(X)$ are characterize by Takagi [10, 11], where Shapiro characterized compact composition operators on $H^p(D)$. Ajay Sharma[9] recently studied compact composition operators on Bergman spaces.

In this paper we characterize Compact and Fredholm Composition Operators on sequence spaces defined by Modulus functions.

## 2 Compact Composition Operators

In this section we first characterize the bounded composition operators on sequence spaces defined by Modulus functions. A necessary and a sufficient condition for a composition operator to be compact is also investigated in this section.

**Theorem 2.1.** *Let $T : N \to N$ be a mapping. Then $C_T : W_\infty(A, f) \to W_\infty(A, f)$ is a bounded operator if and only if there exists $M > 0$ such that*

$$\sum_{m \in T^{-1}(k)} a_{nm} \leq M a_{nk} \text{ for every } n \in N, k \in N$$

**Proof:** Take $x \in W_\infty(A, f)$. Consider

$$\begin{aligned} ||C_T x||_{A,f} &= \sup_n \sum_{k=1}^{\infty} a_{nk} f(|(xoT)(k)|) \\ &= \sup_n \sum_{k=1}^{\infty} \sum_{m \in T^{-1}(k)} a_{nm} f(|(xoT)(m)|) \\ &= \sup_n \sum_{k=1}^{\infty} (\sum_{m \in T^{-1}(k)} a_{nm}) f(|x_k|) \\ &\leq M \sup_n \sum_{k=1}^{\infty} a_{nk} f(|x_k|) \\ &= M ||x||_{A,f} \end{aligned}$$

This proves that $C_T$ is a bounded operator.

Conversely, suppose that $C_T$ is a bounded operator. If $\sup_n a_{nk} = 0$ for every $k \in N$, then there is nothing to prove. Suppose $\sup_n a_{nk_o} > 0$ for some

$k_0 \in N$. Take $x = \frac{e_{k_0}}{\alpha_{k_0}}$, where $\alpha_{k_0} = f^{-1}(\frac{1}{sup a_{n,k_0}})$. Then

$$\begin{aligned} ||x||_{A,f} &= \sup_n \sum_{m=1}^{\infty} a_{nm} f(\frac{e_{k_0}(m)}{\alpha_{k_0}}) \\ &= \frac{\sup_n a_n k_0}{\sup_n a_n k_0} = 1 \end{aligned}$$

But

$$\begin{aligned} ||C_T x||_{A,f} &= \sup_n (\frac{\sum_{m \in T^{-1}(k)} a_{nm}}{\sup_n a_n k_0}) \\ &= \sup_n (b_n \frac{\sum_{m \in T^{-1}(k_0)} c_m}{\sup_n b_n c_{k_o}}) \qquad (1) \end{aligned}$$

Now $C_T$ is bounded. Therefore there exists $M > 0$ such that

$$||C_T x||_{A,f} \leq M ||x||_{A,f}$$

Hence in view of (1),

$$\sum_{m \in T^{-1}(k)} c_m \leq M c_k \ \ \forall \ \ k \in N$$

or

$$\sum_{m \in T^{-1}(k)} b_n c_m \leq M b_n c_k \ \ \forall \ \ n \text{ and } K$$

Hence

$$\sum_{m \in T^{-1}(k)} a_{nm} \leq M a_{nk} \ \ \forall \ \ n \text{ and } k \in N$$

□

**Example 2.2.** *Let* $a_{nk}$ *be an infinite matrix defined by* $a_{nk} = \frac{1}{n} \cdot \frac{1}{k^2}$. *Then*

$$\sup_n \sum_{k=1}^{\infty} a_{nk} = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

Let $T : N \to N$ be defined by $T(1) = 1$ and $T(n) = n-1$ for every $n \geq 2$. Then

$$\begin{aligned} ||C_T x||_{A,f} &= \sup_n \sum_{k=1}^{\infty} a_{nk} f(|(x(T(k))|) \\ &= \sup_n \sum_{k=1}^{\infty} \sum_{m\in T^{-1}(k)} a_{nm} f(|(x(T(m))|) \\ &= \sup_n \sum_{k=1}^{\infty} \sum_{m\in T^{-1}(k)} a_{nm} f(|x_k|) \\ &= \sup_n \sum_{k=1}^{\infty} \sum_{m\in T^{-1}(k)} \frac{1}{n}\frac{1}{m^2} f(|x_k|) \end{aligned}$$

$$\begin{aligned} ||C_T x||_{A,f} &\leq 2\sup_n \sum_{k=1}^{\infty} a_{nk} f(|x_k|) \\ &= 2||x||_{A,f} \quad \text{for every} \quad x \in W_\infty(A,f) \end{aligned}$$

Hence $C_T$ is a bounded operator. □

**Theorem 2.3.** *Let $C_T \in B(W_\infty(A,f))$. Then $C_T$ is compact if and only if the set*

$$S(\epsilon) = \{k : \sum_{m\in T^{-1}(k)} a_{nm} \geq \epsilon a_{nk} \quad \forall \quad n \in N\} \cap \{k : a_{nk} \neq 0$$

*for any $n \in N\}$ is a finite set for each $\epsilon > 0$.*

**Proof:** We first assume that the condition is satisfied. We prove that $C_T$ is a compact operator. Let $\{g^{(p)}\}_{p=1}^{\infty}$ be a bounded sequence in $W_\infty(A,f)$. Then there exists $M > 0$ such that

$$||g^{(p)}||_{A,f} \leq M \quad \text{for every } p \geq 1.$$

But

$$\begin{aligned} ||C_T g^{(p)}||_{A,f} &= \sup_n \sum_{k=1}^{\infty} a_{nk} f(|g^{(p)} oT(k)|) \\ &= \sup_n (\sum_{k=1}^{\infty} \sum_{m\in T^{-1}(k)} a_{nm} f(|g^{(p)}(k)|)) \\ &= \sup_n (\sum_{k=1}^{\infty} ( \sum_{m\in T^{-1}(k)} a_{nm}) f(|g^{(p)}(k)|)) \end{aligned}$$

$$
\begin{aligned}
&= \sup_n \Big( \sum_{k\in S(\epsilon/2)} \sum_{m\in T^{-1}(k)} a_{nm} f(|g^{(p)}(k)|)\Big) \\
&\quad + \Big( \sum_{k\in S'(\epsilon/2)} \sum_{m\in T^{-1}(k)} a_{nm} f(|g^{(p)}(k)|)\Big) \\
&\leq \sup_n \sum_{k\in S(\epsilon/2)} \sum_{m\in T^{-1}(k)} a_{nm} f(|g^{(p)}(k)|) \\
&\quad + \sup_n \sum_{k\in S'(\epsilon/2)} \sum_{m\in T^{-1}(k)} a_{nm} f(|g^{(p)}(k)|) \\
&\leq M \sup_n \sum_{k\in S(\epsilon/2)} a_{nk} f|g^{(p)}(k)| \\
&\quad + \frac{\epsilon}{2} M \sup_n \sum_{k\in S'(\epsilon/2)} a_{nk} f(|g^{(p)}(k)|) \\
&\leq M \sup_n \sum_{k\in S(\epsilon/2)} a_{nk} f(|g^{(p)}(k)|) + M\frac{\epsilon}{2} \qquad (1)
\end{aligned}
$$

Since $S(\frac{\epsilon}{2})$ is a finite set and $\{g^{(p)}\}$ is a bounded sequence, we can find a subsequence $\{g^{(p_r)}\}$ of $\{g^{(p)}\}$ such that

$$\sup_n \sum_{k\in S(\epsilon/2)} a_{nk} f(|g^{(p_r)}(k)|) < \frac{M\epsilon}{2} \quad \forall \;\; r \geq r_0$$

Then using (1), we find that

$$||C_T g^{(p_r)}||_{A,f} < \frac{M\epsilon}{2} + \frac{M\epsilon}{2} = M\epsilon \;\; \forall \;\; r \geq r_0$$

Thus every bounded sequence has a convergent subsequence. This proves that $C_T$ is a compact operator.

Conversely, if the condition of the theorem is not satisfied, then for some $\epsilon > 0$ we can choose an infinite sequence $\{k_r : r \in N\}$ in $S(\epsilon)$ such that

$$\frac{\sum\limits_{m\in T^{-1}(k_r)} a_{nm}}{a_n k_r} \geq \epsilon$$

for infinite many values of $k_r$

Set

$$x^{kr} = f^{-1}\Big(\frac{1}{\sup\limits_n a_n k_r}\Big)$$

Then $||x^{k_r}|| = 1,$ and

$$\begin{aligned} ||C_T x^{(k_r)}|| &= \sup_n \Big( \frac{\sum\limits_{m \in T^{-1}(k_r)} a_{nm}}{\sup\limits_n a_n k_r} \Big) \\ &= \frac{\sup\limits_n \sum\limits_{m \in T^{-1}(k_r)} a_{nm}}{\sup\limits_n a_n k_r} \\ &\geq \frac{\sup\limits_n (\epsilon a_n k_r)}{\sup\limits_n a_n k_r} \\ &= \epsilon \end{aligned}$$

This contradicts the compactness of $C_T$. Hence the condition must be satisfied.

□

# 3 Fredholm Composition Operators

noindent

The main purpose of this section is to characterize Fredholm Composition Operators.

**Theorem 3.1.** *Let $C_T \in B(W_\infty(A, f))$. Then $C_T$ is Fredholm if and only if*
*(i) $N|T(N)$ is a finite set*
*(ii) There exists $\delta > 0$ such that*

$$\sum_{m \in T^{-1}(k)} a_{nm} \geq \delta a_{nk} \text{ for every } n, k \in N$$

*(iii) The set $E = \{n \in N : \sharp T^{-1}(T(n)) \geq 2\}$ is a finite set, where $\sharp(E)$ is the cardinality of the set $E$.*

**Proof:** Suppose $N|T(N)$ is a finite set. Then ker $C_T$ is finite dimensional. Next, if the condition (ii) is true, then we prove that ran $C_T$ is closed. Let $x \in \overline{ranC_T}$. Then there exists a sequence $\{x^{(p)}\}$ in ran $C_T$, such that $x^{(p)} \to x$

since $x^{(p)} \in ranC_T$ we can write $x^{(p)} = C_T y^{(p)}$ for some $y^{(p)} \in W_\infty(A, f)$. Thus $\{C_T y^{(p)}\}_{p=1}^{\infty}$ is a cauchy sequence. Therefore for every $\epsilon > 0$, there exists a positive integer $p_0$ such that

$$||C_T y^{(p)} - C_T y^{(q)}|| < \epsilon \;\; \forall \;\; p, q \geq p_0.$$

In other words,

$$\sup_n \sum_{m=1}^{\infty} a_{nm} f(y^{(p)}(T(m) - y^{(q)}(T(m))) < \epsilon foreveryp, q \geq p_0$$

or equivalently, for all $p, q \geq p_0$, we have

$$\begin{aligned} \epsilon &> \sup_n \sum_{k=1}^{\infty} \sum_{m \in T^{-1}(k)} a_{nm} f(|y^{(p)}(k) - y^{(q)}(k)|) \\ &\geq \delta \sup_n \sum_{k=1}^{\infty} a_{nk} f(|y^{(p)}(k) - y^{(q)}(k)|) \end{aligned}$$

This proves that $\{y^{(p)}\}$ is a Cauchy sequence in $W_\infty(A, f)$. But $W_\infty(A, f)$ is complete. Therefore there exists $y \in W_\infty(A, f)$, such that $y^{(p)} \to y$ as $p \to \infty$. From continuity of $C_T$, we have $C_T y^{(p)} \to C_T y$ as $p \to \infty$ or $x^{(p)} \to C_T y$. Hence $x = C_T y$. This proves that the range of $C_T$ is closed.

Next, if $E$ is a finite set, then obviously ran $C_T$ is finite co-dimensional. Hence $C_T$ is Fredholm.

Conversely, suppose $C_T$ is Fredholm. We prove that conditions (i) - (iii) are true. If the condition (i) is not true, then $e_n \in kerC_T$ for every $n \in N|T(N)$ which shows that ker $C_T$ is infinite dimensional, a contradiction. Hence $N|T(N)$ must be a finite set.

Next, if $E$ is an infinite set then we can choose infinitely many pairs $(m_k, n_k)$ such that

$$T(n_k) = T(m_k)$$

Define

$$K_{m_k,n_k} : W_\infty(A, f) \to C$$

by

$$K_{m_k,n_k}(f) = f(m_k) - f(n_k)$$

Then $K_{m_k,n_k}$ is a linear functional on $W_\infty(A, f)$. Clearly

$$\begin{aligned} C_T^*(K_{m_k,n_k})(f) &= K_{m_k,n_k}(C_T f) \\ &= (C_T f)(m_k) - (C_T f)(n_k) = 0 \quad \forall \;\; f \end{aligned}$$

Therefore $K_{m_k,n_k} \in kerC_T^*$ which proves that ker $C_T^*$ is infinite dimensional. Hence E must be a finite set.

Finally, we prove the condition (ii). If the condition (ii) is false, then for every positive integer $\ell$ there exists $n_\ell$ and $k_\ell$ such that

$$\sum_{m\in T^{-1}(k_\ell)} a_{n_\ell m} \leq \frac{1}{\ell} a_{n_\ell} k_\ell$$

Take $x^\ell = \frac{e_{k_\ell}}{\alpha_{k_\ell}}$, where $\alpha_{k_\ell} = f^{-1}(\frac{1}{supa_{n,k_\ell}})$. Then $||x^\ell|| = 1$. But

$$||C_T x^\ell|| = \sup_n \sum_{m\in T^{-1}(k_\ell)} a_{nm} \frac{1}{supa_n k_\ell} \leq \frac{1}{\ell} \to 0 \text{ as } \ell \to \infty.$$

This shows that $C_T$ is not bounded away from zero and $C_T$ has not closed range. This is a contradiction. Hence the condition must be true. □

**Example 3.2.** *Let $T : N \to N$ be defined by $T(n) = n + 1$ for all $n \in N$ and $(a_{nk})_{n,k=1}^\infty$ be the matrix defined by*

$$a_{nk} = \begin{cases} \frac{1}{n^3}, & \textit{if } k \leq n \\ 0 & \textit{if } \ k > n \end{cases}$$

*Then*

$$\sum_{k=1}^\infty a_{nk} = \sum_{k=1}^\infty \frac{1}{n^3} = \frac{1}{n^2}$$

*or*

$$\sup_n \sum_{k=1}^\infty a_{nk} = \sup_n \{\frac{1}{n^2}\} = 1.$$

Now $N|T(N) = \{1\}$, which is a finite set.

Also

$$\frac{\sum_{m\in T^{-1}(k)} a_{nm}}{a_{nk}} = \frac{a_{n,k-1}}{a_{n,k}} = \frac{\frac{1}{n^3}}{\frac{1}{n^3}} = 1, \quad \text{for} \quad 2 \leq k \leq n.$$

Thus Ran $C_T$ is closed. Also ker $C_T = span\{e_1\}$. Therefore ker $C_T$ is finite dimensional. Clearly Ran $C_T = span(\{e_n : n \in N\} - \{e_1\})$ so that $(RanC_T)^\perp = span\{e_1\}$. Which proves that $RanC_T$ is finite co-dimensional. Hence $C_T$ is Fredholm. □

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