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On New Type of Monads in Topology

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Abstract

The aim of this paper is to present and study a new type of monads, named Pθ-monad, which plays an important role in our approach to p-closed spaces, pre-Urysohn spaces ...etc. Also, we investigated some new properties and relationships between this monad with other types of monads by using some concepts of nonstandard analysis.

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1 Introduction

The concept of point monads, was first introduced by Robinson, A. [8], and axiomatazed by Nelson, E. [7], they has been proved to be a useful tool for characterizing and studying some topological concepts. In 1976 Herrmann, R.A. [5] introduced two new types of monads, namely the θ and α monads in general topology, which are capable of similarly characterizing the various topological concepts associated with quasi-H-closed, nearly compact, Urysohn spaces,…..etc.. In 1982, Mashhour, A.S. & El-Deeb, S.N.[6], defined a new version of nearly open sets which is significant notion to the field of general topology called preopen sets . This work is another attempt of the authors in applying nonstandard analysis in general topology, the previous one was entitled "βθ-monads in general topology" given in [11]. In this paper, we use the notion of preopen sets in topological spaces to introduce and study a new type of monads named Pθ-monad.

2 Basic Backgrounds in General Topology

Throughout this work, (X,τ) or (simply X) denotes a standard topological space on which no separation axioms are assumed unless explicitly stated, we recall the following definitions, notational conventions and characterizations.

The closure (resp., Interior) of a subset A of a space X is denoted by ClA (resp., IntA).

A subset A of a space X is said to be

- preopen set [6] if and only if $A \subset IntClA$.
- preclosed closed set if and only if X\A is preopen set.
- regular closed set if and only if X\A is regular closed.

The intersection of all pre-closed (pre-θ-closed) sets containing A is called pre-closure (pre-θ-closure) and denoted by pClA($pCl_{\theta}A$).

The union of all preopen sets contained in A is called pre-interior and denoted by pIntA.

A subset A of a space X is said to be

- pre-θ-open [2] if for each $x\in A$, there is a preopen subset G of X such that $x\in G\subset$ pClG$\subset$A.
- pre-regular p- open set [2] if and only if A= pIntpClA.

The family of all pre-open sets of a space X is denoted by PO(X).

The family of all pre-closed (pre-regular p- closed)sets of a space X is denoted by PC(X)(PRPC(X)).

Definition 2.1 [3] A space X is called *submaximal* if each dense subset of X is open set.

Definition 2.2 [2] A point $x\in X$ is said to be θ-*accumulation* (resp.,pre-θ accumulation) point of a subset A of a space X if $ClG\cap A\neq\phi$ (resp., $pClG\cap A\neq\phi$ for every $G\in\tau$ (resp., $G\in PO(X)$).

Definition 2.3 [2]

A space X is said to be *p-closed space* if every pre-open cover of X has a finite subfamily whose pre-closure covers X.

Definition 2.4 [3]

A space X is said to be *pre-urysohn* if and only if for each $x,y\in X$, with $x\neq y$, there exists $G,H\in PO(X)$, such that $x\in G$, $y\in H$, and $pClG\cap pClH=\phi$.

Definition 2.5 [2] A space X is said to be *pre*-T_2 if and only if for each distinct points $x,y\in X$, there exists $G,H\in PO(X)$, such that $x\in G$, $y\in H$, and $G\cap H=\phi$.

Definition 2.6 [2] A space X is said to be *locally indiscrete* if every open subset of X is closed set.

Theorem 2.7 [2] Let G be a subset of topological space (X,τ). Then

i) pClG=G∪ClIntG,

ii) pClpIntpClG=pClG,

iii) pClA⊆ClA.

Theorem 2.8 [9] Let A and B be any two subsets of the space X. If $A \cap B = \phi$, then $p\mathrm{Int}A \cap p\mathrm{Cl}B = \phi$.

3 Basic Backgrounds in Nonstandard Analysis

In this section, we use Nelson's nonstandard analysis construction, based on a theory called internal set theory IST.

Recall that for a topological space (X,τ), the monad $\mu(p)$, α-monad $\mu_\alpha(p)$, and θ-monad $\mu_\theta(p)$ at a point p are defined as follows[5]

$\mu(p)= \cap\{{}^*G;\ p\in G\in\tau\}$, α-monad= $\cap\{{}^*(\mathrm{IntClG});\ p\in G\in\tau\}$, and it denoted by $\mu_\alpha(p)$.

θ-monad = $\cap\{{}^*\mathrm{ClG};\ p\in G\in\tau\}$. and it denoted by $\mu_\theta(p)$.

Definition 3.1 [1] A relation r is called concurrent in a standard set U, if r∈U, and if a_1 , a_2 ,...,a_n∈dom(r),then there is an element b such that $(a_i ,b) \in r$, for i=1,2,…,n

Theorem 3.2 (Concurrence relation) [1] Let r be a standard concurrent relation in a standard set U, then there is an element b∈U, such that (a,b) ∈r, for each a∈dom(r).

4 Pθ - Monads

Definition 4.1 Let (X,τ) be a standard topological space. Then the Pθ- monad at the point $a \in X$ is defined as follows Pθ-monad=$\cap\{A;\ A \in \overline{GP(a)}\}$, where $\overline{GP(a)} = \{p\mathrm{Cl}A;\ A \in GP(a)\}$, $GP(a) = \{A;\ a \in AP\mathrm{O}(X)\}$, and it is denoted by $\mu_{p\theta}(a)$.

Proposition 4.2 Let (X,τ) be a standard topological space, and $a \in X$. Then $\mu_{p\theta}(a) = \cap\{p\mathrm{Cl}A\ ;\ A \in GP(a)\ \}$.

Proof. Follows directly from Definition 4.1. □

Proposition 4.3 Let (X,τ) be a standard topological space, then for each $a \in X$, the relation $\mu_{p\theta}(a) \subseteq \mu_\theta(a)$ is true.

Proof. Follows from Theorem 2.7(iii) and definitions of μ_θ and $\mu_{p\theta}$. □

Remark 4.4 The equality of the relation given in Proposition 4.3 is not true in general, such as shown in the following example.

Example 4.5 Let X={a,b,c}, and τ be indiscrete space . Then the family of all preopen sets are PO(X)=P(X) , then, $\mu_\theta(a)=X$, *and* ={{a},{a,b},{a,c}, X}, $\mu_{p\theta}(a) = \{a\}$.

Remark 4.6 Let (X,τ) be a standard topological space, then

i) For the trivial topological space $\mu_{p\theta}(x)=\{x\}$, for each standard $x \in X$.

ii) For the discrete topological space $\mu_{p\theta}(x)=\{x\}$, for each standard $x \in X$.

iii) For the locally indiscrete space, $\mu_{p\theta}(x)=\{x\}$, for each standard $x \in X$.

Proposition 4.7 If X is a submaximal , then $\mu_\theta(a) = \mu_{p\theta}(a)$, for each $a \in X$.

Proof. Follows from Theorem 2.1 and Proposition 4.3. □

Proposition 4.8 Let (X,τ) be a standard topological space, and $a \in X$. If every preopen set is regular closed set, then $\mu_{p\theta}(a) = \cap\{G;\ G \in GP(a)\}$.

Proof. By Theorem 2.7(i) pClG=G∪ClIntG, since G∈, then G is regular closed set. Therefore pClG=G. Using Proposition 4.2, we obtain

$\mu_{p\theta}(a) = \cap\{G; G \in GP(a)\}$. □

Theorem 4.9 Let (X,τ) be a standard topological space. Then the following statements are valid

i) For each $a \in X$, $a \in \mu_{p\theta}(a)$.

ii) For each $a \in X$, $b \in \mu_{p\theta}(a)$ implies $\mu_{p\theta}(b) \subseteq \mu_{p\theta}(a)$.

Proof. i) Is obvious.

ii) Let $x \in \mu_{p\theta}(b)$. Since $b \in \mu_{p\theta}(a)$, then for each standard GPO(X) if $a \in$ pClG, then $b \in$ pClG, and if $b \in$ pClG, then

$$x \in \text{pClG} \tag{1}$$

By transfer axiom, for each GPO(X), and hence we get that the equation (1) holds true. Hence $\mu_{p\theta}(b) \subseteq \mu_{p\theta}(a)$. □

Proposition 4.10 Let (X,τ) be a standard topological space, and $a \in X$. Then $\mu_{p\theta}(a) = \cap\{G;\ G \in PRPC(X,a)\}$.

Proof. Follows directly from Theorem 2.7(ii) and Proposition 4.2. □

Theorem 4.11 Let (X,τ) be a standard topological space, and let $a \in X$. Then there exists a standard pre-open H such that pClH $\subseteq \mu_{p\theta}(a)$.

Proof. Let r(pClG, pClH) be a binary relation defined by r(pClG, pClH) equivalently pClH $\subseteq$ pClG. Then r(pClG, pClH) is concurrent relation. For this, if G_1 , G_2,…, $G_n \in$ PO(X), such that pClH=$\cap$ pClG_i for i=1,2,…,n, then r(pClG, pClH) holds true.

Now, by Theorem 3.2 we get that pClH $\subseteq$ pClG for each G$\in$PO(X). Therefore pClH $\subseteq \mu_{p\theta}$(a). □

Corollary 4.12.

Let (X,τ) be a standard topological space, and let $a \in X$. Then there exists a standard pre-regular-p- open set H such that H $\subseteq \mu_{p\theta}$(a).

Proof. Follows directly from Theorem 4.11 and Theorem 2.7(ii). □

Theorem 4.13 Let A be a standard subset of a standard topological space X. Then A is pre-θ-open set if and only if $\mu_{p\theta}$(a) $\subseteq$A, for each $a \in A$.

Proof. Assume that A is pre-θ-open set, and let $a \in A$. Then there exists a standard pre-open G such that $a \in G \subseteq$ pClG $\subseteq$A,

by transfer axiom pClG $\subseteq$A for each G,A and $a \in$GPO(X).

Now, $\cap${pClG ; $G \in GP(a)$ }$\subseteq$ pClG $\subseteq$A, by Proposition 4.2 we obtain that $\mu_{p\theta}$(a) $\subseteq$A.

Conversely, suppose that $\mu_{p\theta}$(a) $\subseteq$A, then by using Theorem 4.11, for each $a \in A$ that there exists a standard pre-open set G such that pClG $\subseteq \mu_{p\theta}$(a).

Thus a$\in$G$\subset$ pClG$\subset$A, for each standard a. Therefore by transfer axiom we have a$\in$G$\subset$ pClG$\subset$A, for each a. Hence A is pre-θ-open set. □

Proposition 4.14 Let S be a standard non-empty subset of a standard topological space (X,τ). Then S contains the pre-closure of a non-empty pre-open set if and only if $\mu_{p\theta}$(a) $\subseteq$S, for some $a \in S$.

Proof. Assume that S contains the pre-closure of a non-empty pre-open set G. Then by Proposition 4.2 we get $\mu_{p\theta}$(a) $\subseteq$ pClG, for each $G \in \overline{GP(a)}$. Therefore $\mu_{p\theta}$(a) $\subseteq$S for some $a \in S$.

Conversely, assume that $\mu_{p\theta}(a) \subseteq S$, for some $a \in S$. Then by Theorem 4.11, there exists a standard pre-open set H such that $pClH \subseteq \mu_{p\theta}(a)$. By transfer axiom this hold for each $H \in \overline{GP(a)}$. Hence S contain the pre-closure of a non-empty pre-open set H. □

Theorem 4.15 Let A be a standard subset of a standard topological space X. Then A is pre-θ-closed set if and only if $\mu_{p\theta}(a) \cap A = \phi$, for each $a \in X/A$.

Proof. Assume that A is pre-θ-closed set.

Then by Theorem 4.13 we have $\mu_{p\theta}(a) \subseteq X/A$ for each $a \in X \backslash A$, hence $\mu_{p\theta}(a) \cap A = \phi$.

Conversely, assume that $\mu_{p\theta}(a) \cap A = \phi$. Then $\mu_{p\theta}(a) \subseteq X \backslash A$ for each $a \in X \backslash A$. Hence by Theorem 4.13 we get that A is pre-θ-closed set. □

Corollary 4.16 Let A be a standard subset of a space X. Then A is pre-θ-closed set if and only if $\mu_{p\theta}(a) \cap A \neq \phi$ implies $a \in A$.

Proof. It is similar to the proof of Theorem 4.15. □

Theorem 4.17 Let (X,τ) be a standard topological space, and $\mu_{p\theta}(p)$ be the pθ-monad at the point $p \in X$. If $\mu_{p\theta}(p) \subseteq B$, for some internal subset B of X. Then there exists a standard pre-open set G such that $\mu_{p\theta}(p) \subseteq pClG \subseteq B$.

Proof. Suppose that $pClG - B \neq \phi$, for all $G \in PO(X)$, $p \in G$. Then the family $\{pClG - B\}$ has a finite intersection property. Since $pClG_1 - B \cap pClG_2 - B = (pClG_1 \cap pClG_2) - B$, it follows that $\mu_{p\theta}(p) - B \neq \phi$ which is a contradiction. □

Theorem 4.18 A point x is pre-θ-accumulation point of a subset A of a space X, if and only if $\mu_{p\theta}(x)$ contains a point $y \in A$ difference from x.

Proof. If x is a pre-θ-accumulation point of a subset A of a space X , then for each $G\in PO(X)$ which contains x, $pClG\cap A\neq\phi$, this means that there is $y\in pClG\cap A$, *with* $y\neq x$, and each pClG contains a point $y\neq x$. By using Theorem 4.11 we get $y\in \mu_{p\theta}(x)\cap A$, *with* $y\neq x$.

Conversely, assume that $\mu_{p\theta}(x)$ contains a point $y\neq x$ in A. Then for a fixed $G\in PO(X)$, pClG contains a point $y\neq x$. Therefore there is $y\in pClG\cap A$, and by transfer axiom, there is y in standard $pClG\cap A$. Hence $pClG\cap A\neq\phi$,

Therefore x is a pre-θ-accumulation point of a subset A of a space X. □

Theorem 4.19 A point x is θ-accumulation point of a subset A of a space X, if and only if $\mu_\theta(x)$ contains a point $y\in A$ difference from x.

Proof. It is prove is similar to the proof of Theorem 4.18. □

Theorem 4.20 Every pre-θ-accumulation point of a subset A of a standard space X is θ-accumulation point.

Proof. Let $x\in X$ be a standard pre-θ-accumulation point of a subset A of a standard topological space (X,τ). Then by Theorem 4.18 we get $\mu_{p\theta}(x)\cap A\neq\phi$, and then by using Proposition 4.3 we obtain $\mu_\theta(x)\cap A\neq\phi$. Thus by Theorem 4.19, *x* is a θ-accumulation point. □

Theorem 4.21 Let A be a standard subset of a space X. Then

$pCl_\theta A=\{ a\in X;\ \mu_{p\theta}(a)\cap A\neq\phi \}$.

Proof. Let $a\in pCl_\theta A$. Then $a\in F$, for each pre-θ-closed superset of A. If $\mu_{p\theta}(a)\cap A=\phi$, then $\mu_{p\theta}(a)\subseteq X\backslash A$, and by Theorem 4.7, there exists a standard preopen set G such that $\mu_{p\theta}(a)\subseteq pClG\subseteq X\backslash A$. Therefore $pClG\cap A=\phi$, which implies that $a\notin pCl_\theta A$.

Conversely, suppose that $a\in X$ and $\mu_{p\theta}(a) \cap A\neq\phi$. We have to show that $a\in F$, for all pre-θ-closed superset of A. Now, if $a\notin F$, then by Theorem 4.13 we get $\mu_{p\theta}(a) \subseteq X/F$. Therefore $\mu_{p\theta}(a) \cap A=\phi$, which is a contradiction. □

Theorem 4.22 A standard topological space X is p-closed space if and only if $X= \cup\{ \mu_{p\theta}(a); a\in X\}$.

Proof. Assume that X is p-closed space, and let $q\in X$, such that $q\notin\mu_{p\theta}(x)$. Then for each standard $x\in X$, there exists a pre-open set G_x such that $q\notin pCl\, G_x$,

thus $\Gamma=\{ G_x, x\in X , q\notin pCl\, G_x , G_x \in PO(X)\}$ is a pre -open cover of X .

Since X is p-closed space, then there exists a finite subfamily $\{G_1,G_2,\ldots,G_n\}$ such that $X= \cup\{ pCl\, G_i , \text{for } i=1,2,\ldots,n\}$.

This means that, for each standard $x\in X$, implies that $x\in pCl\, G_i$ for some i. Thus by transfer axioms for each $x\in X$, we have $x\in pcl\, G_i$,for some i, which is a contradiction.

Conversely, suppose that X is not p-closed space, and let ρ be a pre-open cover of X such that it has no finite subfamily whose pre-closure covers X. Let $\{G_1,G_2,\ldots,G_n\}\subseteq PO(X)$, define a relation r such that $r(pClG,x)$ iff $x\notin pCl\, G$. Then r is concurrent relation. Then by Theorem 3.2 we get that there is $y\in X$, with $y\in pCl\, G$.

Now, if $x\in X$ such that $x\in pCl\, G$, then $y\notin pClG$, for each standard $G \in PO(X)$, therefore $y\notin\mu_{p\theta}(x)$, which is a contradiction. □

Theorem 4.23 A standard topological space X is pre-Urysohn if and only if $\mu_{p\theta}(x) \cap \mu_{p\theta}(y)=\phi$, for each $x,y\in X$, with $x\neq y$.

Proof. Assume that X is pre-Urysohn , and let $x,y\in X$, with $x\neq y$. Then there exists $G,H\in PO(X)$, and $x\in G$, $y\in H$, with $pClG \cap pClH=\phi$. Now, by Proposition 4.2 we have $\mu_{p\theta}(x) \subseteq pClG$ and $\mu_{p\theta}(y) \subseteq pClH$. Hence $\mu_{p\theta}(x) \cap \mu_{p\theta}(y)=\phi$. Conversely, assume that the condition is valid, Then by Proposition 4.3 for each $x,y \in X$, there exists $D,E \in PO(X)$, such that $x \in pClD \subseteq \mu_{p\theta}(x)$, and $y \in pClE \subseteq \mu_{p\theta}(y)$. Therefore $pClD \cap pClE=\phi$. Hence X is pre-Urysohn space. □

Corollary 4.24 A standard topological space X is pre-Urysohn, p-closed space if and only if $\{\mu_{p\theta}(a);\ a\in X\}$ is a partition of X.

Proof. Follows from Theorem 4.22 and Theorem 4.23. □

Theorem 4.25 A standard space X is pre-T_2 if and only if for each $x,y \in X$, whenever $y\in \mu_{p\theta}(x)$ then $x=y$.

Proof. Assume that X is pre-T_2 and there is $x,y \in X$, such that $y\in \mu_{p\theta}(x)$ and $x\neq y$. Then by Definition 2.5 there exists $G,H \in PO(X)$, such that $x \in G$, $y \in H$, and $G \cap H=\phi$. By Theorem 2.8 $pClG \cap pIntH=\phi$. Since $H\in PO(X)$, then

$pClG \cap H=\phi$, and $\mu_{p\theta}(x) \subseteq pClG$, which imply that $y\in \mu_{p\theta}(x)$, which is a contradiction.

Conversely, assume that the condition valid. Then there is $A \in$ such that $x \in A\subseteq \mu_{p\theta}(x)$. If $y \in A$, then the hypothesis implies that $x=y$. So, if $x,y \in X$, such that $x\neq y$, then $y\notin A$. Therefore there exists $G \in PO(X)$, such that $x \in G$ but $y\notin pClG$. Hence X is pre-T_2 space. □

Remark 4.26 The concepts of compactness and p-closedness are independent, as shown in the following example.

Example 4.27 Let $X=R$ be the set of real numbers**,** with the topology in which non-empty open sets are those subsets of X which contain the point 1. Then X is not compact space.

Since $\{\{x,1\},\ x\in X\}$ is an open cover of X, but has no finite sub cover in this space, every non-empty pre-open set must contain point 1.

Hence X is the only pre-closed set containing any non-empty pre-open set. Therefore $pClG_i=X$ for any $G_i \in PO(X)$, which implies that $\mu_{p\theta}(a)=X$ for any $a\in X$, and $\cup\{\mu_{p\theta}(a); a\in X\}=X$. □

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