Theoretical Mathematics & Applications, vol.2, no.3, 2012, 1-21
ISSN: 1792-9687 (print), 1792-9709 (online)
Scienpress Ltd, 2012

Location Problems of an interface with two layer stationary Navier-Stokes Systems

Mariama Ndiaye[1,2] **Alassane Sy**[1,3] **and Diaraf Seck**[1,4]

Abstract

In this paper, we focus on an interface problem. Two fluids immiscible satisfying stationary Navier-Stokes equations, with transmission boundary conditions on the interface are considered. We use topological optimization tools to get a topological sensitivity of a given cost function, in order to do numerical simulations locating the interface that separates the two fluids. In fact the obtained numerical results permit to see the comportment or the behavior of the fluids when changing the pressure in one compartment.

Mathematics Subject Classification: 76D05, 76N10, 34H05, 54C56
Keywords: Navier-Stokes system, two-layer problem, transmission boundary condition, topological optimization, numerical simulations

---

[1] Laboratoire de Mathématiques de la decision et d'Analyse Numérique (L.M.D.A.N.),
[2] U.F.R: SAT, Université Gaston Berger, BP 234, Saint-Louis Sénégal,
e-mail: mndiak@yahoo.fr
[3] U.F.R: SATIC, Université Alioune Diop de Bambey, BP 30 Bambey Sénégal,
e-mail: azou2sy@hotmail.com/alassane.sy@uadb.edu.sn
[4] FASEG, Université Cheikh Anta Diop de Dakar, BP 5683 Dakar Fann, Sénégal,
e-mail: diaraf.seck@ucad.edu.sn

Article Info: *Received* : July 12, 2012. *Revised* : August 24, 2012
Published online : December 30, 2012

1 Introduction

This work deals about interface problem of two immiscible fluids dynamics, satisfying the stationary Navier-Stokes system with transmissions boundary conditions on the interface. Let us mention that this questioning is an important and interesting one and an increasing attention is done to the fluid interface computation in industrial and environmental applications.

From a mathematical viewpoint, one of the most intriguing unresolved questions concerning the Navier-Stokes equations and closely related to turbulence phenomena is the regularity and uniqueness of the solutions to the initial value problem. More precisely, given a smooth datum at time zero, will the solution of the Navier-Stokes equations continues to be smooth and unique for all time? This question was posed in 1934 by Leray and is still without answer. Let us note that the authors would say that they are not aware that this question is solved.

In the nineteenth century, finding the exact solution to the Navier Stokes equation were studied. In the twentieth century, the concept of weak solution was introduced. Only the existence of the solutions can be ensured. The uniqueness question is among the most important unsolved problems in fluid mechanics. Some particular results on the existence, uniqueness, and regularity of the Navier Stokes equations are nowadays considered as famous and well understood by people who work around these types of problems. They can be found in many references on the mathematical theory of the Navier Stokes equations (see Constantin and Foias [5], Ladyzhenskaya [12], J. L. Lions [13], R. Temam [21], P. L. Lions [14].

In the 2-dimensional evolutive case, the mathematical theory is fairly complete. The weak solutions turn out to be more regular and are, in fact, strong solutions. Moreover, the solutions are unique for a given initial condition and exist for all time.

In the 3-dimensional evolutive case, the mathematical theory is not yet complete. It is known that the weak solutions exists for all time, but it is not known whether they are unique. On the other hand, the strong solution is unique and exists on a certain finite time interval, but it is not known whether they exist for all time (see C. Foias O. Manley R. Rosa R. Temam [7]).

In this paper, we are interested in the steady-state of the Navier Stokes problem. There is non-uniqueness of solutions, in general. Uniqueness occurs only when "the data are small enough, or the viscosity is large enough".

The nature of the interface between two fluids has been the subject of extensive investigation for over two centuries. Young, Laplace, and Gauss, in the early part of the 1800s, considered the interface between two fluids to be represented as a surface of zero thickness endowed with physical properties such as surface tension. In the classical fluid mechanical approach, the interface between two immiscible fluids is modeled as a free boundary that evolves in time. The equations of motion that hold in each fluid are supplemented by boundary conditions at the free surface that involve the physical properties of the interface. Specifically, in the free-boundary formulation it is assumed that the interface has a surface tension, which on applying a stress balance at the interface gives rise to the interfacial boundary condition, see [2]:

$$\sigma.\widehat{n} = |_{-}^{+} = \gamma\kappa\widehat{n}$$

which relates the jump in the stress across the interface to the interfacial curvature. Here σ is the stress tensor, $\widehat{n}$ is the unit vector normal to the interface, γ is the surface tension (here assumed to be constant), and κ is the appropriately signed mean curvature. In addition, an interface between two immiscible fluids is impermeable, in which case conservation of mass across the interface leads to

$$\overrightarrow{u}.\widehat{n} = |_{-} = \overrightarrow{u}.\widehat{n} = |_{+} = U_n$$

where $\overrightarrow{u}$ represents the velocity of the fluid and U_n is the normal velocity of the interface. Finally, for viscous fluids, there is continuity of tangential velocity across the interface

$$[\overrightarrow{u} - (\overrightarrow{u}.\widehat{n})\widehat{n}]|_{-}^{+} = 0$$

The free-boundary description has been a successful model in a wide range of situations. However, there are also important instances where it breaks down.

In this paper, we use topological optimization tools to study the steady Navier-Stokes problem, with transmission boundary condition on the interface between the two fluids. And our aim is to locate the interface that separates the two fluids.

The goal of the topological optimization problem is to find an optimal design with an a priori poor information on the optimal shape of the structure. The topological optimization problem consists in minimizing a functional $j(\Omega) = J(\Omega, u_\Omega)$ where the function u_Ω is defined, for example, on a variable open and bounded subset Ω of $\mathbb{R}^n$. For $\rho > 0$, let $\Omega_\rho = \Omega \backslash \overline{(x_0 + \rho\omega)}$, be the set obtained by removing a small part $x_0 + \rho\omega$ from Ω, where $x_0 \in \Omega$ and $\omega \subset \mathbb{R}^n$ is a fixed open and bounded subset containing the origin. Then, using general adjoint method, an asymptotic expansion of the function will be obtained in the following form:

$$j(\Omega_\rho) = j(\Omega) + f(\rho)g(x_0) + o(f(\rho))$$
$$\lim_{\rho \to 0} f(\rho) = 0, \;\; f(\rho) > 0$$

The topological sensitivity $g(x_0)$ provides information when creating a small hole located at x_0. Hence the function g will be used as descent direction in the optimization process.

Remark 1.1. The physical interpretation of holes depend to the nature of the design. In the case of structural optimization, the insertion of a hole means simply removing some material (see [20], for example). In the case of fluid dynamics, creating a hole means inserting a small obstacle (see [19], for example). The objective here is to see how evolute the fluids when acting on one pressure (for example p_2 varies and p_1 is constant).

The rest of the paper is organized as follows: in Section 2, we recall to the Navier-Stokes equation and some results related to existence and unicity (under useful conditions) of the solution of the problem. In section 3 we formulate the two layer problem and the adapted boundary conditions on the interface. In section 4 we set the topological optimization problem and we derive the topological sensitivity, which is the main theoretical result of the paper and so far its proof. To end the paper, section 5, we use topological gradient to do numerical simulations ($2d$), in order to see variations of δ_j which respects p_2 and therefore to locate the interface between the two fluids.

2 Existence of solution of the Navier-Stokes problem.

The Navier-Stokes equations describing the $n-$dimentional motion of a viscous and incompressible fluid are as follows:

$$\varrho\left(\frac{\partial u_i}{\partial t}+\sum_{j=1}^{n}u_j\frac{\partial u_i}{\partial x_j}\right)-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\sigma_{ij}=\varrho f_i,\quad 1\ \leq\ i,\ j\ \leq\ n, \tag{1}$$

with the incompressibility condition

$$div\ u=\sum_{i=1}^{n}D_{ii}(u)=0, \tag{2}$$

where

$$\left\{\begin{array}{lcl}\sigma_{ij} & = & -P\delta_i^j+2\mu D_{ij}(u)\\ D_{ij}(u) & = & \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\end{array}\right\}\quad 1\ \leq\ i,\ j\ \leq\ n. \tag{3}$$

In these equations, the vector u is the velocity of the fluid, ϱ is its density (assumed to be constant), $\mu>0$ is the viscosity (also assumed to be constant) and P is its pressure; (σ_{ij}) is the stress tensor and the vector f represents a density of body forces per unit mass (gravity for instance). We set

$$p=\frac{P}{\varrho}\quad and\quad \nu=\frac{\mu}{\varrho}.$$

Here p is the kinematic pressure and ν the kinematic viscosity, but for sake of simplicity they will be called pressure and viscosity.

In the sequel, we focus only on the steady case, that is $\frac{\partial u}{\partial t}=0$. Thus the global stationary Navier-Stokes system writes

$$\left\{\begin{array}{c}-\nu\Delta u+\sum_{i=1}^{n}u_iD_iu+\nabla p=f\quad in\ \ \Omega\\ div(u)=0\qquad in\ \ \Omega\\ \gamma(u)=g,\quad on\quad \partial\Omega=\Gamma\end{array}\right. \tag{4}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a Lipschitz continuous boundary Γ and g is a regular given vector function.

In order to write (4) in a variational form, we introduce a trilinear functional,

$$b(u,v,w)=\sum_{i,j=1}^{n}\int_{\Omega}u_iD_iv_jw_idx. \tag{5}$$

We also recall the following spaces

$$\mathcal{V} = \{v \in (\mathcal{D}(\Omega))^n; \quad div\ v = 0\}\ and\ \mathrm{V} = \{v \in (H_0^1(\Omega))^n\ \ div\ v = 0\}$$

The following lemmas give useful properties of b, there proofs can be found in [10, 21].

Lemma 2.1. *The trilinear form b is defined and continuous on $H_0^1(\Omega)^n \times H_0^1(\Omega)^n \times (H_0^1(\Omega)^n \cap L^n(\Omega))$, Ω bounded or unbounded, any dimension of $\mathbb{R}^n$.*

Lemma 2.2. *Let $u \in H^1(\Omega)^n$ with $div\ u = 0$ and $\gamma(u) = 0$ and let v and $w \in H_0^1(\Omega)^n \cap L^n(\Omega)$; then*

$$\begin{aligned} b(u, v, v) &= 0 & (6)\\ b(u, v, w) &= -b(u, w, v) & (7) \end{aligned}$$

And the following estimations holds: (see also [10])

$$\left| \int_\Omega u_i (D_i v_j) w_i dx \right| \leq |u_i|_{L^{\frac{2n}{n-2}}(\Omega)} |D_i v_j|_{L^2(\Omega)} |w_j|_{L^n(\Omega)} \tag{8}$$

$$\|u_i\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq C(\Omega) \|u\|_{H_0^1(\Omega)^n}$$

$$\|D_i v_j\|_{L^2(\Omega)} \leq \|v_j\|_{H_0^1(\Omega)}$$

$$\|w_i\|_{L^n(\Omega)} \leq \|w_i\|_{H_0^1(\Omega) \cap L^n(\Omega)}$$

It follows (8) and the above estimations that:

$$|b(u, v, w)|_{\mathcal{L}_3(\Omega)} \leq C(\Omega) \|u\|_{H_0^1(\Omega)} \|v_j\|_{H_0^1(\Omega)^n} \|w_i\|_{H_0^1(\Omega)} \tag{9}$$

with

$$C(\Omega) = \begin{cases} \frac{2}{3}|\Omega|^{1/6}, & if\ \ n = 3 \\ \frac{|\Omega|^{1/2}}{2}, & if\ \ n = 2 \end{cases}$$

Now let

$$a(u, v) = \int_\Omega \nabla u . \nabla v dx \tag{10}$$

The variational form associated to (4) is: Find $u \in V$ such that

$$\nu a(u, v) + b(u, u, v) = (f, v), \quad \forall\, v \in \widetilde{\mathcal{V}}, \tag{11}$$

where $\widetilde{\mathcal{V}}$ is the closure of $\mathcal{V}$ in $H_0^1(\Omega) \cap L^n(\Omega)$. Given $f \in L^2(\Omega)^n$, it is clear that if u, and p are smooth functions satisfying (4), then u satisfies (11), conversely, if $u \in \mathcal{V}$ satisfying (11) then (4) is satisfied (see the Theorem of De Rham below):

Proposition 2.3 (De Rham)**.** *Let Ω be an open set of $\mathbb{R}^n$ and $h \in (\mathcal{D}'(\Omega))^n$, a necessary and sufficient condition that $h = gradp$ for some $p \in \mathcal{D}'(\Omega)$, is that $< h, v >= 0$.*

Remark 2.4. In the case of incompressible Stoke's system, the pressure can be interpreted as a Lagrange multiplier. In fact the pressure $p \in L^2_loc(\Omega)$. For more details see for instance [6].

The following theorem gives existence result for the Navier-Stokes problem and the proof can be found in [10].

Theorem 2.5. *Let Ω be a bounded domain of $\mathbb{R}^n$ with a Lipschitz continuous boundary Γ. Given a function $f \in (H^{-1}(\Omega))^n$, there exists at least one solution u in V that satisfies (11), and there exists a distribution $p \in L^1_{loc}(\Omega)$ such that (4) is satisfied.*

Remark 2.6. Under the hypothesis of the above lemmas and the theorem, and in addition, if we suppose that $\|u\|_V < \frac{\nu}{\kappa}$, ($\kappa = C(\Omega)$ as defined above) the Navier-Stokes problem (4) admits a unique solution $u \in V$.

3 The two layer problem

Assuming that Ω is halved in two parts Ω_1 and Ω_2 by a manifold Γ_{int} as in Figure 1.

We now consider functions $g_k \in L^2(\Omega_k)^n$ $(k = 1, 2)$. Each domain Ω_k is occupying by a fluid in which the following system holds:

$$\begin{cases} -\nu_k \Delta u_k + \sum_{i=1}^n u_{ki} D_i u_k + \nabla p_k = 0 \quad in \;\; \Omega_k \\ div(u_k) = 0 \qquad in \;\; \Omega_k \\ B_k(u_k) = g_k, \quad on \quad \partial\Omega_k = \Gamma_k \end{cases} \tag{12}$$

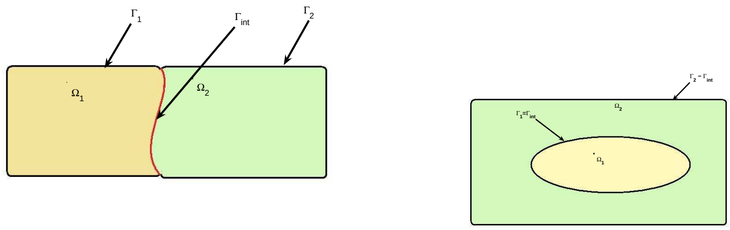


Figure 1: Example of domain Ω halved in two parts Ω_1 and Ω_2

where $B_1,\ B_2$ are Dirichlet or Neumann operators.

We have either the addition transmission boundary conditions

$$\begin{cases} u_1 &= u_2 \quad on \ \ \Gamma_{int} = \Gamma_1 \cap \Gamma_2 \\ \overline{\overline{\sigma_1}}.n &= \qquad \overline{\overline{\sigma_2}}.n \quad on \ \ \Gamma_{int} \end{cases} \tag{13}$$

Without losing generality, we will suppose, in the sequel that $B_1 = B_2 = Id$.

Hence the systems (12)-(13) is an interface problem: Our aim is to look for $(u_1, p_1) \in \mathrm{V} \times L_0^2(\Omega_1)$ and $(u_2, p_2) \in \mathrm{V} \times L_0^2(\Omega_2)$ satisfying

$$\begin{cases} -\nu_1 \Delta u_1 + \sum_{i=1}^{n} u_{1i} D_i u_1 + \nabla p_1 &= \quad 0 \ \ in \ \ \Omega_1 \\ -\nu_2 \Delta u_2 + \sum_{i=1}^{n} u_{2i} D_i u_2 + \nabla p_2 &= \quad 0 \ \ in \ \ \Omega_2 \\ div(u_1) &= 0 \quad in \ \ \Omega_1 \\ div(u_2) &= 0 \quad in \ \ \Omega_2 \\ \gamma_1(u_1) &= g_1, \quad on \quad \Gamma_1 \\ \gamma_2(u_2) &= g_2, \quad on \quad \Gamma_2 \\ u_1 &= u_2 \quad on \ \ \Gamma_{int} \\ \overline{\overline{\sigma_1}}.n &= \overline{\overline{\sigma_2}}.n \quad on \ \ \Gamma_{int} \end{cases} \tag{14}$$

In order to construct an equivalent variational problem, we have to introduce

$$\begin{cases} (i) \quad \text{the space } \mathbf{V} = V \times V \\ (ii) \ \ \text{the space } \mathbf{T} = H^{1/2}(\Gamma_1) \times H^{1/2}(\Gamma_2) \\ (iii) \ \text{the operator} \ \ \gamma = (\gamma_1, -\gamma_2) \end{cases} \tag{15}$$

For this choice, we can present the following trilinear and bilinear forms on $\mathbf{V}$ by:

$$b(u,v,w) = b_1(u_1,v_1,w_1) + b_2(u_2,v_2,w_2), \quad a(u,v) = \nu_1 a_1(u_1,v_1) + \nu_2 a_2(u_2,v_2)$$

where $b_k(.,.,)$ is defined by (5) and $a_k(.,.)$ by (10).

Then the interface problem is the one posed to the space $\mathbf{W}$ defined by

$$\mathbf{W} = \{u = (u_1, u_2) \ \in \ V \times V \ \ such \ that \ \ u_{1|\Gamma_{int}} = u_{2|\Gamma_{int}}\}. \tag{16}$$

equipped with the norm: $\|u\|_{\mathbf{W}} = \max\{|u_1|_V, |u_2|_V\}$ with $|.|_V = |\nabla.|_L^2$. The above variational formulation writes: Find $u \in \mathbf{W}$ such that

$$a(u,v) + b(u,u,v) = (f,v), \quad \forall\, v \ \in \ \widetilde{\mathbf{W}} \tag{17}$$

where $\widetilde{\mathbf{W}}$ is the closure of $\mathbf{W}$. The existence of solution for (17) follows from Theorem 2.5. The unicity follows when $\|u\|_W < \min\{\frac{\nu_1}{\kappa_1}, \frac{\nu_2}{\kappa_2}\}$.

4 Topological asymptotic results

4.1 Setting of the topological optimization problem

For all $\rho > 0$, let $\Omega_\rho = (\Omega_1 \cup \Omega_2) \setminus \omega_\rho$, the domain obtained by removing a small part ω_ρ of radius ρ centered to $x_0 \in \omega_\rho$ from $\Omega = \Omega_1 \cup \Omega_2$ and let $u_\rho = (u_{1,\rho}, u_{2,\rho})$ be the solution of the problem posed on Ω_ρ (see Figure 2): Find $(u_\rho, p_\rho) \in \mathcal{V}_\rho \times L_0^2(\Omega_\rho)$

$$\begin{cases} -\nu\Delta u_\rho + u_\rho \nabla u_\rho + \nabla p_\rho = 0 \quad \Omega_\rho \\ div(u_\rho) = 0 \quad \Omega_\rho \\ \gamma(u_\rho) = g \quad \partial\Omega = \Gamma \\ u_\rho = 0 \quad \partial\omega_\rho \end{cases} \tag{18}$$

The weak formulation of (18) is: find $u_\rho \in \mathcal{V}_\rho$ such that

$$\nu a_\rho(u_\rho, v) + b_\rho(u_\rho, u_\rho, v) = 0, \quad \forall\, v \ \in \ \mathcal{V}_\rho, \tag{19}$$

with

$$a_\rho(u_\rho, v) = \nu \int_{\Omega_\rho} \nabla u_\rho . \nabla v \ \ and \ \ b_\rho(u_\rho, u_\rho, v) = \int_{\Omega_\rho} u_\rho D_j u_{\rho,j}\, v_j dx$$

Where

$$\mathcal{V}_\rho = \{v \ \in \ (\mathcal{D}(\Omega_\rho))^n; \quad div\ v = 0 \ \ and \ \ v = 0 \ \ on \ \ \partial\omega_\rho\}$$

Due to the Theorem 2.5, the problem (18) is well posed and we have the following result.

Proposition 4.1. *Let $u = (u_1, u_2)$ and $u_\rho = (u_{1,\rho}, u_{2,\rho}) \in \mathcal{V}_\rho$ be the solution of (4) and (18), then the following estimation holds*

$$\|u_\rho - u\|_{\mathcal{V}_\rho} = o(f(\rho)),\ with \left\{ \begin{array}{c} f(\rho) = \rho, \quad n = 3 \\ f(\rho) = -\frac{1}{\log \rho}, \quad n = 2 \end{array} \right. \tag{20}$$

4.2 General adjoint method framework

The mathematical framework for domain parametrization introduced by the Murat-Simon work [17] cannot be used here. Alternatively, it is possible however to invoke the adjoint method, as described in [15], in application to topological optimization. A basic feature of the adjoint method is yield of an asymptotic expansion of a functional $J(\Omega, u_\Omega) = J(u)$ which depends of a parameter u_Ω, using a adjoint state v_Ω which does not depend on the parameter. This implies the need to solve a certain system of equations in order to obtain an approximation of the topological gradient $g(x),\ \ \forall\, x \in \Omega$. Accordingly, let $\mathcal{V}$ be a fixed Hilbert space and $\mathcal{L}(\mathcal{V})$ (resp $\mathcal{L}_2(\mathcal{V})$, and $\mathcal{L}_3(\mathcal{V})$) denotes the spaces of linear (resp bilinear and trilinear) forms on $\mathcal{V}$. We are able then to state the following hypotheses:

- **H-1**: There exists a real function f, a trilinear form $\delta_b \in \mathcal{L}_3(\mathcal{V})$, a bilinear form $\delta_a \in \mathcal{L}_2(\mathcal{V})$ and a linear form $\delta_l \in \mathcal{L}(\mathcal{V})$ such that:

$$f(\rho) \longrightarrow 0, \quad \rho \longrightarrow 0^+, \tag{21}$$

$$\|b_\rho - b - f(\rho)\delta_b\|_{\mathcal{L}_3(\mathcal{V})} = o(f(\rho)), \tag{22}$$

$$\|a_\rho - a - f(\rho)\delta_a\|_{\mathcal{L}_2(\mathcal{V})} = o(f(\rho)), \tag{23}$$

$$\|l_\rho - l - f(\rho)\delta_l\|_{\mathcal{L}(\mathcal{V})} = o(f(\rho)). \tag{24}$$

- **H-2**: Consider a cost function $j(\rho) = J(u_\rho)$, where the functional J is differentiable. For $u \in \mathcal{V}$ there exists a linear and continuous form $DJ(u) \in \mathcal{L}(\mathcal{V})$ and δ_J such that:

$$J(u) - J(v) = DJ(u)(u - v) + f(\rho)\delta_J(u) + o(\|u - v\|_{\mathcal{V}}). \tag{25}$$

The Lagrangian of the problem $\mathcal{L}$ is defined by,

$$\mathcal{L}(u,v) = b(u,u,v) + a(u,v) - l(v) + J(u) \quad \forall u \ v \ \in \mathcal{V},$$

and its variation is given, for all $\rho \geq 0$,

$$\mathcal{L}_\rho(u,v) = b_\rho(u,u,v) + a_\rho(u,v) - l_\rho(v) + J_\rho(u) \quad \forall u \ v \ \in \mathcal{V},$$

The next theorem gives an asymptotic expansion for $j(\rho)$.

Theorem 4.2. *If hypotheses (H-1) and (H-2) are satisfied, then*

$$j(\rho) - j(0) = f(\rho)\delta\mathcal{L}(u,v_0) + o(f(\rho)), \tag{26}$$

where u is the solution of (18) with $\rho = 0$, v_0 is the solution to the adjoint problem, find v_0 such that:

$$b(u,u,v_0) + a(u,v_0) = -DJ(u)w \quad \forall w \in \mathcal{V}, \tag{27}$$

and

$$\delta\mathcal{L}(u,v) = \delta_b(u,u,v) + \delta_a(u,v) - \delta_l(v) + \delta_J(u). \tag{28}$$

Proof. The proof is standard in partial differential equations and can be found in [11, 1] □

Variation of the trilinear form

Proposition 4.3. *Let b_ρ and b the trilinear forms defined as above. Then, there exists, a trilinear and continuous form δ_b such that*

$$\|b_\rho - b - f(\rho)\delta_b\|_{\mathcal{L}_3(\Omega)} = o(f(\rho)) \tag{29}$$

where $\mathcal{L}_3(\Omega)$ is the set of trilinear and continuous form on $\mathbf{W} \times L^2(\Omega)$

Proof.

$$b_\rho(u_\rho,u_\rho,v) - b(u,u,v) = b_\rho(u_\rho,u_\rho,v) - b_\rho(u,u_\rho,v) + b_\rho(u,u_\rho,v) - b(u,u,v)$$

$$= b_\rho(u_\rho - u,u_\rho,v) + b_\rho(u,u_\rho,v) - b(u,u,v)$$

$$= b_\rho(u_\rho - u,u_\rho,v) + b_\rho(u,u_\rho,v) - b_\rho(u,u,v) + b_\rho(u,u,v) - b(u,u,v)$$

$$= b_\rho(u_\rho - u,u_\rho,v) + b_\rho(u,u_\rho - u,v) + b_\rho(u,u,v) - b(u,u,v)$$

$$= b_\rho(u_\rho - u, u_\rho, v) + b_\rho(u, u_\rho - u, v) + b_\rho(u, u, v) - b(u, u, v)$$

$$= b_\rho(u_\rho - u, u_\rho, v) + b_\rho(u, u_\rho - u, v) + (b_\rho - b)(u, u, v)$$

Due to the above estimation in Lemma 2.2, we have,

$$|b_\rho(u_\rho - u, u_\rho, v)|_{\mathcal{L}_3(\Omega)} \leq \|u_\rho - u\|_{\mathcal{V}_\rho} \|u_\rho\|_{\mathcal{V}_\rho} \|v\|_{\mathcal{V}} = o(f(\rho))$$

$$|b_\rho(u, u_\rho - u, v)|_{\mathcal{L}_3(\Omega)} \leq \|u_\rho - u\|_{\mathcal{V}_\rho} \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}} = o(f(\rho))$$

We set,

$$b_\rho(u, u, v) - b(u, u, v) = (b_\rho - b)(u, u, v) = f(\rho)\delta_b$$

It follows that

$$\|b_\rho(u_\rho, u_\rho, v) - b(u, u, v)\|_{\mathcal{L}_3(\Omega)} = f(\rho)\delta_b + o(f(\rho)).$$

□

Variation of the bilinear form

Proposition 4.4. *Let a_ρ and a the bilinear forms defined as above. Then, there exists a bilinear and continuous form δ_a such that*

$$\|a_\rho - a - f(\rho)\delta_a\|_{\mathcal{L}_2(\Omega)} = o(f(\rho)) \tag{30}$$

where $\mathcal{L}_2(\Omega)$ is the set of bilinear and continuous form on $\mathcal{V}$

Proof.

$$a_\rho(u_\rho, u_\rho) - a(u, u) = \nu \int_{\Omega_\rho} |\nabla u_\rho|^2 - \nu \int_\Omega |\nabla u|^2 dx$$

$$= \nu \int_\Omega (|\nabla u_\rho|^2 - |\nabla u|^2) dx - \nu \int_{\omega_\rho} |\nabla u|^2 dx$$

$$= \nu \int_\Omega \nabla(u_\rho - u)\nabla(u_\rho + u) dx - \nu \int_{\omega_\rho} |\nabla u|^2 dx$$

$$\left| \int_\Omega \nabla(u_\rho - u) dx \right| \leq \int_\Omega ||\nabla(u_\rho - u)||_{L^2(\Omega)} \leq m(\Omega)||\nabla(u_\rho - u)||_{L^2(\Omega)}$$

$$\leq m(\Omega)||\nabla(u_\rho - u)||_{L^2(\Omega)} \leq m(\Omega)||u_\rho - u||_{H_0^1(\Omega)} = o(f(\rho))$$

It follows that

$$\left|\nu\int_{\Omega}\nabla(u_\rho-u)\nabla(u_\rho+u)dx\right|\leq m(\Omega)||u_\rho-u||_{H_0^1(\Omega)}||u_\rho+u||_{H_0^1(\Omega)}=o(f(\rho))$$

Setting

$$f(\rho)\delta_a=\nu\int_{\omega_\rho}|\nabla u|^2dx,$$

the desired result holds. □

Variation of the cost function

The cost functional

$$J_\Omega(u_1,u_2)=\int_\Omega|u_1-u_2|^2$$

writes

$$J_{\Omega_1\cup\Omega_2}(u_1,u_2)=\int_{\Omega_1\cup\Omega_2}|u_1-u_2|^2=\int_{\Omega_1}|u_1-u_2|^2+\int_{\Omega_2}|u_1-u_2|^2$$

Due to the continuity condition on the interface between Ω_1 and Ω_2 ($u_1=u_2$ on Γ_{int}), it follows that the integral

$$\int_{\Gamma_{int}}|u_1-u_2|=0$$

$$J_{\Omega_\rho}(u_{1,\rho},u_{2,\rho})=\int_\Omega|u_{1,\rho}-u_{2,\rho}|^2 \tag{31}$$

Three cases can presented (see Figure 2):

i) $\omega_\rho\subset\Omega_1$

ii) $\omega_\rho\subset\Omega_2$

iii) $\omega_\rho=\omega_{1,\rho}\cup\omega_{2,\rho}$ with $\omega_{1,\rho}\cap\omega_{2,\rho}=\emptyset$ and $\omega_{1,\rho}\subset\Omega_1$, $\omega_{2,\rho}\subset\Omega_2$

$u_{1,\rho}$ is defined on Ω_1 and $u_{2,\rho}$ on Ω_2 as u_1 and u_2. Thus, we can use the following extension (without lost of generality)

$$u_1(x)=\begin{cases}u_1(x) & x\in\Omega_1\\ 0 & x\in\Omega_2\end{cases},\qquad u_2(x)=\begin{cases}u_2(x) & x\in\Omega_2\\ 0 & x\in\Omega_1\end{cases}$$

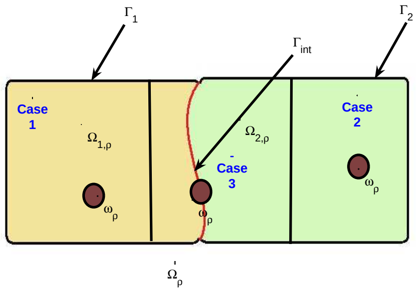


Figure 2: Example of perturbed domains

✓ In the case i), the functional (31) writes

$$J_{\Omega_\rho}(u_{1,\rho}, u_{2,\rho}) = \int_{\Omega_1\setminus\omega_\rho} |u_{1,\rho} - u_{2,\rho}|^2 + \int_{\Omega_2} |u_{1,\rho} - u_{2,\rho}|^2$$

or in Ω_2, $u_{1,\rho} = u_1$ and $u_{2,\rho} = u_2$, so

$$\int_{\Omega_2} |u_{1,\rho} - u_{2,\rho}|^2 = \int_{\Omega_2} |u_1 - u_2|^2$$

Thus,

$$J_{\Omega_\rho}(u_{1,\rho}, u_{2,\rho}) - J_\Omega(u_1, u_2) = \int_{\Omega_1\setminus\omega_\rho} |u_{1,\rho} - u_{2,\rho}|^2 - \int_{\Omega_1} |u_1 - u_2|^2$$

$$= \int_{\Omega_1} |u_{1,\rho} - u_{2,\rho}|^2 - \int_{\Omega_1} |u_1 - u_2|^2 - \int_{\omega_\rho} |u_{1,\rho} - u_{2,\rho}|^2$$

$$= \int_{\Omega_1} \left[|u_{1,\rho} - u_{2,\rho}|^2 - |u_1 - u_2|^2\right] - \int_{\omega_\rho} |u_{1,\rho} - u_{2,\rho}|^2$$

As $u_{1,\rho} \to u_1$ and $u_{2,\rho} \to u_2$ when $\rho \to 0$, follows

$$\lim_{\rho\to 0} |u_{1,\rho} - u_{2,\rho}|^2 = |u_1 - u_2|^2 \Rightarrow \lim_{\rho\to 0} \int_{\Omega_1} |u_{1,\rho} - u_{2,\rho}|^2 = \int_{\Omega_1} |u_1 - u_2|^2 \quad (32)$$

Consequently, the variation of the cost function writes

$$J_{\Omega_\rho}(u_{1,\rho}, u_{2,\rho}) - J_\Omega(u_1, u_2) = \int_{\omega_\rho} |u_{1,\rho} - u_{2,\rho}|^2 + o(f(\rho)). \quad (33)$$

✓ The case ii) is the same as i) (it suffices to replace Ω_1 by Ω_2 in the above calculous) and the result (33) holds.

✓ In the case iii), we have:

$$J_{\Omega_\rho}(u_{1,\rho},u_{2,\rho}) - J_\Omega(u_1,u_2) = \int_{\Omega_1\setminus\omega_{1,\rho}} |u_{1,\rho}-u_{2,\rho}|^2 + \int_{\Omega_1\setminus\omega_{2,\rho}} |u_{1,\rho}-u_{2,\rho}|^2$$
$$- \int_{\Omega_1} |u_1-u_2|^2 - \int_{\Omega_2} |u_1-u_2|^2$$
$$= \underbrace{\int_{\Omega_1} |u_{1,\rho}-u_{2,\rho}|^2 - \int_{\Omega_1} |u_1-u_2|^2}_{=o(f(\rho))\ \ due\ \ to\ \ (32)} - \int_{\omega_{1,\rho}} |u_{1,\rho}-u_{2,\rho}|^2$$
$$+ \underbrace{\int_{\Omega_2} |u_{1,\rho}-u_{2,\rho}|^2 - \int_{\Omega_2} |u_1-u_2|^2}_{=o(f(\rho))\ \ due\ \ to\ \ (32)} - \int_{\omega_{2,\rho}} |u_{1,\rho}-u_{2,\rho}|^2$$

It follows that

$$J_{\Omega_\rho}(u_{1,\rho},u_{2,\rho}) - J_\Omega(u_1,u_2) = -\left[\int_{\omega_{1,\rho}} |u_{1,\rho}-u_{2,\rho}|^2 + \int_{\omega_{2,\rho}} |u_{1,\rho}-u_{2,\rho}|^2\right] + o(f(\rho))$$
$$= \int_{\omega_{1,\rho}\cup\omega_{2,\rho}} |u_{1,\rho}-u_{2,\rho}|^2 + o(f(\rho)) = \int_{\omega_\rho} |u_{1,\rho}-u_{2,\rho}|^2 + o(f(\rho)).$$

Consequently, we have the following result.

Proposition 4.5. *Let $u=(u_1,u_2)$ and $u_\rho=(u_{1,\rho},u_{2,\rho})$ be the solution respectively of systems (14) and (18) and $J(u_\rho)=J_{\Omega_\rho}(u_{1,\rho},u_{2,\rho})$ be the cost functional defined by (31). Then we have the following asymptotic development.*

$$J_{\Omega_\rho}(u_{1,\rho},u_{2,\rho}) - J_\Omega(u_1,u_2) = \int_{\omega_\rho} |u_{1,\rho}-u_{2,\rho}|^2 + o(f(\rho)). \tag{34}$$

Remark 4.6. It is proven in [1] that $\int_{\omega_\rho} |u_\rho - u|^2 dx = o(f(\rho))$.

Due to the general framework of generalized adjoint method in topological optimization, we can derive from Theorem 4.2, the following result which is the main result of this paper.

Theorem 4.7 (Main result)**.** *Let $u=(u_1,u_2)$ and $u_\rho=(u_{1,\rho},u_{2,\rho})$ be the solution respectively of systems (14) and (18) and $J(u_\rho)=J_{\Omega_\rho}(u_{1,\rho},u_{2,\rho})$ be the cost* *functional defined by (31). Then there exist a function* $f(\rho) > 0$, $\lim_{\rho\to 0} f(\rho) = 0$*, a trilinear and continuous form* δ_b *on* $\mathcal{L}_3(\Omega)$*, a bilinear and continuous form for* δ_a *on* $\mathcal{L}_2(\Omega)$ *and a function* δ_j *such that:*

$$J_{\Omega_\rho}(u_\rho) - J_\Omega(u) = f(\rho)(\delta_b(u,u,v_0) + \delta_a(u,v_0) + \delta_J(u)) + o(f(\rho)), \qquad (35)$$

where v_0 *is the solution of the adjoint problem: Find* $v_0 \in \mathfrak{V}$ *such that*

$$a(u,v_0) + b(u,u,v_0) = -DJ(u).\varphi, \quad \forall\, \varphi\ \in \mathfrak{V}.$$

Proof. The lagrangian of the problem $\min J_\Omega(u)$ (J defined by (31)) with u solution of (14) writes:

$$\mathcal{L}(u,v) = J(u) + b(u,u,v) + a(u,v) - (f,v)$$

and its variation which respect ρ

$$\mathcal{L}_\rho(u_\rho,v) = J(u_\rho) + b(u_\rho,u_\rho,v) + a(u_\rho,v) - (f,v)$$

Thus

$$\begin{aligned}\mathcal{L}_\rho(u_\rho,v) - \mathcal{L}(u,v) &= J(u_\rho) + b(u_\rho,u_\rho,v) + a(u_\rho,v) - (f,v)\\ &\quad -(J(u) + b(u,u,v) + a(u,v) - (f,v))\\ &= J(u_\rho) - J(u) + b(u_\rho,u_\rho,v) - b(u,u,v) + a(u_\rho,v) - a(u,v).\end{aligned}$$

Using the above propositions, it follows that:

$$\mathcal{L}_\rho(u_\rho,v) - \mathcal{L}(u,v) = f(\rho)\delta_{\mathcal{L}}(u,v) + o(f(\rho))$$

where $\delta_{\mathcal{L}}(u,v) = \delta_J(u) + \delta_b(u,u,v) + \delta_a(u,v)$.

We now use the fact that the variation of the cost function is equal to the one of the lagrangian, we set $j(\rho) = J_{\Omega_\rho}(u_\rho)$, it follows that

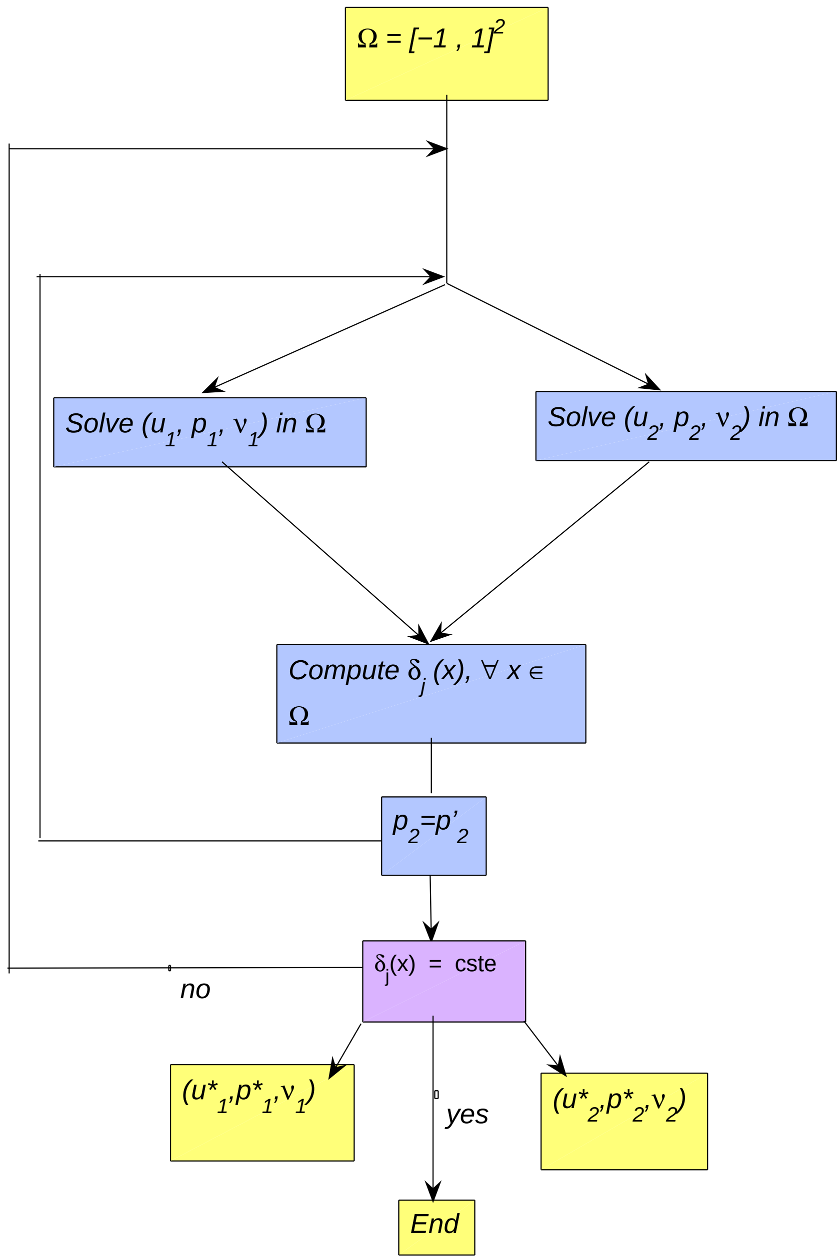
$$j(\rho) - j(0) = f(\rho)\delta_j + o(f(\rho)), \quad (\delta_j = \delta_{\mathcal{L}}(u,v))$$

□

5 Numerical simulations

In this section, we consider a subset $\Omega = [-1, 1]^2$ in which we solve a Navier-Stokes system of two equations $\{(u_1, p_1), (u_2, p_2)\}$ with viscosities $\nu_1 = 1$ and $\nu_2 = 5$ with boundaries conditions $g_1 = g_2 = (1, 1)$ on $\partial\Omega$. We are going to consider two immiscible fluids, and we do not to initialize the interface. It appears naturally because the fluids are in the same domain. After the first step of the resolution, the interface between the two fluids appears in the plot of the topological derivative (Figure 3-(a)). Thus we obtain the solutions $(u_1, p_1), (u_2, p_2)$. So after each step, we fix the pressure p_1 and we augment the pressure p_2 and we observe the evolution of topological derivative with the pressure.

Algorithme 5.1. *The algorithm is summarized on the following figure.*



Proposed algorithm

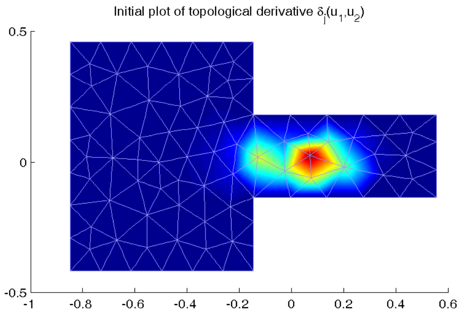


Figure 3: Topological derivative in 2D at the first and last step of optimization process

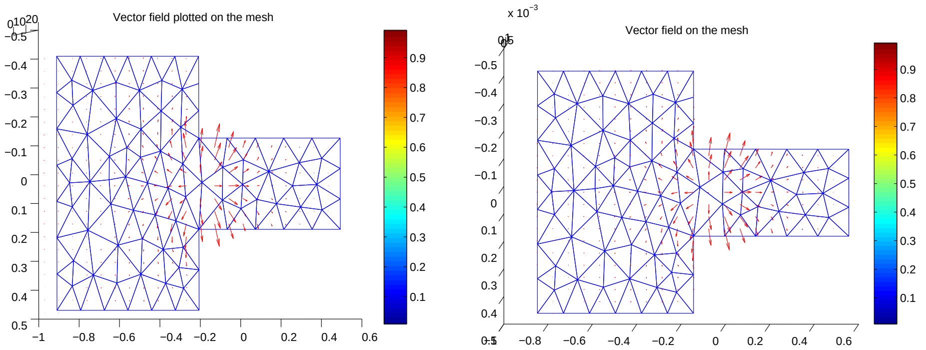


Figure 4: Vector fields on the plotting mesh at the first and last step.

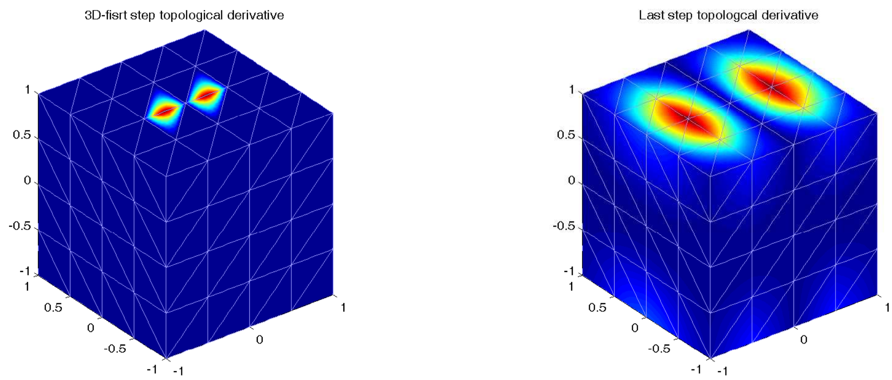


Figure 5: Topological derivative in 3D at the first and last step of optimization process

Remark 5.2. There exists an pressure p_2^* such that the topological derivative still be constant: $\delta_j(u_1^*(x), u_2^*(x)) = -33, 16.10^{-6}\ \ in\ 2D$ and $5, 84.10^{-4}\ \ in\ 3D$ $\forall p_2 \geq p_2^*$.

References

[1] S. Amstutz,The topological asymptotic for the Navier-Stokes equations, *ESIAM COCV*, **11**, (2005), 401-425.

[2] D.M. Anderson and G.B. McFadden, Diffuse-Interface methods in fluid mechanics, *Annu. Rev. Fluid Mech.*, **30**, (1998), 139-165.

[3] V.I. Arnold and B.A. Khesin, *Topological methods in hydrodynamics*, Applied Mathematical Sciences, **125**, Springer-Verlag, New York, 1998.

[4] J. Cèa, Conception optimale, ou identification de forme, calcul rapide de la dérivée directionnelle de la fonction coût, *M.A.A.N.*, **20**(3), (1986), 371-402.

[5] P. Constantin and C. Foias, *Navier Stokes Equations*, University of Chicago Press, 1988.

[6] L.C. Evans, *Partial differential equations*, **19**, A.M.S., 2002.

[7] O. Foias, R. Maniley, R. Rosa and R. Temmam, *Navier-Stokes Equations and Turbulence*, Cambridge University Press, 2004.

[8] S. Friedlander, D. Serre, *Handbook Of Mathematical Fluid Dynamics*, Volume III, Elsevier, 2004.

[9] S. Garreau, P. Guillaume and M. Masmoudi, *The topological asymptotic for PDE systems: the elastic case*, *SIAM Control and optim.*, **39**(6), (2001), 1756-1778.

[10] V. Girault and P. A. Raviart, *Finite element approximation of the Navier-Stokes equations*, Lecture Notes in Mathematics, Springer-Verlag, Berlin Heidelberg, New York, 1979.

[11] Ph. Guillaume and K. SID Idris: The topological expansion for the Dirichlet problem, *SIAM J. Control and optim.*, **41**(4), (2002), 1042-1072.

[12] O. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, revised English edition (translated from the Russian by Richard A. Silverman), Gordon & Breach, New York, 1963.

[13] J.L. Lions, *Quelques methodes de resolution des problemes aux limites non linaires*, Dunod, Paris, 1969.

[14] P.L. Lions, *Mathematical topics in fluid mechanics*, Vol. 1 : Incompressible models, Oxford Lecture Series in Mathematics and its applications, Vol. 3, Oxford University Press, New York, 1996.

[15] M. Masmoudi, *The topological asymptotic*, in computational methods for control applications, H. Kawarada and J. Periaux, eds, GAKUTO Internat. Ser. Math. Sci. Appli. Gakkotōsho, Tokyo, 2002.

[16] Y. Renard and J. Pommier, *Getfem++, An open source generic C++ library for finit element methods,* http://home.gna.org/getfem/.

[17] F. Murat and J. Simon, *Sur le contrôle par un domaine géométrique*, Habilitation de l'Université de Paris, 1976.

[18] J. Simon, Sur les fluides visqueux, incompressibles et non homogènes, *C.R. Acad. Sci de Paris*, **309**, Serie 1, (1989), 447-452.

[19] A. Sy, Fictitious domain approach and level-sets method for Stokes problem, *Internationl Journal of Mathematical Archive*, **2**(12), (2011), 2768-2776.

[20] A. Sy and Y. Renard, A fictitious domain approach for structural optimization with a coupling between shape and topological gradient, *Far East Journal of Mathematical Sciences*, **47**(1), (2010), 33-50.

[21] R. Teman, *Navier Stokes Equations, Studies in Mathematics and its Applications*, North-Holland Publishing Compagny, Amsterdam, New-York, Oxford, 1979.

[22] R. Temam, *Navier Stokes Equations and Nonlinear Functional Analysis*, (CBMSNSF Regional Conf. Ser. in Appl. Math. 66), SIAM, Philadelphia, 2nd ed. published in 1995, North-Holland Publishing Compagny, Amsterdam, New-York, Oxford, 1979.