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Existence results for a class of semilinear elliptic systems

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Abstract

This paper deals with the existence of solutions to a class of semilinear potential elliptic systems of the form

$$\begin{cases} -div(a(x)\nabla u) = \lambda F_u(x,u,v) & \text{in } \Omega, \\ -div(b(x)\nabla v) = \lambda F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where the domain Ω is a bounded domain, the weights $a(x)$, $b(x)$ are measurable nonnegative weights and λ is a positive parameter.

Mathematics Subject Classification: 35J60, 35B30, 35B40
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1 Introduction

In this paper, we deal with a class of semilinear elliptic systems of the form

$$\begin{cases} -div(a(x)\nabla u) = \lambda F_u(x,u,v) & \text{in } \Omega, \\ -div(b(x)\nabla v) = \lambda F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the domain Ω is a bounded domain in $\mathbb{R}^N (N > 2)$, the weights $a(x),\ b(x)$ are measurable nonnegative weights on Ω, $(F_u, F_v) = \nabla F$ stands for the gradients of F in the variables $(u,v) \in \mathbb{R}^2$ and λ is a positive parameter.

Recently, many authors have studied the existence of nontrivial solutions for such problems (see [3, 6, 8, 10,12-14] and their references) because several physical phenomena related equilibrium of continuous media are modeled by these elliptic problems (see [5]).

In [7], N.B. Zographopoulos studied a class of degenerate potential semilinear elliptic systems of the form

$$\begin{cases} -div(a(x)\nabla u) = \lambda\mu(x)|u|^{\gamma-1}|v|^{\delta+1}u & \text{in } \Omega, \\ -div(b(x)\nabla v) = \lambda\mu(x)|u|^{\gamma+1}|v|^{\delta-1}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\lambda > 0$, $\gamma, \delta \geq 0$ and $\mu(x)$ may change sign. He proved the existence of at least one solution for the system (2) under suitable assumption on the data.

In this paper, we consider system (1) and prove under the suitable conditions on nonlinearities F_u and F_v, by using the Minimum principle (see [2, p. 4, Theorem 1.2]) and the Mountain pass theorem of A. Ambrosetti and Robinowitz [4], the system (1) has at least two nontrivial solutions.

Throughout this work, we assume the weights $a,\ b \in L^1_{loc}(\Omega)$, $a^{-s},\ b^{-s} \in L^1(\Omega),\ s \in (\frac{N}{2}, \infty) \cap [1, \infty)$. With the number s we define

$$2_s = \frac{2s}{s+1},\ 2_s^* = \frac{N2_s}{N-2_s} = \frac{N2s}{N(s+1)-2s} > 2.$$

We define the Hilbert spaces $W_0^{1,2}(\Omega, a)$ and $W_0^{1,2}(\Omega, b)$ as the clusures of $C_0^\infty(\Omega)$ with respect to the norms

$$\|u\|_a^2 = \int_\Omega a(x)|\nabla u|^2 dx \qquad \text{for all} \quad u \in C_0^\infty(\Omega),$$

$$\|v\|_b^2 = \int_\Omega b(x)|\nabla v|^2 dx \qquad \text{for all} \quad v \in C_0^\infty(\Omega).$$

Set $W = W_0^{1,2}(\Omega, a) \times W_0^{1,2}(\Omega, b)$. It is clear that W is a Hilbert space under the norm

$$\|(u,v)\|_W = \|u\|_a + \|v\|_b \qquad \text{for all} \quad (u,v) \in W,$$

and with respect to the scalar product

$$\langle \varphi, \psi \rangle_W = \int_\Omega (a(x)\nabla\varphi_1\nabla\psi_1 + b(x)\nabla\varphi_2\nabla\psi_2)dx$$

for all $\phi = (\varphi_1, \varphi_2)$, $\psi = (\psi_1, \psi_2) \in W$.

Then W is a uniformly convex space. Moreover, the continuous embedding

$$W \hookrightarrow (W^{1,2_s})^2$$

holds with $2_s = \dfrac{2s}{s+1}$ (cf. Example1.3, [15]) and we have the Sobolev's embedding $W \hookrightarrow (L^{2_s^*}(\Omega))^2$. We notice that the compact embedding

$$W \hookrightarrow L^r(\Omega) \times L^t(\Omega)$$

holds provided that $1 \leq r, t < 2_s^*$.

Next, we assume that $F(x,t,s)$ is a C^1-functional on $\Omega \times [0,\infty) \times [0,\infty) \to \mathbb{R}$, satisfying the hypotheses below:

($\mathbf{F_1}$) There exist positive constant c_1, c_2 such that

$$|F_t(x,t,s)| \leq c_1 t^\gamma s^{\delta+1} \quad , \quad |F_s(x,t,s)| \leq c_2 t^{\gamma+1} s^\delta$$

for all $(t,s) \in \mathbb{R}^2$, a.e. $x \in \Omega$ and some $\gamma, \delta > 1$ with $\frac{\gamma+1}{p} + \frac{\delta+1}{q} = 1$ and $\gamma + 1 < p < 2_s^*$, $\delta + 1 < q < 2_s^*$.

($\mathbf{F_2}$) There exist positive constant c and $2 < \alpha, \beta < 2_s^*$ such that

$$|F(x,t,s)| \leq c(1 + |t|^\alpha + |s|^\beta).$$

($\mathbf{F_3}$) There exist $R > 0$, θ and θ' with $\frac{1}{2_s^*} < \theta, \theta' < \frac{1}{2}$ such that

$$0 < F(x,t,s) \leq \theta t F_t(x,t,s) + \theta' s F_s(x,t,s).$$

for all $x \in \overline{\Omega}$ and $|t|, |s| \geq \mathbb{R}$.

($\mathbf{F_4}$) There exist $\overline{\alpha} > 2$, $\overline{\beta} > 2$ and $\epsilon > 0$ such that

$$|F(x,t,s)| \leq c(|t|^{\overline{\alpha}} + |s|^{\overline{\beta}})$$

for all $x \in \overline{\Omega}$ and $|t|, |s| \leq \epsilon$.

Definition 1.1. *We say that $(u, v) \in W$ is a weak solution of system (1) if and only if*

$$\int_\Omega (a(x)\nabla u \nabla \varphi + b(x)\nabla v \nabla \psi)dx = \lambda \int_\Omega (F_u(x,u,v)\varphi + F_v(x,u,v)\psi)dx,$$

for all $(\varphi, \psi) \in W$.

The functional corresponding to problem (1) is

$$I_\lambda(u,v) = \frac{1}{2}\int_\Omega (a(x)|\nabla u|^2 + b(x)|\nabla v|^2)dx - \lambda \int_\Omega F(x,u,v)dx. \tag{3}$$

It is easy to see that the functional $I(u,v)$ is well defined and is of class C^1 in W. Thus, weak solutions of (1) are exactly the critical points of the functional I_λ.

Now, we can describe our main results as follows.

Theorem 1.2. *Suppose that the condition ($\mathbf{F_1}$) is satisfied. Then there exists a constant $\underline{\lambda} > 0$ such that for all $0 \leq \lambda < \underline{\lambda}$, system (1) has a weak solution.*

Theorem 1.3. *In addition suppose that the condition ($\mathbf{F_1}$) − ($\mathbf{F_4}$) are satisfied. Then problem (1) has a nontrivial solution.*

2 Proof of Theorem 1.2

Lemma 2.1. *The functional I_λ given by (3) is weakly lower semicontinous in W.*

Proof. Let $\{(u_m, v_m)\}$ be a sequence that converges weakly to (u, v) in W. By the weak lower semicontinuity of the norms in the spaces $W_0^{1,2}(\Omega, a)$ and $W_0^{1,2}(\Omega, b)$ we deduce that

$$\liminf_{m\to\infty} \int_\Omega \Big[a(x)|\nabla u_m|^2 + b(x)|\nabla v_m|^2\Big]dx \geq \int_\Omega \Big[a(x)|\nabla u|^2 + b(x)|\nabla v|^2\Big]dx. \quad (4)$$

We shall show that

$$\lim_{m\to\infty} \int_\Omega F(x, u_m, v_m)dx = \int_\Omega F(x, u, v)dx. \quad (5)$$

Indeed, we have

$$\begin{aligned}
&\Big|\int_\Omega \Big[F(x, u_m, v_m) - F(x, u, v)\Big]dx\Big| \\
&\quad\leq \int_\Omega \Big|F_u(x, u + \theta_{1,m}(u_m - u), v + \theta_{2,m}(v_m - v))\Big||u_m - u|dx \\
&\qquad + \int_\Omega \Big|F_v(x, u + \theta_{1,m}(u_m - u), v + \theta_{2,m}(v_m - v))\Big||v_m - v|dx \\
&\quad\leq c_1 \int_\Omega |u + \theta_{1,m}(u_m - u)|^\gamma |v + \theta_{2,m}(v_m - v)|^{\delta+1}|u_m - u|dx \\
&\qquad + c_2 \int_\Omega |u + \theta_{1,m}(u_m - u)|^{\gamma+1}|v + \theta_{2,m}(v_m - v)|^\delta |v_m - v|dx \\
&\quad\leq c_1\|u + \theta_{1,m}(u_m - u)\|_{L^p}^\gamma \|v + \theta_{2,m}(v_m - v)\|_{L^q}^{\delta+1}\|u_m - u\|_{L^p} \\
&\qquad + c_2\|u + \theta_{1,m}(u_m - u)\|_{L^p}^{\gamma+1}\|v + \theta_{2,m}(v_m - v)\|_{L^q}^\delta \|v_m - v\|_{L^q}. \quad (6)
\end{aligned}$$

Since $2 < \gamma + 1 < p < 2_s^*$ and $2 < \delta + 1 < q < 2_s^*$, the sequence $\{(u_m, v_m)\}$ converges strongly to (u, v) in the space $L^p(\Omega) \times L^q(\Omega)$. It is easy to see that

$$\|u + \theta_{1,m}(u_m - u)\|_{L^p}$$

and

$$\|v + \theta_{2,m}(v_m - v)\|_{L^q}$$

are bounded. Thus, it follows from (6) that relation (5) holds true. Then we have

$$I_\lambda(u, v) \leq \liminf_{m\to\infty} I_\lambda(u_m, v_m).$$

□

Lemma 2.2. *The functional I_λ given by (3) is coercive and bounded below in W.*

Proof. By ($\mathbf{F_1}$), there exists $c_3 > 0$ such that for all $(t,s) \in R^2$ and a.e. $x \in \Omega$ we have

$$|F(x,t,s)| \leq c_3 |t|^{\gamma+1}|s|^{\delta+1}.$$

Using Holder's and Young's inequalities, we obtain

$$\begin{aligned}\int_\Omega F(x,u,v)dx &\leq c_3 \int_\Omega |u|^{\gamma+1}|v|^{\delta+1}dx \\ &\leq c_3\Big(\frac{\gamma+1}{p}\int_\Omega |u|^p dx + \frac{\delta+1}{q}\int_\Omega |v|^q dx\Big) \\ &\leq c_3\Big(\frac{\gamma+1}{p} s\int_\Omega a(x)|\nabla u|^2 dx + \frac{\delta+1}{q} s' \int_\Omega b(x)|\nabla v|^2 dx\Big)\end{aligned}$$

where s and s' are the imbedding constants of $W_0^{1,2}(\Omega,a) \hookrightarrow L^p(\Omega)$ and $W_0^{1,2}(\Omega,b) \hookrightarrow L^q(\Omega)$, respectively. Then we can write

$$I_\lambda(u,v) \geq \Big(\frac{1}{2} - \lambda c\frac{\gamma+1}{p}\Big)\|u\|_a^2 + \Big(\frac{1}{2} - \lambda c\frac{\delta+1}{q}\Big)\|v\|_b^2,$$

where $c = max\{c_3 s, c_3 s'\}$. Let $\underline{\lambda} = \min\Big\{\frac{p}{2c(\gamma+1)}, \frac{q}{2c(\delta+1)}\Big\} > 0$, then for all $0 \leq \lambda < \underline{\lambda}$ we conclude that $I_\lambda(u,v) \to \infty$, provided that $\|(u,v)\| \to \infty$. □

By Lemmas (2.1) and (2.2), applying the Minimum principle, the functional I_λ attains its minimum, and thus system (1) admits at least one weak solution.

3 Proof of Theorem 1.3

Lemma 3.1. *The functional I_λ given by (3) satisfies the Palais-Smale condition in W.*

Proof. Let $\{(u_m, v_m)\}$ be a Palais-Smale sequence for the functional I_λ, thus there exists $c_4 > 0$ such that

$$|I_\lambda(u_m,v_m)| \leq c_4 \qquad \text{for any} \quad m \in N \tag{7}$$

and there exists a strictly decreasing sequence $\{\epsilon_m\}_{m=1}^{\infty}$, $\lim_{m\to\infty}\epsilon_m = 0$, such that

$$|\langle I'_\lambda(u_m, v_m), (\xi, \eta)\rangle| \le \epsilon_m \|(\xi, \eta)\|, \tag{8}$$

for any $m \in N$ and for any $(\xi, \eta) \in W$.

By Lemma (2.2), we deduce that I_λ is coercive, relation (7) implies that the sequence $\{(u_m, v_m)\}$ is bounded in W. Since W is a Hilbert space, there exists $(u, v) \in W$ such that, passing to subsequence, still denote by $\{(u_m, v_m)\}$, it converges weakly to (u, v) in W and strongly in $L^p(\Omega) \times L^q(\Omega)$. Choosing $(\xi, \eta) = (u_m - u, 0)$ in (7), we have

$$\Big| \int_\Omega a(x)|\nabla u_m|\nabla(u_m - u) - \lambda \int_\Omega F_u(x, u_m, v_m)(u_m - u)\Big| \le \epsilon_m \|u_m - u\|. \tag{9}$$

Using the condition $(\mathbf{F_1})$ combined with Holder's inequality we conclude that

$$\begin{aligned}\int_\Omega F_u(x, u_m, v_m)|u_m - u|dx &\le c_1 \int_\Omega |u_m|^\gamma |v_m|^{\delta+1}|u_m - u|dx \\ &\le c_1 \|u_m\|_{L^p}^\gamma \|v_m\|_{L^q}^{\delta+1} \|u_m - u\|_{L^p}.\end{aligned}$$

It follows from relations (9) and (10) that

$$\lim_{m\to\infty} \int_\Omega a(x)|\nabla u_m|\nabla(u_m - u)dx = 0$$

subtracting

$$\int_\Omega a(x)|\nabla u|(\nabla u_m - \nabla u)dx,$$

we obtain

$$0 = \lim_{m\to\infty} \int_\Omega a(x)(|\nabla u_m| - |\nabla u|)(\nabla u_m - \nabla u)dx \ge \lim_{m\to\infty} (\|u_m\|_a - \|u\|_a)^2 \ge 0$$

which implies that $\|u_m\|_a \to \|u\|_a$. The uniform convexity of $W_0^{1,2}(\Omega, a)$ yields that u_m converges strongly to u in $W_0^{1,2}(\Omega, a)$.
Similarly, we obtain $v_m \to v$ in $W_0^{1,2}(\Omega, b)$ as $n \to \infty$. $\square$

By Lemma (3.1), we obtain that the functional I_λ satisfies (PS)-condition (compactness condition). Now we verify that the functional I_λ has the geometry of the Mountain pass theorem.

Lemma 3.2. *Under assumption* $(\mathbf{F_1}) - (\mathbf{F_4})$, *the functional* I_λ *satisfies* (**i**) *There exists* $\rho,\ \sigma > 0$ *such that* $\|(u,v)\|_H = \rho$ *implies* $I(u,v) \geq \sigma > 0$.

(**ii**) *There exists* $(z_1, z_2) \in W$ *such that* $\|(z_1, z_2)\|_H > \rho$ *and* $I(z_1, z_2) \leq 0$.

Proof. (**i**) From ($\mathbf{F_2}$) and ($\mathbf{F_4}$), one obtains

$$|F(x,u,v)| \leq c(|u|^{\alpha} + |v|^{\beta} + |u|^{\overline{\alpha}} + |v|^{\overline{\beta}})$$

for all $x \in \overline{\Omega}$ and $(u,v) \in R^2$ where $2 < \alpha, \overline{\alpha}, \beta, \overline{\beta} < 2_s^*$. By Sobolev embedding we obtain

$$\int_{\Omega} F(x,u,v)dx \leq c(\|u\|_a^{\alpha} + \|v\|_b^{\beta} + \|u\|_a^{\overline{\alpha}} + \|v\|_b^{\overline{\beta}}).$$

So, we can estimate the functional $I_\lambda(u,v)$ by

$$I_\lambda(u,v) \geq \frac{1}{2}(\|u\|_a^2 + \|v\|_b^2) - c(\|u\|_a^{\alpha} + \|v\|_b^{\beta} + \|u\|_a^{\overline{\alpha}} + \|v\|_b^{\overline{\beta}})$$

which implies that there exist $\sigma, \rho > 0$ such that $I_\lambda(u,v) \geq \sigma > 0$ for $\|u\|_a + \|v\|_b = \rho$.

(**ii**) Using ($\mathbf{F_3}$), we have

$$\begin{aligned} \frac{d}{dt}F(x, t^{\theta}u, t^{\theta'}v) &= \theta u F_u(x, t^{\theta}u, t^{\theta'}v)t^{\theta-1} + \theta' v F_v(x, t^{\theta}u, t^{\theta'}v)t^{\theta'-1} \\ &\geq \frac{1}{t}F(x, t^{\theta}u, t^{\theta'}v) \end{aligned}$$

which implies that there exists some function $K(x,u,v)$ such that

$$F(x, t^{\theta}u, t^{\theta'}v) \geq tK(x,u,v). \tag{10}$$

From (11), we obtain

$$\begin{aligned} I(t^{\theta}u, t^{\theta'}v) &= \frac{1}{2}(t^{2\theta}\|u\|_a^2 + t^{2\theta'}\|v\|_b^2) - \int_{\Omega} F(x, t^{\theta}u, t^{\theta'}v)dx \\ &\leq \frac{1}{2}(t^{2\theta}\|u\|_a^2 + t^{2\theta'}\|v\|_b^2) - t\int_{\Omega} K(x,u,v)dx. \end{aligned}$$

Since $2\theta, 2\theta' < 1$, we conclude that

$$I(t^{\theta}u, t^{\theta'}v) \to -\infty \quad as \quad t \to +\infty,$$

and thus there exists a constant t_0 such that $I(t_0^{\theta}u, t_0^{\theta'}v) < 0$. □

Consequently, the functional I_λ has a nonzero critical point and the nonzero critical point of I_λ is precisely the nontrivial solution of problem (1).

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] M. Struwe, *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Fourth Edition, Springer-Verlag, Berlin, 2008.

[3] N.T. Chung and H.Q. Toan, On a class of degenerate and singular elliptic systems in bounded domain, *J. Math. Anal. Appl.*, **360**, (2009), 422-431.

[4] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14**, (1973), 349-381.

[5] R. Dautray and J.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology I: Physical Origins and Classical Methods*, Springer-Verlag, Berlin, 1985.

[6] G. Zhang and Y. Wang, Some existence results for a class of degenerate semilinear elliptic systems, *J. Math. Anal. Appl.*, **333**, (2007), 904-918.

[7] N.B. Zographopoulos, On a class of degenarate potential elliptic system, *Nonlinear Diff. Equ. Appl.*, **11**, (2004), 191-199.

[8] N.B. Zographopoulos, p-Laplacian systems on resonance, *Appl. Anal.*, **83**, (2004), 509-519.

[9] M. Willem, *Minimax Theorems*, Birkhauser, Boston, 1996.

[10] D.G. Costa, On a class of elliptic systems in R^N, *Electron. J. Differential Equations*, **07**, (1994).

[11] G.Zhang and Y. Wang, On a class of Schrodinger system with discontinuous nonlinearities, *J. Math. Anal. Appl.*, **11**, (2004), 191-199.

[12] P. Caldiroli and R. Musina, On a variational degenerate elliptic problem, *Nonlinear Diff. Equ. Appl.*, **7**, (2000), 187-199.

[13] L. Boccardo and D.G. De Figueiredo, Some remarks on a system of quasilinear elliptic equations, *Nonlinear Diff. Equ. Appl.*, **9**, (2002), 309-323.

[14] A. Djellit and S. Tas, Existence of solutions for a class of elliptic systems in R^N involving the p-Laplacian, *Electron. J. Differential Equations*, **56**, (2003), 1-8.

[15] P. Drabek, A. Kufner and F. Nicolosi, *Quasilinear elliptic equations with degenarate and singularities*, vol. 5 of de Gruyter Series in Nonlinear Analysis and Applications, Walter de Gruyter and Co., Berlin, 1997.