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The Stability of Non-smooth Slow-varying Systems

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Abstract

In this paper, we studied the stabilty of general situation of non-smooth slow-varying systems. We gave an exponential stability theorem for linear time-varying systems with undifferential right-hand, which only satisfies Lipschitz condition. Furthermore, the upper bound of ε is given.

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1 Introduction

The interest in the study of the nonsmooth analysis for discontinuous systems is essentially motivated by many practical problems. The main reason is there are many systems which have nonsmooth dynamics, such as the systems

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have Coulomb friction, contact interaction, and variable structure. So it is essential for us to analyze these systems and address such issues as the existence of equilibria, their stability, and qualitative dynamics. Such systems arise in a large variety of important engineering applications such as the control of friction and pendulums([1],[2],[3],[4]). It is therefore of great interest in control engineering to develop methods for determining whether slowly time-varying systems are uniformly globally asymptotically stable. As we know, there is a great different nature between a general slow-varying system

$$\dot{x} = f(x,t) \tag{1}$$

and a invariant system.We can not use eigenvalues, eigenvectors to describe the solutions. So many author's work focused on specific slowly varying linear and nonlinear systems.

Consider time-varying linear system

$$\dot{x} = A(t)x \tag{2}$$

where all the elements $a_{i,j}(t)(i,j = 1,2,\cdots,n)$ of $A(t)$ are continuous and bounded, and all the eigenvalues of $A(t)$ have strict negative real parts, but this does't ensure that the system (2) is asymptotically stable. In 1963, H.H.Rosonblock[4] pointed out that for some special $A(t)$, if $|\dot{a}_{i,j}(t)|$ is small enough, then the system (2) is stable. In 2001, Wang Yibing and Han Zengjin [5] studied the stability of general nonlinear slow-varying system, and obtained an exponential stability theorem. But in their results, they need $A(t)$ is differential. In this paper, we will consider that $A(t)$ is not differential , then the Lyapunov function $V(t,x)$ is also not differential. That is to say, it is enough ensure the system (2) is stable if $A(t)$ is only Lipschitz.

2 Main Results

Consider initial value problem of the following general nonlinear slow-varying system

$$\dot{x} = f(t,x(t)), x(0) = x_0, t \geq 0 \tag{3}$$

where $x(t) \in R^n$. We assume:

(H_1) $f : [0,+\infty) \times R^n$ is continuous differentiable function, $D_x f(t,x)$ is uniformly continuous about x on $[0,+\infty)\times\Omega$, and Ω is an compact neighborhood of $x=0$.

(H_2) $f(t,x)=0, t\geq 0$ if and only if $x=0$.

(H_3) There exist positive constants σ and M such that

$$Re\lambda(D_x f(t,0))\leq -\sigma<0, \|D_x f(t,0)\|\leq M, t>0.$$

According to condition (H_3), it is easy for us to get: for any $k(0<k<\sigma)$, there must exist $K>0$ which depends on σ, M, k, n such that

$$\|e^{D_x f(t,0)s}\|\leq Ke^{-ks}, s>0.$$

Theorem 2.1. *Suppose the system (3) satisfies conditions* (H_1),(H_2),(H_3)*. Let* $A(t)=D_x f(t,0)$*. If* $A(t)$ *satisfies Lipschitz condition, and for any small enough* $\varepsilon>0$,

$$\sup_{h>0}\|\frac{A(t+h)-A(t)}{h}\|<\varepsilon<\frac{\sigma^2}{2K^2+K^2\sigma},$$

then the trivial solution $x=0$ *of system (3) is exponential stable.*

In the following discussion, we always assume that the system satisfies the condition (H_1),(H_2),(H_3). Now, for $x\in R^n$, we can get

$$f(t,x)=\int_0^1\frac{d}{d\theta}f(t,\theta x)d\theta. \tag{4}$$

Let

$$\tilde{A}(t,x)=D_x f(t,x), \tilde{C}(t,x)=\int_0^1(\tilde{A}(t,\theta x)-\tilde{A}(t,0))d\theta.$$

By (4) we can get

$$f(t,x)=\tilde{A}(t,0)x(t)+\tilde{C}(t,x(t))x(t).$$

Let

$$A(t)=\tilde{A}(t,0), C(t)=\tilde{C}(t,0).$$

Then the system (3) can be rewritten as

$$\dot{x}(t)=A(t)x(t)+C(t)x(t), x(0)=x_0, t\geq 0. \tag{5}$$

Corollary 2.2. *For any given one constant* $\eta > 0$, *there must exist a* $\tilde{\delta} > 0$ *such that* $\|C(t)\| \leq \eta$ *when* $\|x(t)\| \leq \tilde{\delta}, t \geq 0$.

Proof. By the condition (H_1),we know, for any given $\eta > 0$, there exists one $\delta > 0$ such that $\|\tilde{A}(t,x) - \tilde{A}(t,0)\| \leq \eta$ if $\|x\| \leq \delta$. Then for any $\theta \in [0,1]$, we know $\|\tilde{A}(t,x) - \tilde{A}(t,0)\| \leq \eta$ if $\|x\| \leq \delta$.Thus

$$\|\tilde{C}(t,x(t))\| = \|\int_0^1 (\tilde{A}(t,\theta x) - \tilde{A}(t,0))d\theta\| \leq \int_0^1 \|\tilde{A}(t,\theta x) - \tilde{A}(t,0)\|d\theta \leq \eta,$$

that is to say, for any $t \geq 0$, if $\|x(t)\| \leq \tilde{\delta}$, then we can get $\|C(t)\| \leq \eta$, the proof is completed. □

Now consider Lyapunov matrix equation

$$A^T(t)R(t) + R^T(t)A(t) = -I \tag{6}$$

Owing to $Re\lambda(A(t)) \leq -\sigma$, for each $t \geq 0$, the matrix function

$$R(t) = \int_0^{+\infty} e^{A^T(t)s}e^{A(t)s}ds \tag{7}$$

is continuous and the only definite solution of (6), and if $A(t)$ satisfies the Lipschitz condition, then $R(t)$ also. Now we prove it by mean value theorem.

For any $t_1, t_2 > 0$, if $\|A(t_1) - A(t_2)\| \leq L|t_1 - t_2|$, then

$$\begin{aligned} \|R(t_1) - R(t_2)\| &= \|\int_0^{+\infty} e^{A^T(t_1)s}e^{A(t_1)s} - e^{A^T(t_2)s}e^{A(t_2)s}ds\| \\ &\leq \int_0^{+\infty} (\|e^{A^T(t_1)s}[e^{A(t_1)s} - e^{A(t_2)s}]\| + \|[e^{A^T(t_1)s} - e^{A^T(t_2)s}]e^{A(t_2)s}\|)ds. \end{aligned}$$

By the mean value theorem, there exist $t_3, t_4 \in (t_1, t_2)$, such that the above equation can be reduced to

$$\begin{aligned} &\int_0^{+\infty} (\|e^{A^T(t_1)s}e^{A(t_3)s}[A(t_1)s - A(t_2)s]\| + \|e^{A^T(t_4)s}[A^T(t_1)s - A^T(t_2)s]e^{A(t_2)s}\|)ds \\ &\leq \int_0^{+\infty} 2K^2 e^{-\sigma s} s ds \cdot \|A(t_1) - A(t_2)\| \\ &\leq \int_0^{+\infty} 2K^2 e^{-\sigma s} sL|t_1 - t_2|ds \leq \frac{2K^2L}{\sigma^2}|t_1 - t_2|, \end{aligned}$$

because of $\|e^{A(t)s}\| \leq Ke^{-\frac{\sigma}{2}s}$. So we prove

$$\|R(t_1) - R(t_2)\| \leq \frac{2K^2L}{\sigma^2}|t_1 - t_2|.$$

Construct the Lyapunov function as follow

$$V(t,x) = x^T R(t) x,$$

thus $V(t,x)$ satisfies the Lipschitz condition by the reason of function $R(t)$.

Corollary 2.3. *There exist constants $b \geq a > 0$ such that*

$$a\|x\|^2 \leq V(t,x) \leq b\|x\|^2, t \geq 0, x \in R^n.$$

Proof. Suppose all the eigenvalues, eigenvectors of $A(t)$ are $\lambda_i(t), \xi_i(t)(i = 1,2,\cdots,n)$ respectively. then

$$\bar{\xi}_i^T(t)R(t)\xi_i(t) = \int_0^{+\infty} \bar{\xi}_i^T(t)\xi_i(t)e^{2Re\lambda_i(t)s}ds = -\frac{\bar{\xi}_i^T(t)\xi_i(t)}{2Re\lambda_i(t)},$$

that is to say $\dfrac{-1}{2Re\lambda_i(t)}$ is a eigenvalues of $R(t)$, then we let

$$0 < a^{\frac{1}{n}} \leq \min_{i=1,2,\cdots,n}\{\frac{-1}{2Re\lambda_i(t)}\}, b^{\frac{1}{n}} \geq \max_{i=1,2,\cdots,n}\{\frac{-1}{2Re\lambda_i(t)}\},$$

the Corollary can be proved. □

Proof of Theorem 2.1 First, from Corollary 2.2, for any $\delta \in (0,\tilde{\delta}]$, there must exist a $t_a > 0$ such that $\|x(t)\| \leq \delta, \quad \forall\, t \in [0,t_a]$, otherwise, assume $\|x(t_a)\| = \delta$, for $x(t+h) = x(t) + h\dot{x}(t) + o(h)$, consider

$$\begin{aligned} &V(t,x(t+h)) - V(t,x(t)) \\ &\quad = x^T(t)(R(t+h) - R(t))x(t) + hx^T(t)R(t+h)\dot{x}(t) + h\dot{x}^T(t)R(t+h)x(t) + o(h), \end{aligned}$$

owing to $\|R(t+h) - R(t)\| \leq \frac{2K^2}{\sigma^2}\|A(t+h) - A(t)\|$, so we can get

$$\begin{aligned} &DV(t,x(t)) \\ &\quad \leq -\|x(t)\|^2 + \frac{2K^2}{\sigma^2}\lim_{h\to 0}\sup_{h>0}\|\frac{A(t+h) - A(t)}{h}| \cdot \|x(t)\|^2 + 2\|R(t)\| \cdot \|C(t)\| \cdot \|x(t)\|^2 \\ &\quad \leq -(1 - \frac{2K^2}{\sigma^2}\varepsilon - \frac{K^2}{\sigma}\varepsilon)\|x(t)\|^2. \end{aligned}$$

By Corollary 2.3, we know

$$V(t_a,x(t_a)) \geq a\|x(t_a)\|^2 = a\delta^2, \quad V(0,x_0) \leq \frac{K^2}{\sigma^2}\|x_0\|^2 < b\delta_1^2 = \frac{a\delta^2}{2}.$$

On the other hand, there exists a $t_b \in (0, t_a)$ such that

$$V(t_b, x(t_b)) = \frac{a\delta^2}{2}, \quad V(t, x(t)) > \frac{a\delta^2}{2}, \quad t \in (t_b, t_a),$$

thus we can get

$$\|x(t)\|^2 > \frac{a\delta^2}{2}, \quad t \in (t_b, t_a).$$

But

$$V(t_a, x(t_a)) - V(t_b, x(t_b)) \leq -(1 - \frac{2K^2}{\sigma^2}\varepsilon - \frac{K^2}{\sigma}\varepsilon) \int_{t_b}^{t_a} \|x(t)\|^2 dt,$$

then

$$V(t_a, x(t_a)) \leq \frac{a\delta^2}{2} - \frac{a\delta^2 \mu(t_a - t_b)}{2b} < \frac{a\delta^2}{2},$$

where $\mu = 1 - \frac{2K^2}{\sigma^2}\varepsilon - \frac{K^2}{\sigma}\varepsilon$, that is to say

$$a\delta^2 = a\|x(t_a)\|^2 \leq V(t_a, x(t_a)) < \frac{a\delta^2}{2},$$

Obviously, this is a contradiction, so there does not exist such t_a, thus we can obtain

$$DV(t, x(t)) \leq -\frac{\mu}{b} V(t, x(t)), \quad t \geq 0.$$

So the trivial solution of system (3) is exponentially stable. And also it is easy for us upper bound of ϵ is $\frac{\sigma^2}{2K^2(1+\sigma)}$. $\square$

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