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Theoretical Properties of Weighted Generalized Rayleigh and Related Distributions

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Abstract

In this paper, a new class of weighted generalization of the Rayleigh distribution is constructed and studied. The statistical properties of these distributions including the behavior of hazard or failure rate and reverse hazard functions, moments, moment generating function, mean, variance, coefficient of variation, coefficient of skewness, coefficient of kurtosis are obtained. Other important properties including entropy (Shannon, beta and generalized) which are measures of the uncertainty in these distributions are also presented.

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1 Introduction

The use and application of weighted distributions in research related to reliability, bio-medicine, ecology and several other areas are of tremendous practical importance in mathematics, probability and statistics. These distributions arise naturally as a result of observations generated from a stochastic process and recorded with some weight function. The concept of weighted distributions has been employed in wide variety applications in reliability and survival analysis, analysis of family data, meta-analysis, ecology, and forestry. Several authors have presented important results on length-biased distributions and on weighted distributions in general. Rao [11] unified the concept of weighted distributions. Vardi [14] derived the nonparametric maximum likelihood estimate (NPMLE) of a lifetime distribution in the presence of length bias and established convergence to a pinned Gaussian process with a simple covariance function under mild conditions. Rao [10] identified the various sampling situations that can be modeled by what he called weighted distributions, extending the idea of the methods of ascertainment upon estimation of frequencies by Fisher. Patil and Rao [7], [8] investigated the applications of the weighted distributions. Statistical applications of weighted distributions, especially to the analysis of data relating to human population and ecology can be found in Patil and Rao [7]. For additional and important results on weighted distributions, see Gupta and Keating [4], Oluyede [6], Patil and Ord [9], Zelen and Feinleib [15] and references therein.

Consider the following general class of weight functions given by

$$w(x) = x^k e^{tx} F^i(x) \overline{F}^j(x), \tag{1}$$

where $\bar{F}(x) = 1 - F(x)$ and $F(x)$ is the survival or reliability function and cumulative distribution function (cdf) of the random variable X. Now setting $t = 0$; $k = j = i = 0$; $t = i = j = 0$; $(k = t = 0, i \to i - 1, j = n - i)$; $k = t = i = 0$ and $k = t = j = 0$ in the weight function, one at a time, implies probability weighted moments, moment-generating functions, moments, order statistics, proportional hazards and proportional reversed hazards, respectively.

Let X be a non-negative random variable (rv) with its natural probability density function (pdf) $f(x; \theta)$, where the natural parameter is $\theta \in \Omega$ (Ω is the parameter space). Suppose a realization x of X under $f(x;\theta)$ enters the investigator's record with probability proportional to $w(x;\beta)$, so that the recording (weight) function $w(x;\beta)$ is a non-negative function with the parameter β representing the recording (sighting) mechanism. Clearly, the recorded x is not an observation on X, but on the rv X_w, having a pdf

$$f_w(x;\theta,\beta) = \frac{w(x,\beta)f(x;\theta)}{\omega}, \tag{2}$$

where ω is the normalizing factor obtained to make the total probability equal to unity by choosing $0 < \omega = E[w(X,\beta)] < \infty$. The random variable X_w is called the weighted version of X, and its distribution is related to that of X and is called the weighted distribution with weight function w.

The generalized Rayleigh distribution (GRD) is considered to be a very useful life distribution. It presents a flexible family in the varieties of shapes and is suitable for modeling data with different types of hazard rate function: increasing, decreasing and upside down bathtub shape (UBT). The main objective of this article is to explore the properties of weighted generalized Rayleigh distribution (WGRD).

This article is organized as follows. Section 2 contains the some basic definitions, utility notions and useful functions including the weighted generalization. The probability density function (pdf), cumulative distribution function (cdf), hazard function and reverse hazard function of the WGRD are derived. In section 3, moments and related measures are derived. Measures of uncertainty are presented in section 4, followed by concluding remarks.

2 Basic Utility Notions

In this section, some basic utility notions and results on the WGRD are presented. Suppose the distribution of a continuous random variable X has the parameter set $\theta^* = \{\theta_1, \theta_2, \cdots, \theta_n\}$. Let the pdf of the rv X be given by $f(x;\theta^*)$. The corresponding cdf is defined to be

$$F(x;\theta^*) = \int_{-\infty}^{x} f(t;\theta^*)\, dt. \tag{3}$$

The hazard function of X can be interpreted as the instantaneous failure rate or the conditional probability density of failure at time x, given that the unit has survived until time x, see Shaked and Shanthikumar [13]. The hazard function $h(x;\theta^*)$ is defined to be

$$h(x;\theta^*) = \lim_{\Delta x \to 0} \frac{P\left(x \leq X \leq x + \Delta x\right)}{\Delta x[1 - F(x;\theta^*)]} = \frac{-\bar{F}'(x;\theta^*)}{\bar{F}(x;\theta^*)} = \frac{f(x;\theta^*)}{1 - F(x;\theta^*)}, \tag{4}$$

where $\bar{F}(x;\theta^*)$ is the survival or reliability function. The concept of reverse hazard rate was introduced as the hazard rate in the negative direction and received minimal attention, if any, in the literature. Keilson and Sumita [5] demonstrated the importance of the reverse hazard rate and reverse hazard orderings. Shaked and Shanthikumar [13] presented results on reverse hazard rate. See Chandra and Roy [2], Block and Savits [1] for additional details. We present a formal definition of the reverse hazard function of a distribution function F. See Ross [12], Chandra and Roy [2] for additional details. The reverse Hazard function can be interpreted as an approximate probability of a failure in $[x, x + dx]$, given that the failure had occurred in $[0, x]$.

Definition 2.1. *Let (a, b), $-\infty \leq a < b < \infty$, be an interval of support for F. Then the reverse hazard function of X (or F) at $t > a$ is denoted by $\tau_F(t)$ and is defined as*

$$\tau(t;\theta^*) = \frac{d}{dt} \log F(t;\theta^*) = \frac{f(t;\theta^*)}{F(t;\theta^*)}. \tag{5}$$

Some useful functions that are employed in subsequent sections are given below. The gamma function is given by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{6}$$

The digamma function is defined by

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{7}$$

where

$$\Gamma'(x) = \int_0^\infty t^{x-1} (\log t) e^{-t}\, dt$$

is the first derivative of the gamma function. The second derivative of the gamma function is

$$\Gamma''(x) = \int_0^\infty t^{x-1} (\log t)^2 e^{-t}\, dt.$$

Definition 2.2. *The n^{th}-order derivative of gamma function is given by:*

$$\Gamma^{(n)}(s) = \int_0^\infty z^{s-1}(\log z)^n \exp(-z)\, dz. \tag{8}$$

This derivative will be used frequently in this paper. The lower incomplete gamma function and the upper incomplete gamma function are

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\, dt \quad \text{and} \quad \Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\, dt, \tag{9}$$

respectively.

2.1 Weighted Generalized Rayleigh Distribution

First consider the following two parameter probability density function (pdf) given by

$$f(x;\theta,k) = \frac{k}{\theta^{\frac{1}{k}}\Gamma(\frac{1}{k})} \exp\left(-\frac{x^k}{\theta}\right) \qquad \text{for } x \geq 0,\ k > 0 \text{ and } \theta > 0. \tag{10}$$

The corresponding moments are given by

$$E(X^m) = \int_0^\infty x^m \frac{k}{\theta^{\frac{1}{k}}\Gamma\left(\frac{1}{k}\right)} \exp\left(-\frac{x^k}{\theta}\right) dx. \tag{11}$$

Let $c = \dfrac{k}{\theta^{\frac{1}{k}}\,\Gamma(\frac{1}{k})}$ and $z = \frac{x^k}{\theta}$, then $x = \theta^{\frac{1}{k}} z^{\frac{1}{k}}$, $dx = \frac{1}{k}\theta^{\frac{1}{k}} z^{\frac{1}{k}-1}\, dz$, and

$$\begin{aligned}
E(X^m) &= c\int_0^\infty \left(\theta^{\frac{m}{k}} z^{\frac{m}{k}}\right) \exp(-z) \left(\tfrac{1}{k}\theta^{\frac{1}{k}} z^{\frac{1}{k}-1}\right) dz \\
&= \frac{c\,\theta^{\frac{m+1}{k}}}{k} \int_0^\infty z^{\frac{m+1}{k}-1} \exp(-z)\, dz \\
&= \frac{k}{\theta^{\frac{1}{k}}\,\Gamma(\frac{1}{k})} \frac{\theta^{\frac{m+1}{k}}\,\Gamma(\frac{m+1}{k})}{k} \\
&= \frac{\theta^{\frac{m}{k}}\,\Gamma(\frac{m+1}{k})}{\Gamma(\frac{1}{k})}.
\end{aligned}$$

Now consider the following weighted (length-biased) version of the *pdf* given above in equation (10):

$$f_l(x;\theta,k) = \frac{xf(x)}{E(X)} = \frac{kx}{\theta^{\frac{2}{k}}\Gamma(\frac{2}{k})} \exp\left(-\frac{x^k}{\theta}\right) \quad \text{for } x \geq 0,\ k > 0, \text{ and } \theta > 0. \tag{12}$$

With $k = 2$ we obtain the usual Rayleigh probability density function (*pdf*). The weighted *pdf* corresponding to $f_l(x)$ with weighted function $w(x;m) = x^m$ and for $x \geq 0$, $k > 0$, and $\theta > 0$ is given by

$$\begin{aligned} g_w(x;\theta,k,m) &= \frac{x^m f_l(x)}{E(X^m)} \\ &= \frac{kx^{m+1}}{\theta^{\frac{m+2}{k}}\Gamma(\frac{m+2}{k})} \exp\left(-\frac{x^k}{\theta}\right). \end{aligned}$$

Note that

$$\lim_{x\to 0} g_w(x;\theta,k,m) = 0, \quad \text{and} \quad \lim_{x\to\infty} g_w(x;\theta,k,m) = 0. \tag{13}$$

The cdf corresponding to the weighted generalized Rayleigh distribution (WGRD) $g_w(x;\theta,k,m)$ is given by:

$$G_W(x;\theta,k,m) = \int_0^x g_w(y;\theta,k,m)\, dy. \tag{14}$$

Let $c_0 = \dfrac{k}{\theta^{\frac{m+2}{k}}\Gamma(\frac{m+2}{k})}$ and $z = \frac{y^k}{\theta}$, then $y = \theta^{\frac{1}{k}} z^{\frac{1}{k}}$ and $dy = \frac{\theta^{\frac{1}{k}} z^{\frac{1}{k}-1}}{k}\, dz$. We have

$$\begin{aligned} G_W(x;\theta,k,m) &= c_0 \int_0^x \left(\theta^{\frac{m+1}{k}} z^{\frac{m+1}{k}}\right) \exp(-z) \left(\frac{1}{k}\theta^{\frac{1}{k}} z^{\frac{1}{k}-1}\right) dz \\ &= \frac{c_0\theta^{\frac{m+2}{k}}}{k} \int_0^x z^{\frac{m+2}{k}-1} \exp(-z)\, dz. \end{aligned} \tag{15}$$

Since the lower incomplete gamma function is defined as

$$\gamma(s,x) = \int_0^x t^{s-1} \exp(-t)\, dt, \tag{16}$$

we have:

$$\begin{aligned} G_W(x;\theta,k,m) &= \frac{c_0\theta^{\frac{m+2}{k}}}{k} \int_0^x z^{\frac{m+2}{k}-1} \exp(-z)\, dz \\ &= \frac{k}{\theta^{\frac{m+2}{k}}\Gamma(\frac{m+2}{k})} \frac{\theta^{\frac{m+2}{k}}}{k} \gamma(\tfrac{m+2}{k}, x) \\ &= \frac{\gamma(\frac{m+2}{k}, x)}{\Gamma(\frac{m+2}{k})}. \end{aligned} \tag{17}$$

The survival function is given by

$$\bar{G}_W(x;\theta,k,m) = 1 - G_W(x;\theta,k,m) = 1 - \frac{\gamma(\frac{m+2}{k},x)}{\Gamma(\frac{m+2}{k})}. \quad (18)$$

The corresponding hazard function is given by

$$\begin{aligned} h_{G_W}(x;\theta,k,m) &= \frac{g_w(x;\theta,k,m)}{\bar{G}_W(x;\theta,k,m)} \\ &= \left(\frac{kx^{m+1}\exp(-\frac{x^k}{\theta})}{\theta^{\frac{m+2}{k}}\Gamma(\frac{m+2}{k})}\right)\Big/\left(1 - \frac{\gamma(\frac{m+2}{k},x)}{\Gamma(\frac{m+2}{k})}\right) \\ &= \frac{kx^{m+1}\exp(-\frac{x^k}{\theta})}{\theta^{\frac{m+2}{k}}\left[\Gamma(\frac{m+2}{k}) - \gamma(\frac{m+2}{k},x)\right]}. \end{aligned} \quad (19)$$

The reverse hazard function $\tau_{G_W}(x;\theta,k,m)$ is given by

$$\begin{aligned} \tau_{G_W}(x;\theta,k,m) &= \frac{g_w(x;\theta,k,m)}{G_w(x;\theta,k,m)} \\ &= \frac{\frac{kx^{m+1}\exp(-\frac{x^k}{\theta})}{\theta^{\frac{m+2}{k}}\Gamma(\frac{m+2}{k})}}{\frac{\gamma(\frac{m+2}{k},\frac{x^k}{\theta})}{\Gamma(\frac{m+2}{k})}} \\ &= \frac{kx^{m+1}\exp(-\frac{x^k}{\theta})}{\theta^{\frac{m+2}{k}}\gamma(\frac{m+2}{k},\frac{x^k}{\theta})}. \end{aligned} \quad (20)$$

We study the behavior of the hazard function of the WGRD via the following lemma, due to Glaser [3].

Lemma 2.3. *Let $f(x)$ be a twice differentiable probability density function of a continuous random variable X. Define $\eta(x) = \frac{-f'(x)}{f(x)}$, where $f'(x)$ is the first derivative of $f(x)$ with respect to x. Furthermore, suppose the first derivative of $\eta(x)$ exist.*

1. If $\eta'(x) < 0$, for all $x > 0$, then the hazard function is monotonically decreasing (DHR).

2. If $\eta'(x) > 0$, for all $x > 0$, then the hazard function is monotonically increasing (IHR).

3. Suppose there exist x_0 such that $\eta'(x) > 0$, for all $0 < x < x_0$, $\eta'(x_0) = 0$ and $\eta'(x) < 0$ for all $x > x_0$. In addition, $\lim_{x\to 0} f(x) = 0$, then the hazard function is upside down bathtub shape (UBT).

4. Suppose there exist x_0 such that $\eta'(x) < 0$, for all $0 < x < x_0$, $\eta'(x_0) = 0$ and $\eta'(x) > 0$ for all $x > x_0$. In addition, $\lim_{x\to 0} f(x) = \infty$, then the hazard function is bathtub shape (BT).

Now consider the weighted distribution discussed above. We compute the quantity $\eta_{G_W}(x;\theta,k,m)(x) = \frac{-g'_w(x;\theta,k,m)}{g_w(x;\theta,k,m)}$, and apply Glaser [3] result. Note that

$$\begin{aligned}
\eta_{G_W}(x;\theta,k,m) &= -\frac{g'_W(x;\theta,k,m)}{g_W(x;\theta,k,m)} \\
&= -\frac{\Big[kx^{m+1}\exp(-\frac{x^k}{\theta})\Big]'}{cx^{m+1}\exp(-\frac{x^k}{\theta})} \\
&= \frac{kx^k-(m+1)\theta}{\theta x}.
\end{aligned}$$

The derivative $\eta'_{G_W}(x)$ is given by

$$\begin{aligned}
\eta'_{G_W}(x) &= \frac{d}{dx}\Big(-\frac{m+1}{x}+\frac{k}{\theta}x^{k-1}\Big) \\
&= \frac{m+1}{x^2}+\frac{k(k-1)}{\theta}x^{k-2}. \qquad (21)
\end{aligned}$$

- If $k \geq 1$, then $\eta'_{G_W}(x) > 0$; and the hazard function is monotonically increasing.
- If $0 < k < 1$, then $\eta'_{G_W}(x) = \frac{m+1}{x^2} - \frac{k(1-k)}{\theta}x^{k-2}$. There exists $x_0 = \sqrt[k]{\frac{\theta(m+1)}{k(1-k)}}$ such that $\eta'_{G_W}(x) > 0$, for all $0 < x < x_0$, $\eta'_{G_W}(x_0) = 0$ and $\eta'_{G_W}(x) < 0$ for all $x > x_0$. In addition, $\lim_{x\to 0} g_w(x) = 0$, consequently, the hazard function is upside down bathtub shape (UBT).

3 Moment Generating Function and Moments

Under the density function $f_l(x) = \frac{xf(x)}{E(x)} = \frac{kx}{\theta^{\frac{2}{k}}\Gamma(\frac{2}{k})}\exp(-\frac{x^k}{\theta})$, we have

$$\begin{aligned}
E(e^{tX^k}) &= \int_0^\infty e^{tx^k}\frac{kx}{\theta^{\frac{2}{k}}\Gamma(\frac{2}{k})}\exp(-\frac{x^k}{\theta})\,dx \\
&= c_l\int_0^\infty x\exp\Big[-x^k\Big(\frac{1}{\theta}-t\Big)\Big]\,dx, \qquad (22)
\end{aligned}$$

where $c_l = \frac{k}{\theta^{\frac{2}{k}}\,\Gamma(\frac{2}{k})}$. Let $z = x^k(\frac{1}{\theta} - t)$, where $\theta t < 1$, then we have $x = \frac{z^{\frac{1}{k}}}{(\frac{1}{\theta}-t)^{\frac{1}{k}}}$ and $dx = \frac{z^{\frac{1}{k}-1}}{k(\frac{1}{\theta}-t)^{\frac{1}{k}}}\,dz$, so that

$$\begin{aligned} E(e^{tX^k}) &= c_l \int_0^\infty \frac{z^{\frac{1}{k}}}{(\frac{1}{\theta}-t)^{\frac{1}{k}}}\, \exp(-z)\, \frac{z^{\frac{1}{k}-1}}{k(\frac{1}{\theta}-t)^{\frac{1}{k}}}\,dz \\ &= \frac{c_l}{k(\frac{1}{\theta}-t)^{\frac{2}{k}}} \int_0^\infty z^{\frac{2}{k}-1} \exp(-z)\,dz \qquad (23) \\ &= \frac{k}{\theta^{\frac{2}{k}}\,\Gamma(\frac{2}{k})}\, \frac{\Gamma(\frac{2}{k})}{k(\frac{1}{\theta}-t)^{\frac{2}{k}}} \\ &= \frac{1}{(1-\theta t)^{\frac{2}{k}}}. \qquad (24) \end{aligned}$$

We also present the raw moments, mean, variance, coefficients of variation, skewness and kurtosis for the WGRD. Now, under the weighted generalized Rayleigh *pdf* $g_w(x;\theta,k,m)$, the n^{th} raw moment is given by:

$$E(X^n) = \int_0^\infty c_0 x^n x^{m+1} \exp\left(-\frac{x^k}{\theta}\right)dx = c_0 \int_0^\infty x^{n+m+1} \exp\left(-\frac{x^k}{\theta}\right)dx, \qquad (25)$$

where $c_0 = \frac{k}{\theta^{\frac{m+2}{k}}\Gamma(\frac{m+2}{k})}$. Let $z = x^k$, then

$$\begin{aligned} E(X^n) &= c_0 \int_0^\infty z^{\frac{m+n+1}{k}} \exp\left(-\frac{z}{\theta}\right)\left(\frac{1}{k} z^{\frac{1}{k}-1}\right)dz \\ &= \frac{c_0}{k}\int_0^\infty z^{\frac{m+n+2}{k}-1}\exp(-\frac{z}{\theta})\,dz \\ &= \frac{k}{\theta^{\frac{m+2}{k}}\Gamma(\frac{m+2}{k})}\frac{1}{k}\left[\theta^{\frac{m+n+2}{k}}\Gamma(\frac{m+n+2}{k})\right] \\ &= \frac{\theta^{\frac{n}{k}}\,\Gamma(\frac{m+n+2}{k})}{\Gamma(\frac{m+2}{k})}. \qquad (26) \end{aligned}$$

Let $\Gamma_0 = \Gamma(\frac{m+2}{k})$ and $\Gamma_j = \Gamma(\frac{m+j+2}{k})$, then $E(X^n)$ can be rewritten as $\frac{\theta^{\frac{n}{k}}\Gamma_n}{\Gamma_0}$. The variance σ^2, coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) are given below. The variance is

$$\sigma^2 = \frac{\theta^{\frac{2}{k}}\Gamma(\frac{m+4}{k})}{\Gamma(\frac{m+2}{k})} - \frac{\theta^{\frac{2}{k}}\Gamma^2(\frac{m+3}{k})}{\Gamma^2(\frac{m+2}{k})}. \qquad (27)$$

The coefficient of variation (CV) is given by

$$\begin{aligned} CV &= \frac{\theta^{\frac{1}{k}}\left[\Gamma(\frac{m+2}{k})\Gamma(\frac{m+4}{k})-\Gamma^2(\frac{m+3}{k})\right]^{1/2}/\Gamma(\frac{m+2}{k})}{\theta^{\frac{1}{k}}\Gamma(\frac{m+3}{k})/\Gamma(\frac{m+2}{k})} \\ &= \frac{(\Gamma_0\Gamma_2-\Gamma_1^2)^{\frac{1}{2}}}{\Gamma_1}. \end{aligned} \tag{28}$$

Note that

$$\mu_3 = \frac{\theta^{\frac{3}{k}}\Gamma(\frac{m+5}{k})}{\Gamma(\frac{m+2}{k})} - 3\frac{\theta^{\frac{1}{k}}\Gamma(\frac{m+3}{k})}{\Gamma(\frac{m+2}{k})}\frac{\theta^{\frac{2}{k}}\Gamma(\frac{m+4}{k})}{\Gamma(\frac{m+2}{k})} + 2\frac{\theta^{\frac{3}{k}}\Gamma^3(\frac{m+3}{k})}{\Gamma^3(\frac{m+2}{k})},$$

and

$$\sigma^3 = \frac{\theta^{\frac{3}{k}}\left[\Gamma(\frac{m+2}{k})\Gamma(\frac{m+4}{k})-\Gamma^2(\frac{m+3}{k})\right]^{\frac{3}{2}}}{\Gamma^3(\frac{m+2}{k})}.$$

The coefficient of skewness (CS) is given by

$$\begin{aligned} CS &= \frac{\mu_3}{\sigma^3} \\ &= \frac{\Gamma_0^2\Gamma_3-3\Gamma_0\Gamma_1\Gamma_2+2\Gamma_1^3}{[\Gamma_0\Gamma_2-\Gamma_1^2]^{\frac{3}{2}}}. \end{aligned} \tag{29}$$

The coefficient of kurtosis (CK) is defined as $CK = \dfrac{\mu_4}{\sigma_4}$ which can be readily computed. In fact,

$$\begin{aligned} \mu_4 &= \frac{\theta^{\frac{4}{k}}\Gamma(\frac{m+6}{k})}{\Gamma(\frac{m+2}{k})} - 4\frac{\theta^{\frac{1}{k}}\Gamma(\frac{m+3}{k})}{\Gamma(\frac{m+2}{k})}\frac{\theta^{\frac{3}{k}}\Gamma(\frac{m+5}{k})}{\Gamma(\frac{m+2}{k})} + 6\frac{\theta^{\frac{2}{k}}\Gamma^2(\frac{m+3}{k})}{\Gamma(\frac{m+2}{k})}\frac{\theta^{\frac{2}{k}}\Gamma(\frac{m+3}{k})}{\Gamma(\frac{m+2}{k})} - 3\frac{\theta^{\frac{4}{k}}\Gamma^4(\frac{m+3}{k})}{\Gamma^4(\frac{m+2}{k})} \\ &= \theta^{\frac{4}{k}}\left(\frac{\Gamma_4}{\Gamma_0} - 4\frac{\Gamma_1\Gamma_3}{\Gamma_0^2} + 6\frac{\Gamma_1^2\Gamma_2}{\Gamma_0^3} - 3\frac{\Gamma_1^4}{\Gamma_0^4}\right). \end{aligned} \tag{30}$$

Also σ_4 is

$$\sigma_4 = \left(\sigma^2\right)^2 = \left(\frac{\theta^{\frac{2}{k}}\Gamma_2}{\Gamma_0} - \frac{\theta^{\frac{2}{k}}\Gamma_1^2}{\Gamma_0^2}\right)^2 = \frac{\theta^{\frac{4}{k}}}{\Gamma_0^4}\left(\Gamma_2\Gamma_0-\Gamma_1^2\right)^2, \tag{31}$$

so that the coefficient of kurtosis (CK) is given by

$$\begin{aligned} CK &= \frac{\theta^{\frac{4}{k}}\left(\frac{\Gamma_4}{\Gamma_0} - \frac{4\Gamma_3\Gamma_1}{\Gamma_0^2} + \frac{6\Gamma_1^2\Gamma_2}{\Gamma_0^3} - \frac{3\Gamma_1^4}{\Gamma_0^4}\right)}{\frac{\theta^{\frac{4}{k}}}{\Gamma_0^4}\left(\Gamma_2\Gamma_0-\Gamma_1^2\right)^2} \\ &= \frac{\Gamma_4\Gamma_0^3-4\Gamma_1\Gamma_3\Gamma_0^2+6\Gamma_1^2\Gamma_2\Gamma_0-3\Gamma_1^4}{\left(\Gamma_2\Gamma_0-\Gamma_1^2\right)^2}. \end{aligned} \tag{32}$$

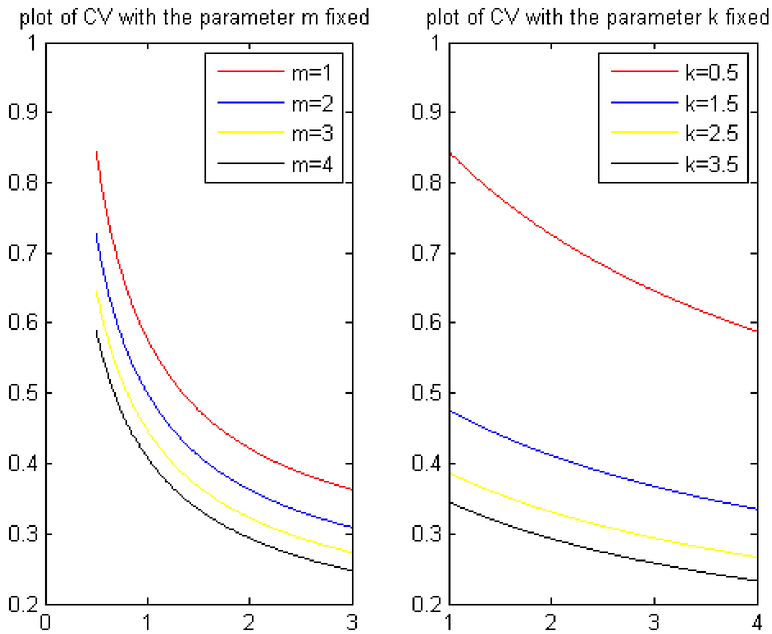

Figure 1: Plot of coefficient of variation with fixed m or k

The graphs of the coefficients of variation (CV), skewness (CS) and kurtosis (K) are given in Figure 1, Figure 2 and Figure 3 when the parameters θ, and k are fixed.

4 Some Measures of Uncertainty for WGRD

The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. In this section, we present some useful and important measures of uncertainty including Shannon entropy, $\beta-$entropy, and generalized entropy for the WGRD.

4.1 Shannon Entropy

Shannon entropy of a random variable X is a measure of the uncertainty

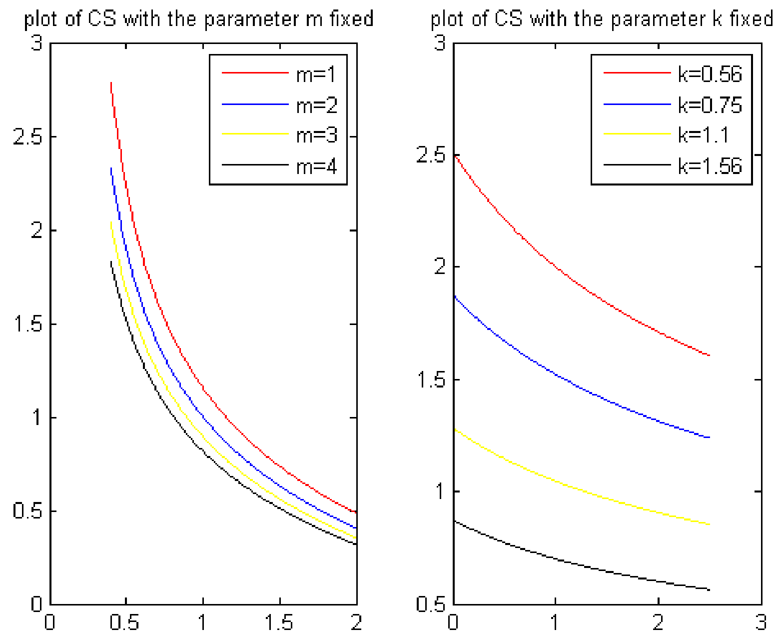

Figure 2: Plot of coefficient of skewness with fixed m or k

and is given by $E_F(-log(f(X))$, where $f(x)$ is the pdf of the random variable X. Under the WGRD, Shannon entropy is given as follows:

$$H(g_w) = E\left[-\log g_w(X)\right] = -\int_0^{\infty} \left[\log g_w(x)\right] g_w(x;\theta,k,m)\, dx. \tag{33}$$

Note that

$$\begin{aligned}\log g_w(x) &= \log \frac{k x^{m+1} \exp\left(-\frac{x^k}{\theta}\right)}{\theta^{\frac{m+2}{k}}\Gamma(\frac{m+2}{k})} \\ &= \log c_0 + (m+1)\log x - \frac{x^k}{\theta},\end{aligned} \tag{34}$$

where $c_0 = \frac{k}{\theta^{\frac{m+2}{k}}\Gamma(\frac{m+2}{k})}$, so that

$$\begin{aligned}E\left[-\log g_w(X)\right] &= \int_0^{\infty} \left[\log g_w(x)\right] g_w(x) dx \\ &= \int_0^{\infty} \left[\log c_0 + (m+1)\log x - \frac{x^k}{\theta}\right] c_0\, x^{m+1} \exp\left(-\frac{x^k}{\theta}\right) dx\end{aligned}$$

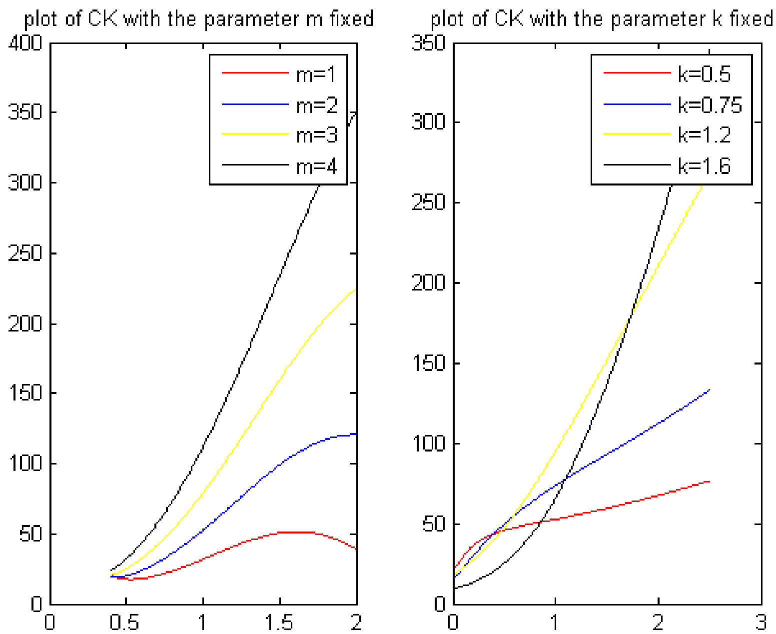

Figure 3: Plot of coefficient of kurtosis with fixed m or k

$$\begin{aligned}
&= -c_0(m+1)\int_0^\infty x^{m+1}\exp(-\frac{x^k}{\theta})\log x\,dx \\
&- c_0\log c_0\int_0^\infty x^{m+1}\exp(-\frac{x^k}{\theta})\,dx \\
&+ \frac{c_0}{\theta}\int_0^\infty x^{k+m+1}\exp(-\frac{x^k}{\theta})\log x\,dx.
\end{aligned}$$

Let $z=\frac{x^k}{\theta}$, then $x=\theta^{\frac{1}{k}}z^{\frac{1}{k}}$, $dx=\frac{1}{k}\theta^{\frac{1}{k}}z^{\frac{1}{k}-1}$, $x^{m+1}=\theta^{\frac{m+1}{k}}z^{\frac{m+1}{k}}$ and $\log x=\log(\theta^{\frac{1}{k}}z^{\frac{1}{k}})=\frac{1}{k}\log\theta+\frac{1}{k}\log z$, so that

$$\begin{aligned}
c_0(m+1)\int_0^\infty x^{m+1}e^{-\frac{x^k}{\theta}}\log x\,dx &= \frac{c_0(m+1)\theta^{\frac{m+2}{k}}}{k^2}\int_0^\infty z^{\frac{m+2}{k}-1}(\log(\theta z))\,e^{-z}\,dz \\
&= \frac{(m+1)\log\theta}{k}+\frac{(m+1)\Gamma_0^{(1)}}{k\Gamma_0}. \qquad (35)
\end{aligned}$$

Also, note that

$$\begin{aligned}
\int_0^\infty c_0\log c_0\,x^{m+1}\exp\left(-\frac{x^k}{\theta}\right)dx &= c_0\log c_0\int_0^\infty \theta^{\frac{m+1}{k}}z^{\frac{m+1}{k}}e^{-z}\left(\frac{1}{k}\theta^{\frac{1}{k}}z^{\frac{1}{k}-1}\right)dz \\
&= \frac{c_0\theta^{\frac{m+2}{k}}\log c_0}{k}\int_0^\infty z^{\frac{m+2}{k}-1}e^{-z}\,dz \\
&= \log k-(\tfrac{m+2}{k})\log\theta-\log\Gamma_0, \qquad (36)
\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \frac{c_0}{\theta}\, x^{k+m+1}\exp\left(-\frac{x^k}{\theta}\right)dx &= \frac{c_0}{\theta}\int_0^\infty x^{k+m+1}\exp(-\frac{x^k}{\theta})\,dx\\ &= \frac{c_0}{\theta}\int_0^\infty \theta^{\frac{m+k+1}{k}} z^{\frac{m+k+1}{k}}\exp(-z)\left(\frac{1}{k}\theta^{\frac{1}{k}} z^{\frac{1}{k}-1}\right)dz\\ &= \frac{c_0\theta^{\frac{m+k+2}{k}}}{k\theta}\int_0^\infty z^{\frac{m+2}{k}}\exp(-z)\,dz\\ &= \frac{k}{\theta^{\frac{m+2}{k}}\Gamma\left(\frac{m+2}{k}\right)}\frac{\theta^{\frac{m+2}{k}}}{k}\Gamma(\tfrac{m+k+2}{k})\\ &= \frac{\Gamma_k}{\Gamma_0}. \qquad (37)\end{aligned}$$

Therefore, Shannon Entropy $H(g_w)$ is given by

$$\begin{aligned}E\left[-\log g_w(X)\right] &= -\int_0^\infty [\log g_w(x)]\,g_w(x)\,dx\\ &= -\frac{(m+1)\log\theta}{k} - \frac{(m+1)\Gamma_0^{(1)}}{k\Gamma_0} - \log k\\ &+ \frac{(m+2)\log\theta}{k} + \log\Gamma_0 + \frac{\Gamma_k}{\Gamma_0}\\ &= \frac{\log\theta}{k} - \log k + \log\Gamma_0 - \frac{(m+1)\Gamma_0^{(1)}}{k\,\Gamma_0} + \frac{\Gamma_k}{\Gamma_0}. \qquad (38)\end{aligned}$$

4.2 β-Entropy

In this section, we present β-entropy for the WGRD, which is given by

$$\begin{aligned}H_\beta(g_w) &= \frac{1}{\beta-1}\left[1-\int_0^\infty g_w^\beta(x)\,dx\right]\\ &= \frac{1}{\beta-1}\left[1-\int_0^\infty c_0^\beta\, x^{\beta(m+1)}\exp\left(-\frac{\beta x^k}{\theta}\right)dx\right], \qquad (39)\end{aligned}$$

where $c_0 = \frac{k}{\theta^{\frac{m+2}{k}}\Gamma(\frac{m+2}{k})}$. Let $z = \frac{\beta x^k}{\theta}$, then $x = (\frac{\theta}{\beta})^{\frac{1}{k}} z^{\frac{1}{k}}$, $dx = \frac{1}{k} z^{\frac{1}{k}-1}(\frac{\theta}{\beta})^{\frac{1}{k}}\,dz$

and $x^{\beta(m+1)} = (\frac{\theta}{\beta})^{\frac{\beta(m+1)}{k}} z^{\frac{\beta(m+1)}{k}}$. We have:

$$\begin{aligned}
\int_0^\infty g_w^\beta(x)dx &= c_0^\beta \int_0^\infty (\frac{\theta}{\beta})^{\frac{\beta(m+1)}{k}} z^{\frac{\beta(m+1)}{k}} \exp(-z) \left(\frac{1}{k} z^{\frac{1}{k}-1} (\frac{\theta}{\beta})^{\frac{1}{k}}\right) dz \\
&= \frac{c_0^\beta}{k} (\frac{\theta}{\beta})^{\frac{\beta(m+1)+1}{k}} \int_0^\infty z^{\frac{\beta(m+1)+1}{k}-1} \exp(-z)\, dz \\
&= \frac{c_0^\beta}{k} (\frac{\theta}{\beta})^{\frac{\beta(m+1)+1}{k}} \Gamma\left(\frac{\beta(m+1)+1}{k}\right) \\
&= \frac{k^{\beta-1}\, \theta^{\frac{1-\beta}{k}}}{\Gamma^\beta(\frac{m+2}{k})\, \beta^{\frac{\beta(m+1)+1}{k}}} \, \Gamma\left(\frac{\beta(m+1)+1}{k}\right). \qquad (40)
\end{aligned}$$

Therefore, β-entropy for the WGRD is given by:

$$\begin{aligned}
H_\beta(g_w) &= \frac{1}{\beta-1}\left[1 - \int_0^\infty g_w^\beta(x)dx\right] \\
&= \frac{1}{\beta-1}\left[1 - \frac{k^{\beta-1}\, \theta^{\frac{1-\beta}{k}}}{\Gamma^\beta(\frac{m+2}{k})\, \beta^{\frac{\beta(m+1)+1}{k}}} \, \Gamma\left(\frac{\beta(m+1)+1}{k}\right)\right], \qquad (41)
\end{aligned}$$

for $\beta \neq 1$.

4.3 Generalized Entropy

In this section, we present the generalized entropy, which is defined by

$$I_{G_W}(\alpha) = \frac{v_\alpha \mu^{-\alpha} - 1}{\alpha(\alpha-1)} \quad \alpha \neq 0, 1, \qquad (42)$$

where $v_\alpha = \int x^\alpha \, dF(x)$ and $F(x)$ is the cumulative distribution function (*cdf*) for the random variable X. Since

$$v_\alpha = \frac{\theta^{\frac{\alpha}{k}} \Gamma(\frac{m+\alpha+2}{k})}{\Gamma(\frac{m+2}{k})}, \qquad (43)$$

we have

$$\begin{aligned}
I_{G_W}(\alpha) &= \frac{1}{\alpha(\alpha-1)}\left[\frac{\theta^{\frac{\alpha}{k}} \Gamma(\frac{m+\alpha+2}{k})}{\Gamma(\frac{m+2}{k})} \, \frac{\theta^{-\frac{\alpha}{k}} \Gamma^{-\alpha}(\frac{m+3}{k})}{\Gamma^{-\alpha}(\frac{m+2}{k})} - 1\right] \\
&= \frac{1}{\alpha(\alpha-1)}\left[\frac{\Gamma(\frac{m+\alpha+2}{k})\Gamma^{-\alpha}(\frac{m+3}{k})}{\Gamma^{1-\alpha}(\frac{m+2}{k}) - 1}\right] \\
&= \frac{1}{\alpha(\alpha-1)}\left(\frac{\Gamma_\alpha \Gamma_1^{-\alpha}}{\Gamma_0^{1-\alpha}} - 1\right), \qquad (44)
\end{aligned}$$

for $\alpha \neq 0, 1$.

The mean logarithmic deviation (MLD) index is given by

$$\begin{aligned} I_{G_W}(0) &= \lim_{\alpha\to 0} I(\alpha) = \log\mu - v_0 \\ &= \log\frac{\theta^{\frac{1}{k}}\Gamma(\frac{m+3}{k})}{\Gamma(\frac{m+2}{k})} - 1 \\ &= \frac{1}{k}\log\theta + \log\Gamma\left(\frac{m+3}{k}\right) - \log\Gamma\left(\frac{m+2}{k}\right) - 1. \end{aligned} \tag{45}$$

The Theil index is given by

$$\begin{aligned} I_{G_W}(1) &= \lim_{\alpha\to 1} I(\alpha) \\ &= \frac{\mu}{v_1} - \log\mu \\ &= \frac{\theta^{\frac{1}{k}}\,\Gamma(\frac{m+3}{k})/\Gamma(\frac{m+2}{k})}{\theta^{\frac{1}{k}}\,\Gamma(\frac{m+3}{k})/\Gamma(\frac{m+2}{k})} - \log\mu \\ &= 1 - \frac{1}{k}\log\theta - \log\Gamma\left(\frac{m+3}{k}\right) + \log\Gamma\left(\frac{m+2}{k}\right). \end{aligned} \tag{46}$$

5 Concluding Remarks

Some properties of the generalized Rayleigh distribution and weighted generalized Rayleigh distributions are presented. The probability density function, cumulative distribution function, the failure rate function or hazard function and the reverse hazard function are presented. The behavior of the hazard function was also established, indicating that this generalization covers a wide range of possibilities as a lifetime model. Entropy measures, including Shannon entropy, $\beta-$entropy, and generalized entropy for the WGRD are also derived.

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