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Some iteration methods for common fixed points of a finite family of strictly pseudocontractive mappings

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Abstract

In this paper, we introduce new implicit and explicit iteration methods based on the Krasnoselskii-Mann iteration method and a contraction for finding a common fixed point of a finite family of strictly pseudocontractive self-mappings of a closed convex subset in real Hilbert spaces. An extension to the problem of convex optimization is showed.

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1 Introduction and preliminaries

Let C be a nonempty closed and convex subset of a real Hilbert space H with inner product $\langle ., . \rangle$ and norm $\|.\|$ and let T be a $\tilde{\gamma}$-strictly pseudocontactive

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and self-mapping of C, i.e.,

$$||Tx - Ty||^2 = ||x - y||^2 + \tilde{\gamma}||(I - T)x - (y - Ty)||^2$$

and $T : C \to C$, respectively, for all $x, y \in C$, where $\tilde{\gamma}$ is a fixed number in [0, 1). When $\tilde{\gamma} = 0, T$ is called nonexpansive. Denote the set of fixed points of T by $Fix(T)$, i.e., $Fix(T) := \{x \in C : x = Tx\}$, and the projection of $x \in H$ onto C by $PC(x)$. Note that in a Banach sapce E, T is a $\tilde{\gamma}$ -strictly pseudocontactive, if

$$\langle Tx - Ty, j(x - y)\rangle \leq ||x - y||^2 - \tilde{\gamma}||(I - T)x - (y - Ty)||^2,$$

where $j(x-y) \in J(x-y)$, and J is the normalized duality mapping of E, i.e., $J : E \to E^*$ and satisfies the condition $\langle x, J(x)\rangle = ||x||^2$ for all $x \in E$.

Let $\{T_i\}_{i=1}^N, 1 \leq N < \infty$, be a N $\tilde{\gamma}$-strictly pseudocontactive and self-mappings T_i of C. In this paper, we sssume that $\cap_{i=1}^N Fi(T_i) \neq \emptyset$ and intro-duce some new iteration methods for finding an element $p^* \in \cap_{i=1}^N Fix(T_i)$.

The class of strictly pseudocontractive mappings has been studied intensively by several authors (see for example [1]- [18] and references therein). Clearly this class of mappings includes the class of nonexpansive mappings.

In order to study the fixed point problem for a nonexpansive self-mapping T of a slosed convex subset C in a real Hilbert space, one recent way is to construct the iterative scheme, the so-called viscosity iteration method:

$$x_{k+1} = \lambda_k f(x_k) + (1 - \lambda_k)Tx_k, k \geq 0, \tag{1.1}$$

proposed firstly by Moudafi [19], where $\lambda_k \in (0, 1)$ and f is a contraction of C with constant $\tilde{\alpha} \in [0, 1)$. In particular, under the conditions:
$(L1)$ $\lim_{k\to\infty} \lambda_k = 0$;
$(L2)$ $\sum\limits_{k=0}^{\infty} \lambda_k = \infty$; and
$(L3)$ $\sum\limits_{k=0}^{\infty} |\lambda_{k+1} - \lambda_k| < \infty$; or
$(L4)$ $\lim_{k\to\infty} \dfrac{\lambda_{k+1}}{\lambda_k} = 1$,
in papers [20, 32], Xu proved that the sequence $\{xk\}$ generated by (1.1) converges strongly to a fixed point p^* of T, which is the unique solution of the following variational inequality:

$$\langle F(p^*), p^* - p\rangle \leq 0 \quad \forall p \in Fix(T), \tag{1.2}$$

where $F = I - f$. In 2006, related to a certain optimization problem, Marino and Xu [5, 21] introduced the following general iterative scheme for the fixed point problem of a nonexpansive mapping:

$$x_{k+1} = \lambda_k \omega f(x_k) + (1 - \lambda_k A)Tx_k, \quad k \geq 0, \tag{1.3}$$

where A is a strongly positive bounded linear operator, $\lambda_k \in (0, 1)$ and $\omega > 0$. They proved that the sequence $\{x_k\}$ generated by (1.3) converges strongly to the unique solution of the variational inequality (1.2) with $F = A - \omega f$. Further, algorithm (1.3) was extended in 2009 by Cho et al. [9] to the class of k-strictly pseudocontractive mappings as follows:

Theorem 1.1. *Let C be a closed convex subset of a real Hilbert space H such that $C \pm C \subset C$, and $T : C \to H$ be a $\gamma-$strictly pseudocontractive mapping with $Fix(T) \neq \emptyset$ for some $\gamma \in [0, 1)$. Let $A : C \to H$ be a strongly positive bounded linear operator with coefficient $\tilde{\gamma}$ and f is a contraction of C with the contractive constant $\tilde{\alpha} \in [0, 1)$ such that $0 < \omega < \tilde{\gamma}/\tilde{\alpha}$. Let $x_0 \in C$ and let $\{x_k\}$ be a sequence in C generated by*

$$x_{k+1} = \lambda_k \omega f(x_k) + (1 - \lambda_k A)P_C S x_k, k \geq 0, \tag{1.4}$$

where $S := \gamma I + (1 - \gamma)T$ and P_C is the metric projection of H onto C. Let $\{\lambda_k\}$ with $\lambda_k \in (0, 1)$ be satisfy conditions $(L1), L(2)$ and $(L3)$. Then $\{x_k\}$ defined by (1.4) converges strongly to a fixed point $p*$ of T, which is the unique solution of variational inequality (1.2) with $F = A - \omega f$.

In 2010, to remove condition $(L3)$ in [9] and [20] as well in [22], Jung [16] studied the following composite iterative scheme.

Theorem 1.2. *Let C be a closed convex subset of a real Hilbert space H such that $C \pm C \subset C$, and $T : C \to H$ be a $\gamma-$strictly pseudocontractive mapping with $Fix(T) \neq \emptyset$ for some $\gamma \in [0, 1)$. Let $A : C \to H$ be a strongly positive bounded linear operator with coefficient $\tilde{\gamma}$ and f is a contraction of C with the contractive constant $\tilde{\alpha} \in [0, 1)$ such that $0 < \omega < \tilde{\gamma}/\tilde{\alpha}$. Let $x_0 \in C$ and let $\{x_k\}$ be a sequence in C generated by*

$$\begin{aligned} y_k &= \beta_k x_k + (1 - \beta_k)P_C S x_k, \\ x_{k+1} &= \lambda_k \omega f(x_k) + (1 - \lambda_k A)y_k, k \geq 0, \end{aligned} \tag{1.5}$$

where $S := \gamma I + (1-\gamma)T$ *and* P_C *is the metric projection of* H *onto* C. *Let* $\{\lambda_k\}$ *with* $\lambda_k \in (0,1)$ *be satisfy conditions* $(L1)$ *and* $(L2)$ *and let* $\{\beta_k\}$ *be satisfy the condition* $0 < \liminf_{k\to\infty} \gamma_k \le \limsup_{k\to\infty} \gamma_k < 1$. *Then* $\{x_k\}$, *defined by (1.5), converges strongly to a fixed point* p^* *of* T, *which is the unique solution of variational inequality (1.2) with* $F = A - \omega f$.

In 2011, Jung [17] proposed an extension of (1.5) in the combination with Halpern [22] and Wittmann [23] methods. Note that the results in [9] and [16] are applicable to find $p^* \in \cap_{i=1}^N Fix(T_i)$ by putting $T = \sum\limits_{i=1}^N \omega_i T_i$ where $\omega_i > 0$ for all $i = 1, ..., N$ and $\sum\limits_{i=1}^N \omega_i = 1$ with $\gamma = \max\{\tilde{\gamma}_i : i = 1, ..., N\}$.

For finding an element $p \in \cap_{i=1}^N Fix(T_i)$, when each T_i is a nonexpansive self-mapping of C, Xu and Ori introduced in [24] the following implicit iteration process. For $x_0 \in C$ and $\{\beta_k\}_{k=1}^\infty \subset (0,1)$, the sequence $\{x_k\}$ is generated as follows:

$$\begin{aligned} & x_1 = \beta_1 x_0 + (1-\beta_1)T_1 x_1, \\ & x_2 = \beta_2 x_1 + (1-\beta_2)T_2 x_2, \\ & \vdots \\ & x_N = \beta_N x_{N-1} + (1-\beta_N)T_N x_N, \\ & x_{N+1} = \beta_{N+1} x_N + (1-\beta_{N+1})T_1 x_{N+1}, \\ & \vdots \end{aligned}$$

The compact expression of the method is the form

$$x_k = \beta_k x_{k-1} + (1-\beta_k)T_{[k]} x_k, \quad k \ge 1, \tag{1.6}$$

where $T_{[n]} = T_{n mod N}$, for integer $n \ge 1$, with the mod function taking values in the set $\{1, 2, ..., N\}$. They proved the following result.

Theorem 1.3. *Let* H *be a real Hilbert space and* C *be a nonempty closed convex subset of* H. *Let* $\{T_i\}_{i=1}^N$ *be* N *nonexpansive self-mappings of* C *such that* $\cap_{i=1}^N Fix(T_i) \neq \emptyset$, *where* $Fix(T_i) = \{x \in C : T_i x = x\}$. *Let* $x_0 \in C$ *and* $\{\beta_k\}_{k=1}^\infty$ *be a sequence in* $(0,1)$ *such that* $\lim_{k\to\infty} \beta_k = 0$. *Then, the sequence* $\{x_k\}$ *defined implicitly by (1.6) converges weakly to a common fixed point of the mappings* $\{T_i\}_{i=1}^N$.

Further, Zeng and Yao [25] gave a modification of (1.6) based on an L-Lipschitz continuous and $\eta-$strong monotone mapping F, i.e., F satisfies the following conditions:

$$||F(x) - F(y)|| \leq L||x - y||;$$
$$\langle F(x) - F(y), x - y\rangle \geq \eta||x - y||^2,$$

where L and η are fixed positive numbers. For an arbitrary initial point $x_0 \in H$, the sequence $\{x_k\}_{i=1}^{\infty}$ is generated as follows:

$$\begin{aligned}
&x_1 = \beta_1 x_0 + (1 - \beta_1)[T_1 x_1 - \lambda_1 \mu F(T_1 x_1)],\\
&x_2 = \beta_2 x_1 + (1 - \beta_2)[T_2 x_2 - \lambda_2 \mu F(T_2 x_2)],\\
&\vdots\\
&x_N = \beta_N x_{N-1} + (1 - \beta_N)[T_N x_N - \lambda_N \mu F(T_N x_N)],\\
&x_{N+1} = \beta_N + 1 x_N + (1 - \beta_{N+1})[T_1 x_{N+1} - \lambda_{N+1} \mu F(T_1 x_{N+1})],\\
&\vdots
\end{aligned}$$

The scheme is written in a compact form as

$$x_k = \beta_k x_{k-1} + (1 - \beta_k)[T_{[k]} x_k - \lambda_k F(T_{[k]} x_k)], \quad k \geq 1. \tag{1.7}$$

They proved the following result.

Theorem 1.4. *Let H be a real Hilbert space and $F : H \to H$ be a mapping such that for some constants $L, \eta > 0, F$ is L-Lipschitz continuous and η-strongly monotone. Let $\{T_i\}_{i=1}^{N}$ be N nonexpansive self-mappings of H such that $\cap_{k=1}^{N} Fix(T_i) \neq \emptyset$, Let $\mu \in (0, 2\eta/L^2), x_0 \in H, \{\mu_k\}_{k=1}^{\infty} \subset [0, 1)$ and $\{\beta_k\}_{k=1}^{i} nfty \subset (0, 1)$ satisfying the conditions: $\Sigma_{k=1}^{\infty} \lambda_k < \infty$ and $\alpha \leq \beta_k \leq \beta, \ \ k \geq 1$ for some $\alpha, \beta \in (0, 1)$. Then, the sequence $\{x_k\}$ defined by (1.7) converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^{N}$. Moreover, the convergence is strong if and only if $\liminf_{k\to\infty} d(x_k, \cap_{i=1}^{N} Fix(T_i)) = 0$ where $d(x, D) = \min_{y\in D} ||x - y||$ for a closed convex subset D in H.*

Next, Zhou and Chang [26] proved the strong convergence of (1.6) in a Banach space setting under the condition: each T_i is a semicompact nonexpansive self-mapping of C. Chidume and Shahzad in [27] proved the above result under the condition that just one of the mappings is semicompact.

Very recently, Buong and Anh in [28] introduced the strong convergence implicit algorithm:

$$\begin{aligned} &x_t = T^t x_t, T^t = T_0^t T^t N_1 ... T_1^t, \\ &T_0^t = I - \lambda_t \mu F, \\ &T_i^t = (1-\beta_t^i)I + \beta_t^i T_i, i = 1, \dots, N. \end{aligned} \tag{1.8}$$

They proved the following result.

Theorem 1.5. *Let H be a real Hilbert space and let $F : H \to H$ be a mapping such that for some constants $L, \eta > 0, F$ is L-Lipschitz continuous and η-strongly monotone. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-mappings of H such that $\cap_{i=1}^N Fix(T_i) \neq \emptyset$. Let $\mu(0, 2\eta/L^2)$ and let $t \in (0,1), \{\lambda_t\}, \{\beta_t^i\} \subset (0,1)$, such that $\lambda_t \to 0$, as $t \to 0$ and $0 < \liminf_{t\to 0} \beta_t^i \leq \limsup_{t \to 0} \beta_t^i < 1, \quad i = 1, ..., N$.*
Then, the net x_t defined by (1.8) converges strongly to the unique element $p*$ solving the following variational inequality:

$$p* \in \cap_{i=1}^N Fix(T_i) : \langle F(p*), p * -p\rangle \leq 0 \quad \forall p \cap_{i=1}^N Fix(T_i). \tag{1.9}$$

He et al. [29] have proposed the following explixit iteration method

$$x_{k+1} = (1-\alpha_k)x_k + \alpha_k T_N ... T_2 T_1 x_k \tag{1.10}$$

and proved the following result.

Theorem 1.6. *Let E be a uniformly convex Banach space with a Frechet differentiable norm, let C be a nonempty closed convex subset of E and let $\{T_i\}_{i=1}^N$ be N nonexpansive self-mappings of C such that $\cap_{i=1}^N Fix(T_i) \neq \emptyset$, where $Fix(T_i) = \{x \in C : T_i x = x\}$. Let $x_0 \in C$ and $\{\alpha_k\}_{k=1}^\infty$ be a sequence in $(0,1)$ such that the following conditions hold:*
(i) $\Sigma_{k=0}^\infty \alpha_k (1-\alpha_k) = \infty$ and
(ii) $\Sigma_{k=0}^\infty \alpha_k D_\rho(T_N ... T_2 T_1, T_i) < \infty$ for every $\rho > 0$ and $i = 1, ..., N$, where $D_\rho(T_N ..., T_2 T_1) = \sup\{||T_N ... T_2 T_1 x - T_i x|| : ||x|| \leq \rho\}$.

Then, the sequence $\{x_k\}$ generated by (1.10) converges weakly to a point $\cap_{i=1}^N Fix(T_i)$.

We want to note that since ak $\alpha_k \in (0,1)$, we have $0 < \alpha_k(1-\alpha_k) \leq \alpha_k$. So, from condition (i) in Theorem 1.6 it follows that $\Sigma_{k=0}^{\infty}\alpha_k = \infty$. On the other hand, we have, from condition (ii) in Theorem 1.6, that if there exists $i_0 \in 1,...,N$ such that $D\rho(T_N...T_2T_1,T_{i_0}) > 0$, then $\Sigma_{k=0}^{\infty}\alpha_k < \infty$,and hence, we obtain a contradiction. So, in order to have no contradiction, a question is posed: when $D_\rho(T_N....T_2T_1,T_i) = 0$ for all $i = 1,...,N$, for every $\rho > 0$.
In the case that T_i is a strictly pseudocontactive self-mapping of C, Osilike [2] obtained a weak convergence theorem for (1.6). Wang et al. [6] obtained strong convergence result for a modification of (1.6) to the case with the condition that one of the strictly pseudocontactive self-mappings $\{T_i\}$ is demicompact. They proved the following result.

Theorem 1.7. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^{N}$ be N strictly pseudocontractive self-mappings of C such that $\cap_{i=1}^{N}Fix(T_i) \neq \emptyset$, where $Fix(T_i) = \{x \in C : T_ix = x\}$. Let $x_0 \in C$ and $\{u_k\}$ be a bounded sequence in C, let $\{\alpha_k\}, \{\beta_k\}$ and $\{\gamma_k\}$ be three sequencesin $[0,1]$ satisfying the following conditions:*
(i) $\alpha_k + \beta_k + \gamma_k = 1$ for all $k \geq 1$;
(ii) $\beta_k \in (\rho_1;\rho_2)$ for some $\rho_1, \rho_2 \in (0;1)$;
(iii) $\Sigma_{k=1}^{\infty}\gamma_k < \infty$; and
(iv) there exists $i_0 \in \{1,2,...,N\}$ such that T_{i_0} is demicompact.
Then the implicit iterative sequence $\{x_k\}$ defined by

$$x_k = \alpha_k x_{k-1} + \beta_k T_{[k]} x_k + \gamma_k u_k$$

converges strongly to a common fixed point of the maps $\{T_i\}_{i=1}^{N}$.

Next, Li et al. [12] gave a modification the algorithm for a Banach space. Some modifications of Mann iteration method for finding a fixed point of a compact or demicompact, strictly pseudocontractive self-mapping T of a closed convex subset in a Banach space were studied in [1], [3], [4, 35] and [18].
Motivated by the above results, in this paper, without the condition that one of the mappings is semicompact, we develop (1.8) and (1.10) to the case that each T_i is a $\tilde{\gamma}_i$ -strictly pseudocontractive mapping and then introduce two new explicit iteration methods based on the Krasnoselskii-Mann iteration method and a contraction self-mapping f of C, for example, $f(x) = P_C(\tilde{\alpha}x)$ with

$\tilde{\alpha} \in [0, 1)$ for any $x \in C$ or $f(x) = u$, a fixed point $u \in C$, for all $x \in C$. The implicit algorithm is contructed as follows:

$$x_t = T^t x_t, T^t := T_0^t T_N^t ....T_1^t \text{ or} T^t := T_N^t ...T_1^t T_0^t, \tag{1.11}$$

for $t \in (0, 1)$, where T_i^t are defined by

$$\begin{aligned} T_0^t &= (1 - \lambda_t \mu)I + \lambda_t \mu f, \\ T_i^t &= (1 - \beta_t^i)I + \beta_t^i T_i, i = 1, ..., N, \end{aligned} \tag{1.12}$$

$\mu \in (0, 2(1 - \tilde{\alpha})/(1 + \tilde{\alpha})^2)$, the sequences of real numbers: $\{\lambda_t\} \in (0, 1)$ satisfying the following condition $t \to 0$ as $t \to 0$ and $\{\beta_t^i\} \subset (\alpha, \beta)$ for all $t \in (0, 1), 1 \leq i \leq N$, and some $\alpha, \beta \in (0, 1 - \gamma)$ with $\gamma = \max_{1 \leq i \leq N} \tilde{\gamma}_i$, The explicit iteration schemes are generated by:

$$\begin{aligned} &x_1 \in C, \text{any element}, \\ &x_{k+1} = (1 - \gamma_k)x_k + \gamma_k T^k x_k, \quad k \geq 1, \end{aligned} \tag{1.13}$$

where $T^k = T_N^k ...T_1^k T_0^k$ or $T^k = T_0^k T_N^k ...T_1^k$, each T_i^k is defined by (1.12) with $t = t_k$ and, for the sake of simplicity, $T_i^{t_k}, \lambda_{t_k}$ and $\beta_{t_k}^i$ are replaced by $T_i^k \lambda_k$ and β_k^i, respectively, the sequence of real numbers $\{\gamma_k\} \subset (a, b)$ for some $a, b \in (0, 1)$, and $\{\lambda_k\}, \{\beta_k^i\}$ satisfy the conditions

$$\lambda_k \to 0, \Sigma_{k \geq 1} \lambda_k = \infty, |\beta_{k+1}^i - \beta_k^i| \to 0, k \to \infty \ \ \forall i = 1, ..., N. \tag{1.14}$$

We need the following facts to prove strong convergence theorems for (1.11)-(1.12) and (1.13) with (1.14) in the next section, Section 2, and show an extension to the problem of convex optimization in Section 3.

Lemma 1.8. [10] *(i)* $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle$ *and for any fixed* $t \in [0, 1]$ *(ii)* $||(1 - t)x + ty||^2 = (1 - t)||x||^2 + t||y||^2 - (1 - t)t||x - y||^2, \ \ \forall x, y \in H.$

Lemma 1.9. [30] $||T^\lambda x - T^\lambda y|| \leq (1 - \lambda\tau)||x - y||$ *for a fixed number* $\mu \in (0, 2\eta/L^2), \lambda \in (0, 1)$, *where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)} \in (0, 1)$,

$$T^\lambda x = (I - \lambda\mu F)T_x,$$

F is L-Lipschitz continuous and η-strongly monotone, and T is a nonexpansive mapping of H.

Lemma 1.10. (Demiclosedness Principle) [8] *Assume that T is a strictly pseu-docontractive self-mapping of a closed convex subset K of a Hibert space H. If T has a fixed point, then $I - T$ is demiclosed; that is, whenever $\{x_k\}$ is a sequence in K weakly converging to some $x \in K$ and the sequence $\{(I-T)x_k\}$ strongly converges to some y, it follows that $(I-T)x = y$.*

Lemma 1.11. [31]. Let $\{x_k\}$ and $\{z_k\}$ be bounded sequences in a Banach space E such that $x_{k+1} = (1-\beta_k)x_k + \beta_k z_k$ for $k \geq 1$ where $\{\beta_k\}$ is in $[0,1]$ such that $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$. Assume that $\limsup_{k\to\infty} ||z_{k+1} - z_k|| - ||x_{k+1} - x_k|| \leq 0$. Then $\lim_{k\to\infty} ||x_k - z_k|| = 0$.

Lemma 1.12. [20] *Let a_k be a sequence of nonnegative real numbers satisfying the following conditions $a_{k+1} \leq (1 - b_k)a_k + b_k c_k$, where b_k and c_k are sequences of real numbers such that*
(i) $b_k \in [0,1]$ *and* $\Sigma_{k=1}^{\infty} b_k = \infty$
(ii) $\limsup_{k\to\infty} c_k \leq 0$.
Then, $\lim_{k\to\infty} a_k = 0$.

2 Main results

Now, we are in a position to prove the following results.

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $f : C \to C$ be a contraction with the contractive constant $\tilde{\alpha} \in [0,1)$. Let $\{T\}_{i=1}^{N}$ be N $\tilde{\gamma}$-strictly pseudocontractive self-mapping T_i of C such that $\cap_{n=1}^{N} Fix(T_i) \neq \emptyset$. Let $\mu \in (0, 2(1-\tilde{\alpha})/(1+\tilde{\alpha})^2)$ and let $\{\lambda_t\} \in (0,1), \{\beta_t^i\} \in (0, 1-\gamma)$ with $\gamma = \max_{1\leq i\leq N} \tilde{\gamma}_i$ for each $t \in (0,1)$ such that $\lambda_t \to 0$, as $t \to 0$ and $0 < \liminf_{t\to 0} \beta_t^i \leq \limsup_{t\to 0} \beta_t^i < 1 - \gamma, i = 1, ..., N$. Then, the net $\{x_t\}$ defined by (1.11)-(1.12) converges strongly to the unique element $p*$ in (1.9) with $F = I - f$.*

Proof. First, we consider the case that $T^t := T_0^t T_N^t....T_1^t$. Since T_i and f are the self-mappings of C, T^t is also a self-mapping of C. We note that the mapping T_0^t can be rewritten as $T_0^t = (I - \lambda_t \mu F)$ with $F = I - f$, which is $(1+\tilde{\alpha})$-Lipschitz continuous and $(1-\tilde{\alpha})$-strongly monotone. We also note that the mappings T_i^t, for $\{\beta_t^i\} \in (0, 1-\gamma) \subseteq (0, 1-\gamma_i)$, are nonexpansive. Indeed, by (ii) of Lemma 1.1 and the property of T_i, we have that

$$\begin{aligned}
||T_x^t - T_i^t y||^2 &= ||(1-\beta_t^i)(x-y) + \beta_t^i(T_i x - T_i y)||^2 \\
&= (1-\beta_t^i)||x-y||^2 + \beta_t^i||T_i x - T_i y||^2 \\
&- (1-\beta_t^i)\beta_t^i||x - y - (T_i^t x - T_i^t y)||^2 \\
&\leq (1-\beta_t^i)||x-y||^2 + \beta_t^i[||x-y||^2 \\
&+ \tilde{\gamma}_i||x - y - (T_i^t x - T_i^t y)||^2] \\
&- (1-\beta_t^i)\beta_t^i||x - y - (T_i^t x - T_i^t y)||^2 \\
&= ||x-y||^2 - (1-\tilde{\gamma}_i - \beta_t^i)\beta_t^i||x - y - (T_i^t x - T_i^t y)||^2 \\
&= ||x-y||^2,
\end{aligned}$$

because $\beta_t^i > 0$ and $1 - \tilde{\gamma}_i - \beta_t^i > 0$. So, T_i^t is nonexpansive for each $t \in (0, 1)$. By using Lemma 1.2 with $T = I$, we obtain that

$$\begin{aligned}
||T_t x - T_t y|| &\leq (1-\lambda_t\tau)||T_N^t...T_1^t x - T_N^t...T_1^t y|| \\
&\leq (1-\lambda_t\tau)||T_i^t...T_1^t x - T_i^t...T_1^t y|| \\
&\leq (1-\lambda_t\tau)||T_i^t x - T_1^t y|| \leq (1-\lambda_t\tau)||x-y|| \forall x, y \in C
\end{aligned}$$

So, T^t is a contraction of C. By Banach's Contraction Principle, there exists a unique element $x_t \in C$ such that $x_t = T^t x_t$ for all $t \in (0, 1)$.
Next, we show that $\{x_t\}$ is bounded. Indeed, for a fixed point $p \in \cap_{i=1}^N Fix(T_i)$, we have that $T_i^t p = p$ for $i = 1, ..., N$, and hence

$$\begin{aligned}
||x_t - p|| &= ||T^t x_t - p|| = ||T^t x_t - T_N^t...T_1^t p|| \\
&= ||(I - \lambda_t\mu F)T_N^t...T_1^t x_t - (I - \lambda_t\mu F)T_N^t...T_1^t p - \lambda_t\mu F(p)|| \\
&\leq (1-\lambda_t\tau)||T_N^t...T_1^t x_t - T_{N-1}^t...T_1^t p|| + \lambda_t\mu||F(p)|| \\
&\leq (1-\lambda_t\tau)||T_{N-1}^t...T_1^t x_t - T_{N-1}^t...T_1^t p|| + \lambda_t\mu||F(p)|| \\
&\leq (1-\lambda_t\tau)||T_i^t...T_1^t x_t - T_i^t...T_1^t p|| + \lambda_t\mu||F(p)|| \\
&\leq (1-\lambda_t\tau)||T_1^t...T_1^t p|| + \lambda_t\mu||F(p)|| \\
&\leq (1-\lambda||x_t - p|| + \lambda_t\mu||F(p)||
\end{aligned}$$

Therefore,

$$||x_t - p|| \leq \mu/\tau||F(p)||$$

that implies the boundedness of x_t. So, are the nets $F(y_t^N), y_t^i, i = 1, ..., N$, where we put

$$y_t^1 := T_1^t x_t, y_t^i := T_i^t y_t^{i-1}, i = 2, ..., N. \quad (2.15)$$

Then, from (1.11) with $T^t = T_0^t T_N^t ... T_1^t$, it follows that

$$x_t = (I - \lambda_t \mu F) y_i^N. \quad (2.16)$$

Moreover,

$$\begin{aligned} ||x_t - p||^2 &= ||(I - \lambda_t \mu F) y_t^N - p||^2 \\ &= ||y_t^N - p||^2 - 2\lambda_t \mu \langle F(y_t^N), y_t^N - p \rangle + \lambda_t^2 \mu^2 ||F(y_t^N)||^2 \\ &\leq ||y_t^{N-1} - p||^2 - 2\lambda_t \mu \langle F(y_t^N), y_t^N - p \rangle + \lambda_t^2 \mu^2 ||F(y_t^N)||^2 \\ &\cdots \\ &\leq ||y_t^1 - p||^2 - 2\lambda_t \mu \langle F(y_t^N), y_t^N - p \rangle + \lambda_t^2 \mu^2 ||F(y_t^N)||^2 \\ &\leq ||x_t - p||^2 - 2\lambda_t \mu \langle F(y_t^N), y_t^N - p \rangle + \lambda_t^2 \mu^2 ||F(y_t^N)||^2 \end{aligned}$$

Thus,

$$(1 - \tilde{\alpha})||y_t^N - p||^2 + \langle F(y), y_t^N - p \rangle \leq \lambda_t \mu/2||F(y_t^N)||^2 \quad (2.17)$$

Further, for the sake of simplicity, we put $y_t^0 = x_t$ and prove that

$$||y_t^{i-1} - T_i y_t^{i-1}|| \to 0$$

as $t \to 0$ for $i = 1, ..., N$.

Let $t_k \subset (0, 1)$ be an arbitrary sequence converging to zero as $k \to \infty$ and $x_k := x_{t_k}$. We have to prove that $||y_k^{i-1} - T_i y_k^{i-1}|| \to 0$, where y_k^i are defined by (2.1) with $t = t_k$ and $y_k^i = y_{t_k}^i$. Let x_l be a subsequence of x_k and x_{k_j} be a subsequence of x_l such that

$$\limsup ||y_k^{i-1} - T_i y_k^{i-1}|| = \lim ||y_l^{i-1} - T_i y_l^{i-1}||.$$

and

$$\limsup ||x_k - p|| = \lim ||x_{k_j} - p||.$$

From (2.2) and Lemma 1.1, it implies that

$$\begin{aligned}
||x_{k_j} - p||^2 &= ||(I - \lambda_{k_j}\mu F)y^N_{k_j} - p||^2 \\
&\le ||y^N_{k_j} - p|| - 2\lambda_{k_j}\mu\langle F(y^N_{k_j}), x_{k_j} - p\rangle \\
&= ||T^{k_j}_N y^{N-1}_{k_j} - T^{t_kj}_N p||^2 - 2\gamma_{k_j}\mu\langle F(y^N_{k_j}), x_{k_j} - p|| \\
&\le ||y^{N-1}_{k_j} - p|| - 2\lambda_{k_j}\mu\langle F(y^N_{k_j}), x_{k_j} - p\rangle \\
&\le .... \le ||y^1_{k_j} - p|| - 2\lambda_{k_j}\mu\langle F(y^N_{k_j}), x_{k_j} - p\rangle \\
&= ||T^{t_kj}_1 x_{k_j} - T^{t_kj}_1 p||^2 - 2\gamma_{k_j}\mu\langle F(y^N_{k_j}), x_{k_j} - p\rangle \\
&\le ||x_{k_j} - p||^2 - 2\lambda_{k_j}\mu\langle F(y^N_{k_j}), x_{k_j} - p\rangle
\end{aligned}$$

Hence,

$$\lim ||x_k - p|| = \lim ||y^i_{k_j} - p||, i = 1, ..., N. \tag{2.18}$$

By Lemma 1.1 and that T^t_j t are nonexpansive for $l = i-1, i-2, ..., 1$,

$$\begin{aligned}
||y^i_{k_j} - p||^2 &= (1 - \beta^i_{k_j})||y^i_{k_j} - p||^2 + \beta^i_{k_j}||T_i y^{i-1}_{k_j} - p||^2 \\
&- \beta^i_{k_j}(1 - \beta^i_{k_j})||y^{i-1}_{k_j} - T_i y^{i-1}_{k_j}||^2 \\
&\le (1 - \beta^i_{k_j})||y^{i-1}_{k_j} - p||^2 + \beta^i_{k_j}||y^{i-1}_{k_j} - p||^2 \\
&- \beta^i_{k_j}(1 - \beta^i_{k_j}) - \tilde{\gamma}_i||y^{i-1}_{k_j} - T_i y^{i-1}_{k_j}||^2 \\
&= ||y^i_{k_j} - p||^2 - \beta^i_{k_j}(1 - \beta^i_{k_j}) - \tilde{\gamma}_i||y^{i-1}_{k_j} - T_i y^{i-1}_{k_j}||^2 \\
&\le ... = ||y^0_{k_j} - p||^2 - \beta^i_{k_j}(1 - \beta^i_{k_j}) - \tilde{\gamma}_i||y^{i-1}_{k_j} - T_i y^{i-1}_{k_j}||^2 \\
&= ||x_{k_j} - p||^2 - \beta^i_{k_j}(1 - \beta^i_{k_j}) - \tilde{\gamma}_i||y^{i-1}_{k_j} - T_i y^{i-1}_{k_j}||^2, i = 1, ..., N
\end{aligned}$$

Without loss of generality, we can assume that $\alpha \le \beta^i_t \le \beta$ for some $\alpha, \beta \in (0, 1-\gamma)$. Then, we have

$$\alpha(1 - \gamma - \beta)||y^{i-1}_{k_j} - T_i y^{i-1}_{k_j}||^2 \le ||x_{kj} - p||^2 - ||y^i_{k_j} - p||^2.$$

This together with (2.4) implies that

$$\lim_{j\to\infty} ||y^{i-1}_{k_j} - T_i y^{i-1}_{k_j}||^2 = 0, i = 1, ..., N.$$

It means that $||y^{i-1}_t - T_i y^{i-1}_j||^2 \to 0$ as $t \to 0$ for $i = 1, ..., N$.
Next, we show that $||x_t - T_i x_t|| \to 0$ as $t \to 0$. Indeed, in the case that $i = 1$ we have $y^0_t = x_t$. So, $||x_t - T_1 x_t|| \to 0$ as $t \to 0$. Further, since

$$||y^1_t - T_1 x_t|| = ||y^1_t - T_1 y^0_t|| = (1 - \beta^1_t)||y^0_t - T_1 y^0_t||$$

and $||y_t^0 - T_1 y_t^0|| \to 0$, we have that $||y_t^1 - T_1 x_t|| \to 0$. Therefore, from

$$||x_t - y_t^1|| \leq ||x_t - T_1 x_t|| + ||T_1 x_t - y_t^1||$$

it follows that $||x_t - y_1^t|| \to 0$ as $t \to 0$. On the other hand, since

$$||y_t^2 - T_2 y_t^1|| = (1 - \beta_t^2)||y_t^1 - T_2 y_y^1|| \to 0$$

and

$$\begin{aligned} ||y_t^2 - x_t|| &\leq (1 - \beta_t^2)||y_t^1 - x_t|| + \beta_t^2||T_2 y_t^1 - x_t|| \\ &\leq (1 - \beta_t^2)||y_t^1 - x_t|| + \beta_t^2||T_2 y_t^1 - y_t|| + ||y_t^1 - x_t|| \end{aligned}$$

we obtain that $||y_t^2 - x_t|| \to 0$ as $t \to 0$. Now, from

$$\begin{aligned} ||x_t - T_2 x_t|| &\leq ||x_t - y_t^2|| + ||y_t^2 - T_2 y_t^1|| + ||T_2 y_t^1 - T_2 x_t|| \\ &\leq ||x_t - y_t^2|| + ||y_t^2 - T_2 y_t^1|| + L_2||y_t^1 - x_t||, \end{aligned}$$

where $L_2 = (1+\tilde{\gamma}_2)/(1-\tilde{\gamma}_2)$ (see [4, 35]), and $||x_t - y_t^2||, ||y_t^2 - T_2 y_t^1||, ||y_t^1 - x_t|| \to 0$, it follows that $||x_t - T_2 x_t|| \to 0$. Similarly, we obtain that $||x_t - T_i x_t|| \to 0$, for $i, ..., N$ and $||y_t^N - x_t|| \to 0$ as $t \to 0$.

Let $\{x_k\}$ be any sequence of $\{x_t\}$ converging weakly to $\tilde{p}$ as $k \to \infty$. Then, $||x_k - T_i x_k|| \to 0$, for $i = 1, ..., N$ and $\{y_k^N\}$ also converges weakly to $\tilde{p}$. By Lemma 1.3, we have that $\tilde{p} \in \cap_{i=1}^N Fix(T_i)$ and from (2.3), it follows that

$$\langle F(p), p - \tilde{p}\rangle \geq 0 \quad \forall p \in \cap_{i=1}^N Fix(T_i)$$

Since $p, \tilde{p} \in \cap_{i=1}^N Fix(T_i)$, a closed convex subset in H (see [4, 35]), by replacing p by $tp + (1-t)\tilde{p}$ in the last inequality, dividing by t and taking $t \to 0$ in the just obtained inequality, we obtain

$$\langle F(\tilde{p}), p - \tilde{p}\rangle \geq 0 \quad \forall p \in \cap_{i=1}^N Fix(T_i)$$

The uniqueness of $p*$ in (1.4) guarantees that $\tilde{p} = p*$. Again, replacing p in (2.3) by $p*$, we obtain the strong convergence for $\{x_t\}$. The case that $T^t = T_N^t ... T_1^t T_0^t$ is similarly proved. This completes the proof. □

Theorem 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $f : C \to C$ be a contraction with the contractive constant $\tilde{\alpha} \in [0, 1)$. Let $\{T\}_{i=1}^N$ be N $\tilde{\gamma}_i$-strictly pseudocontractive self-mapping T_i of C such that*

$\cap_{i=1}^{N} Fix(T_i) \neq \emptyset$. *Assume that* $\mu \in (0, 2(1-\tilde{\alpha})/(1+\tilde{\alpha})^2)$, *the sequences of real numbers* $\{\lambda_k\} \subset (a, b)$ *for some* $\alpha, \beta \in (0, 1)$, *and* $\{\lambda_k\} \in (0, 1), \{\beta_k^i\} \in (\alpha, \beta)$ *for some* $\alpha, \beta \in (0, 1)$ *satisfy conditions (1.14). Then, the sequences* $\{x_k\}$ *defined by (1.13) converge strongly to the unique element* $p*$ *in (1.9) with* $F = I - f$.

Proof. First, consider the case that $T^k = T_N^k ... T_1^k T_0^k$. Put

$$\begin{aligned} y_k^0 &= (1 - \lambda_k \mu) x_k + \lambda_k \mu f(x_k), \\ y_k^i &= (1 - \beta_k^i) y_k^{i-1} + \beta_k^i T_i y_k^{i-1}, i = 1, ..., N. \end{aligned}$$

Then, from (1.13) it follows that

$$x_{k+1} = (1 - \gamma_k) x_k + \gamma_k y_k^N.$$

We prove that $\{x_k\}$ is bounded. Since $T_i^k := (1-\beta_k^i)I + \beta_k^i T_i$ with $\beta_k^i \in (0, \gamma) \subseteq (0, 1 - \gamma_i)$, for $k \geq 1$, is a nonexpansive self-mapping of C and $T_i^k p = p$ for each $p \in \cap_{i=1}^{N} Fix(T_i)$, we have that

$$||y_k^i - p|| = ||T_i^k y_k^{i-1} - T_k p|| \leq ||y_k^{i-1} - p|| \quad \forall i = 1, ..., N, \tag{2.19}$$

and

$$\begin{aligned} ||y_k^1 - p|| &= ||T_1^k y_k^0 - T_1^k p|| \\ &\leq ||y_k^0 - p|| = ||(I - \lambda_k \mu F) x_k - p|| \\ &\leq (1 - \lambda_k \tau) ||x_k - p|| + \lambda_k \mu ||F(p)|| \end{aligned} \tag{2.20}$$

for $k \geq 1$. Further, we have also from (1.13), (2.5) and (2.6) that

$$\begin{aligned} ||x_{k+1} - p|| &\leq (1 - \gamma_k) ||x_k - p|| + \gamma_k ||y_k^N - p|| \\ &\leq (1 - \gamma_k) ||x_k - p|| + \gamma_k ||T_N^k y_k^{N-1} - T_N^k p|| \\ &\leq (1 - \gamma_k) ||x_k - p|| + \gamma_k ||y_k^{N-1} - p|| \\ &\leq (1 - \gamma_k) ||x_k - p|| + \gamma_k ||y_k^0 - p|| \\ &\leq (1 - \gamma_k) ||x_k - p|| + \gamma_k [(1 - \lambda_k \tau) ||x_k - p|| + \lambda_k \mu ||F(p)||] \\ &\leq (1 - \gamma_k \lambda_k \tau) ||x_k - p|| + \gamma_k \lambda_k \mu ||F(p)||. \end{aligned}$$

Put $Mp = \max\{||x_1 - p||, \mu ||F(p)||/\tau\}$. Then, $||x_1 - p|| \leq M_p$. So, if $||x_k - p|| = M_p$ then $||y_k^i - p|| \leq M_p$ for $i = 1, ..., N$, and hence

$$||x_{k+1} - p|| \leq (1 - \gamma_k \lambda_k \tau) M_p + \gamma_k \lambda_k \tau M_p = M_p.$$

Therefore, by induction, the sequence $\{x_k\}$ is bounded. So, are the sequences $\{F(x_k)\}, \{y_i^i\}$, and $\{T_i y_k^{i-1}\}, i = 1, 2, ..., N$. Without loss of generality, we assume that they are bounded by a positive constant M_1.
Next, we have, from (1.13) and the nonexpansive property of T_i^k for $k \geq 1$, that

$$\begin{aligned}
||y_{k+1}^N - y_{k+1}^N|| &= ||T_N^{k+1} y_{k+1}^{N-1} - T_N^k y_k^{N-1}|| \\
&\leq ||T_N^{k+1} y_{k+1}^{N-1} - T_N^k y_k^{N-1}|| + ||T_N^{k+1} y_k^{N-1} - T_N^k y_k^{N-1}|| \\
&\leq ||y_{N-1}^{k+1} y_k^{N-1}|| + 2M_1 |\beta_{k+1}^N - \beta_k^N| \\
&\leq ||y_{k+1}^i - y_k^i|| + 2M_1 \sum_{j=i+1}^{N} |\beta_{k+1}^j - \beta_k^j| \\
&\leq ||y_{k+1}^0 y_k^0|| + 2M_1 \sum_{i=1}^{N} |\beta_{k+1}^i - \beta_k^i| \\
&\leq ||x_{k+1} - x_k|| + M_1(\lambda_{k+1} + \lambda_k)\mu + 2M_1 \sum_{i=1}^{N} |\beta_{k+1}^i - \beta_k^i|
\end{aligned}$$

So, we obtain that

$$||y_{k+1}^N - y_k^N|| - ||x_{k+1} - x_k|| + M_1(\lambda_{k+1} + \lambda_k)\mu + 2M_1 \sum_{i=1}^{N} |\beta_{k+1}^i - \beta_k^i|$$

This together with $\lambda_k \to 0$ and $|\beta_{k+1}^i - \beta_k^i| \to 0$ for $i = 1, ..., N$, implies that

$$\limsup_{k\to\infty} ||y_{k+1}^N - y_k^N|| - ||x_{k+1} - x_k|| \leq 0.$$

By Lemma 1.4, $||x_k - y_k^N|| \to 0$ as $k \to \infty$. Therefore, $||x_{k+1} - x_k|| = (1 - \gamma_k)||x_k - y_k^N|| \to 0$.
Further, we shall prove that $||x_k - T_i x_k|| \to 0$ for $i = 1, ..., N$. As in the proof of Theorem 2.1, first, we prove that $||y_k^{i-1} - T_i y_k^{i-1}|| \to 0$. Let $\{x_l\}$ be a subsequence of $\{x_k\}$ and let $\{x_{k_j}\}$ be a subsequence of $\{x_l\}$ such that

$$\begin{aligned}
&\limsup_{k\to\infty} ||y_k^{i-1} - T_i y_k^{i-1}|| = \lim_{l\to\infty} ||y_l^{i-1} - T_i y_l^{i-1}||, \\
&\limsup_{l\to\infty} ||x_l - p|| = \lim_{j\to\infty} ||x_{k_j} - p||.
\end{aligned}$$

It is clear from (2.5) and (2.6) that

$$\begin{aligned}
||x_{k_j} - p|| &\leq ||x_{k_j} - y_{k_j}^N|| + ||y_{k_j}^N - p|| \\
&\leq ||x_{k_j} - y_{k_j}^N|| + ||y_{k_j}^i - p|| \\
&\leq ||x_{k_j} - y_{k_j}^N|| + ||x_{k_j} - p|| + \lambda_{k_j} M_1 \mu
\end{aligned}$$

Therefore,

$$\lim_{j\to\infty} ||x_{k_i} - p|| = \lim_{j\to\infty} ||y^i_{k_j} - p||, i = 1, ..., N. \tag{2.21}$$

Next, again by Lemma 1.1, we obtain that

$$\begin{aligned} ||y^i_{k_j} - p||^2 &= (1-\beta^i_{k_j})||y^{i-1}_{k_j} - p||^2 + \beta^i_{k_j}||T_i y^{i-1}_{k_j} - p||^2 \\ &- (1-\beta^i_{k_j})\beta^i_{k_j}||y^{i-1}_{k_j} - T_i y^{i-1}_{k_j}||^2 \\ &\le (1-\beta^i_{k_j})||y^{i-1}_{k_j} - p||^2 + \beta^i_{k_j}||y^{i-1}_{k_j} - p||^2 \\ &- (1-\tilde{\gamma}_i - \beta^i_{k_j})\beta^i_{k_j}||y^{i-1}_{k_j} - T_i y^{i-1}_{k_j}||^2 \\ &= ||y^i_{k_j} - p||^2 - (1-\tilde{\gamma}_i - \beta^i_{k_j})\beta^i_{k_j}||y^{i-1}_{k_j} - T_i y^{i-1}_{k_j}||^2 \\ &\le ||y^i_0 - p||^2 - (1-\tilde{\gamma}_i - \beta^i_{k_j})\beta^i_{k_j}||y^{i-1}_{k_j} - T_i y^{i-1}_{k_j}||^2 \\ &= ||x_{k_j} - p||^2 + M_1(\lambda_{k+1} + \lambda_k)\mu \\ &- (1-\tilde{\gamma}_i - \beta^i_{k_j})\beta^i_{k_j}||y^{i-1}_{k_j} - T_i y^{i-1}_{k_j}||^2 \end{aligned}$$

Hence,

$$\alpha(1-\gamma-\beta)||y^{i-1}_{k_j} - T_i y^{i-1}_{k_j}||^2 \le ||x_{k_j} - p||^2 - ||y^i_{k_j} - p||^2$$

which together with (2.7) implies that $||y^{i-1}_{k_j} - T_i y^{i-1}_{k_j}|| \to 0$ as $j \to \infty$. It means that $||y^{i-1}_k - T_i y^{i-1}_k|| \to 0$ for $i = 1, ..., N$. Now, we prove that $||x_k - T_i x_k|| \to 0$ as $k \to \infty$ for $i = 1, ..., N$.

In the case that $i = 1$ we have that $||y^0_k - x_k|| = \lambda_k\mu||F(x_k)|| \le \lambda_k\mu M_1 \to 0$ and hence, by $||T_i x - T_i y|| \le L_i||x - y||$ where $L_i = (1+\tilde{\gamma}_i)/(1-\tilde{\gamma}_i)$, we obtain that

$$\begin{aligned} ||x_k - T_1 x_k|| &\le ||x_k - y^0_k|| + ||y^0_k - T_1 y^0_k|| + ||T_1 y^0_k - T_1 x_k|| \\ &\le (1+L_1)||x_k - y^0_k|| + ||y^0_k - T_1 y^0_k|| \end{aligned}$$

which converges to zero, as $k \to \infty$, because $||x_k - y^0_k||$ and $||y^0_k - T_1 y^0_k||$ tend to zero. In the case that $i = 2$, from $||y^1_k - T_2 y^1_k|| \to 0$ and that $||y^1_k - x_k|| \le ||x_k - y^0_k|| + ||y^1_k - y^0_k|| = ||x_k - y^0_k|| + \beta^1_k||y^0_k - T_1 y^0_k|| \to 0$, it follows that $||x_k - T_2 x_k|| \to 0$. By the similar argument, we obtain that $||x_k - T_i x_k|| \to 0$ for $i = 1, ..., N$.

Next, we show that

$$\limsup_{k\to\infty} \langle F(p*), p * - x_k\rangle \le 0.$$

Indeed, let $\{x_{k_j}\}$ be a subsequence of $\{x_k\}$ that converges weakly to $\tilde{p}$ such that

$$\limsup_{k\to\infty} \langle F(p*), p * - x_k\rangle = \lim_{j\to\infty} \langle F(p*), p * - x_{k_j}\rangle$$

Then, $||x_{k_j} - T_i x_{k_j}|| \to 0$. So, by Lemma 1.3, $\tilde{p} \in C$. Therefore, from (1.4), it implies (2.8).
Finally, by the convexity of $||.||^2$, (2.5) and (2.6), we have that

$$\begin{aligned}
||x_{k+1} - p*|| &= ||(1-\gamma_k)x_k + \gamma_k y_k^N - p*||^2 \\
&\leq (1-\gamma_k)||x_k - p*|| + \gamma_k||y_k^N - p*||^2 \\
&\leq (1-\gamma_k)||x_k - p*|| + \gamma_k||y_k^i - p*||^2 \\
&\leq (1-\gamma_k)||x_k - p*|| + \gamma_k||y_k^0 - p*||^2 \\
&\leq (1-\gamma_k)||x_k - p*|| + \gamma_k||(I - \lambda_k\mu F)x_k - p*||^2 \\
&\leq (1-\gamma_k)||x_k - p*|| \\
&+ \gamma_k||(I - \lambda_k\mu F)x_k - (I - \lambda_k\mu F)p* - \lambda_k\mu F(p*)||^2 \\
&\leq (1-\gamma_k)||x_k - p*|| + \gamma_k(1-\lambda_k\tau)||x_k - p*||^2 \\
&- 2\lambda_k\mu\langle F(p*), x_k - p* - \lambda_k\mu F(x_k)\rangle \\
&\leq (1-\gamma_k\lambda_k\mu)||x_k - p*||^2 \\
&\gamma_k\lambda_k\mu[\frac{2\mu}{\tau}\langle F(p*), x_k, p* - x_k\rangle + \lambda_k\frac{2\mu}{\tau}\langle||F(p*)||M_1]
\end{aligned}$$

Using Lemma 2.5 with $a_k = ||x_k - p*||, b_k = \gamma_k\lambda_k\tau$ and

$$c_k = \frac{2\mu}{\tau}\langle F(p*), x_k, p* - x_k\rangle + \lambda_k\frac{2\mu}{\tau}||F(p*)||M_1$$

with $\lambda_k \to 0$ and (2.8), we obtain that $||x_k - p*|| \to 0$.
Note that the strong convergence of algorithm (1.13), when $T_k = T_0^k T_N^k ... T_1^k$ is similarly proved as that for (1.11)-(1.12) and (1.13) with $T_k = T_N^k T_1^k ... T_0^k$ by putting $y_k^0 = x_k$ and $y_k^i = T_i^k y_k^{i-1}$. Then, $x_{k+1} = (1-\gamma_k)x_k + \gamma_k T_0^k y_k^N$. This completes the proof. □

3 Extension

Let $T_i : H \to H, i = 1, ..., N$, be N $\tilde{\gamma}_i$ -strictly pseudocontractive mappings such that $\cap_{i=1}^N Fix(T_i) \neq \emptyset$ and let φ be a Frechet differentiable convex function on H with the L-Lipschitz continuous and η-strong monotone derivative $F = \varphi'$. The optimization problem is formulated as follows: find an element $p* \in \cap_{i=1}^N Fix(T_i)$ such that

$$\varphi(p*) = \min_{p \in \cap_{i=1}^N Fix(T_i)} \varphi(p). \tag{3.22}$$

Problem (3.1) was posed and studied firstly in [33] by Deutsch and Yamada, when each mapping T_i is nonexpansive. It is well-known that (3.1) is equivalent to variational ineqaulity (1.9). In the case that each T_i is nonexpansive, Yamada [30] proposed the following iterative algorithm

$$u_{k+1} = T_{[k+1]}u_k - \lambda_{k+1}\mu F(T_{[k+1]}u_k), \tag{3.23}$$

where $\mu \in (0, 2\eta/L_2)$ and $\{\lambda_k\} \subset (0, 1)$, and proved that under conditions (L1), (L2) and (L5): $\sum |\lambda_k - \lambda_{k+N}| < \infty$, the sequence $\{u_k\}$ in (3.2) converges strongly to $p*$ in (1.9). Further, Xu and Kim in [34], by replacing condition (L5) by (L6): $\lim(\lambda_k - \lambda_{k+N})/\lambda_{k+N} = 0$, proved the following result.

Theorem 3.1. *Let H be a real Hilbert space and $F : H \to H$ be a mapping such that for some constants $L, \eta > 0, F$ is L-Lipschitz continuous and η-strongly monotone. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-mappings of H such that $\cap_{i=1}^N Fix(T_i) \neq \emptyset$, $\mu \in (0, 2\eta/L^2)$ and let conditions (L1), (L2), and (L6) be satisfied. Assume in addition that*

$$\begin{aligned} \cap_{i=1}^N Fix(T_i) &= Fix(T_1T_2...T_N) \\ &= Fix(T_NT_1T_2...T_{N-1}) \\ &= .... = Fix(T_2T_3...T_NT_1). \end{aligned} \tag{3.24}$$

Then, the sequence $\{u_k\}$ defined by (3.2) converges strongly to the unique element $p*$ in (1.9).

It is not hard to see that (L5) implies (L6), if $\lim \lambda_k/\lambda_{k+N}$ exists. However, in general, conditions (L5) and (L6) are not comparable: neither of them implies the other (see [33] for details).
Recently, Zeng et al. [4, 35] proposed the the following iterative scheme:

$$u_{k+1} = T_{[k+1]}u_k - \lambda_{k+1}\eta_{k+1}F(T_{[k+1]}u_k) \tag{3.25}$$

with the variable parameter μ_k and proved the following result.

Theorem 3.2. *Let H be a real Hilbert space and $F : H \to H$ be a mapping such that for some constants $L, \eta > 0, F$ is L-Lipschitz continuous and η-strongly monotone. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps of H such that $\cap_{i=1}^N Fix(T_i) \neq \emptyset$ and let $\mu_k \in (0, 2\eta/L^2)$. Assume conditions (L1), (L2) and hold:*
$(i) \sum \lambda_k = \infty$ *where* $\{\lambda_k\} \subset (0,1)$;
$(ii) |\mu_k - \eta/L^2| \leq \sqrt{\eta^2 - cL^2}/L^2$, *for some* $c \in (0, \eta^2/L^2)$;
$(iii) \lim(\mu_{k+N} - (\lambda_k/\lambda_{k+N})\mu_k) = 0$;
Assume in addition that (3.3) holds. If

$$\limsup_{k\to\infty} \langle T_{[k+N]}...T_{[k+1]}u_k - u_{k+N}, T_{[k+N]}...T_{[k+1]}u_k - u_k\rangle \leq 0, \tag{3.26}$$

then, the sequence $\{u_k\}$ defined by (3.4) converges strongly to the unique element $p*$ in (1.9).

They also showed that conditions (L1), (L2) and (L4) are sufficient for u_k to be bounded and

$$\lim_{k\to\infty} ||u_k T_{[k+1]}...T_{[k+1]}u_k|| = 0,$$

So, (3.5) is satisfied.
Meantimes, Liou et al. [36], following [37], defined, for each k, mappings

$$\begin{aligned}
&U_{k1} = \alpha_{k_1}T_1 + (1-\alpha_{k1})I,\\
&U_{k2} = \alpha_{k2}T_2U_{k1} + (1-\alpha_{k1})I,\\
&\vdots\\
&U_{k,N-1} = \alpha_{k,N-1}T_{N-1}U_{k,N-2} + (1-\alpha_{k,N-1})I,\\
&W_k := U_{kN} = \alpha_{kN}T_NU_{k,N-1} + (1-\alpha_{kN})I,
\end{aligned}$$

and proved the following result.

Theorem 3.3. *Let H be a real Hilbert space and $F : H \to H$ be a mapping such that for some constants $L, \eta > 0, F$ is L-Lipschitz continuous and η-strongly monotone. Let $\{T_i\}_{i-1}^N$ be N nonexpansive self-mappings of H such that $\cap_{i=1}^N Fix(T_i) \neq \emptyset$, $\mu \in (0, 2\eta/L^2)$ and let conditions (L1) and (L2) be satisfied. Assume that the sequences $\{\alpha_k i\}_{i-1}^N$ satisfy $\lim_{k\to\infty}(\alpha_k i - \alpha_{k,i-1}) = 0$ for all $i = 1, 2, ..., N$. Then, the sequence $\{x_k\}$ defined by*

$$x_{k+1} = \beta x_k + (1-\beta)[W_kx_k - \lambda_k\mu F(W_kx_k)], k \geq 0,$$

for an arbitrary initial point $x_0 \in H$, converges strongly to $p*$ in (1.9).

When $Fx = Ax - u$, where A is a self-adjoint bounded linear mapping such that A is strongly positive, i.e.,

$$\langle Ax, x\rangle \geq \eta ||x||^2, \forall x \in H$$

and u is some fixed element in H, Xu introduced in [20, 32] the following iteration process:

$$u_{k+1} = (I - \lambda_{k+1}A)T_{[k+1]}u_k + \lambda_{k+1}u, \tag{3.27}$$

and proved the following result.

Theorem 3.4. *Let Conditions (L1), (L2) and (L3) or (L4) be satisfied. Assume in addition that (3.3) holds. Then the sequence $\{u_k\}$ generated by algorithm (3.6) converges strongly to the unique solution of (1.9) with $Fx = Ax - u$.*

Clearly, from the proof of Theorem 3.2, we obtain the following result.

Theorem 3.5. *Let H be a real Hilbert space and let $\{T\}_{i-1}^N$ be N $\tilde{\gamma}_i$ -strictly pseudocontractive self-mapping T i of H such that $\cap_{i-1}^N Fix(T_i) \neq \infty$. Let $F : H \to H$ be a mapping such that for some constants $L, \eta > 0, F$ is L-Lipschitz continuous and η-strongly monotone. Assume that $\mu \in (0, 2\eta/L^2)$, the se-quences of real numbers $\{\gamma_k\} \subset (a, b)$ for some $a, b \in (0, 1)$, and $\{\lambda_k\} \in (0, 1), \{\beta_k^i\} \subset (\alpha, \beta)$ for som $(\alpha, \beta(0, 1)$ satisfy the conditions (1.14). Then, the sequences $\{x_k\}$ generated by*

$$x_1 \in H, \text{ any element,}$$

$$x_{k+1} = (1 - \gamma_k)x_k + \gamma_k T_k x_k, k \geq 1,$$

where $T^k = T_N^k...T_1^kT_0^k$ or $T^k = T_0^kT_N^k...T_i^k, T_i^t$, $i = 0, 1, ..., N$, are defined by (1.12) with f replaced by $I - F$, converge strongly to the unique element $p*$ in (1.9).

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