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Endomorphism Rings of Essentially Pseudo Injective Modules

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Abstract

In this paper some of the results on the endomorphism rings of essentially pseudo injective modules have been obtained. In particular, it is proved that for a uniform essentially pseudo injective module M, the socle of M is essential in M iff Jacobson radical of endomorphism ring of M is equal to the set of all homomorphisms from socle of M to M. It has been shown that the endomorphism ring of an essentially pseudo injective uniform module is local and the mono-endomorphism of an essentially pseudo injective uniform module is an automorphism. Finally, we found a characterization for a uniform module M to be essentially pseudo injective in terms of its injective hull.

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1 Introduction

Throughout this paper the basic ring R is supposed to be ring with unity and all modules are unitary left R-modules.

A submodule N of a module M is called essential in M and is denoted as $N \subseteq_e M$, if $N \cap L = 0$ where $L \subseteq M$ implies that $L = 0$. A nonzero module is called uniform if every nonzero submodule is essential in it.

A ring R is called *regular* (in the sense of Von-Neumann) if for each $r \in R$ there exists $x \in R$ such that $r = rxr$. The Jacobson radical of a module M, denoted by $J(M)$ is the intersection of all maximal submodules of M. An R-module M is called local if it has a unique maximal submodule which contains every proper submodules of M. If M is an R-module then socle of M denoted by $Soc(M)$ is defined as intersection of essential submodules of M. An R-module M is called π-injective if for all submodules U, V of M with $U \cap V = 0$, there exists $f \in S$ with $U \subset Kerf$ and $V \subset Ker(1-f)$. We call a module M, a duo module if every submodule of M is fully invariant. An R-module M is called extending module if every submodule of M is essential in a direct summand of M.

2 Preliminary Notes

Definition 1. *A module M is said to be essentially pseudo injective if for any module A, any essential monomorphism $g : A \to M$ and monomorphism $f : A \to M$ there exists $h \in End(M)$ such that $f = hog$.*

Definition 2. *A submodule T of a module M is said to be essentially pseudo stable if for any essential monomorphism $g : A \to M$ and monomorphism $f : A \to M$ with $Img + Imf \subseteq T$, there exists $h \in End(M)$ such that $f = hog$ then $h(T) \subseteq T$.*

3 Main Results

Proposition 3.1. *Let M be an essentially pseudo injective uniform module. Then every monomorphism in $End(M)$ is an automorphism.*

Proof. Let $g : M \to M$ be any monomorphism then $Img \neq 0$ and since M is uniform, g is an essential monomorphism. By essential pseudo injectivity of M, $I_M : M \to M$ can be extended to a homomorphism $h : M \to M$ such that $hog = I_M \Rightarrow h$ is onto.
Again h is one-one as $kerh = 0$, for if $kerh \neq 0$ then $Img \cap kerh \neq 0 \Rightarrow \exists$ $0 \neq x \in Img \cap kerh \Rightarrow x \in Img$ and $x \in kerh \Rightarrow g(y) = x$ for some $y \in M$ and $h(x) = 0$.
Now $h(x) = 0 \Rightarrow hog(y) = 0 \Rightarrow I_M(y) = 0 \Rightarrow y = 0$
$\Rightarrow x = 0$, a contradiction.
So $kerh = 0$. Hence h is an isomorphism and therefore $h^{-1} = g$ is an automorphism.

□

Proposition 3.2. *If S is the endomorphism ring of an essentially pseudo injective uniform module M then S is local.*

Proof. If $\alpha \in S$ then $ker\alpha \subseteq M$. Either $ker\alpha = 0$ or $ker\alpha \neq 0$.
Case(1) Let $ker\alpha = 0$ then α is a mono-endomorphism of M which by Proposition 1 implies that α is an isomorphism, hence α is invertible.
Case(2) If $ker\alpha \neq 0$. Consider the map $g = (I_M - \alpha) : M \to M$. Claim that $kerg = 0$. Let $kerg \neq 0$ then $ker\alpha \cap kerg \neq 0 \Rightarrow \exists\ 0 \neq x \in ker\alpha \cap kerg$, then $x \in ker\alpha$ and $x \in kerg \Rightarrow \alpha(x) = 0$ and $g(x) = 0 \Rightarrow (I_M - \alpha)(x) = 0$ $\Rightarrow I_M(x) = 0 \Rightarrow x = 0$, a contradiction.
So, $kerg = 0 \Rightarrow g$ is a mono-endomorphism of M $\Rightarrow g$ is an isomorphism $\Rightarrow g = (I_M - \alpha)$ is invertible.
Thus for any $\alpha \in S$ either α or $I_M - \alpha$ is mono-endomorphism of M.
So, S is local.

□

Proposition 3.3. *Let M be an essentially pseudo injective uniform module. Let S be the endomorphism ring of M and $J(S)$ be the Jacobson radical of S. Let $T = \{\alpha \in S | ker\alpha$ is essential in $M\}$ then*

(a) *$T = J(S)$*
(b) *$J(S) \subseteq Hom(Soc(M), M)$.*
(c) *$S/J(S)$ is Von-Neumann regular ring.*

Proof. **(a)** Let $\alpha \in T$. Consider the map $g = (1-\alpha) : M \to M$, then $kerg = 0$ follows from the proof of Proposition 2. So g is an isomorphism. Whence $(1-\alpha)$ is an isomorphism. As $(1-\alpha)$ is invertible $\Rightarrow \alpha \in J(S)$. So, $T \subseteq J(S)$. Conversly, let $\alpha \in J(S)$ and let $Ker\alpha \cap K = 0$ for some $K \subseteq M$. Let $\nu : K \to M$ be the inclusion map. There are two possibilities either $Im(\alpha o \nu) = 0$ or $Im(\alpha o \nu) \neq 0$.
Case 1. If $Im(\alpha o \nu) = 0$ then $\alpha o \nu(K) = 0 \Rightarrow \alpha(K) = 0 \Rightarrow K \subseteq Ker\alpha$ $\Rightarrow K = 0$. This shows that $Ker\alpha$ is an essential submodule of M and so $\alpha \in T$.
Case 2. If $Im(\alpha o \nu) \neq 0 \Rightarrow Im(\alpha o \nu)$ is an essential submodule of M. So by essential pseudo injectivity of M $\exists$ $\psi \in End(M)$ such that $\nu = \psi o \alpha o \nu$ i.e. $\nu(K) = \psi o \alpha o \nu(K) \Rightarrow (1 - \psi o \alpha)\nu(K) = 0 \Rightarrow \nu(K) = 0 \Rightarrow K = 0$. Thus $Ker\alpha$ is an essential submodule of $M \Rightarrow \alpha \in T$. Thus, $J(S) \subseteq T$.

(b) Let $\alpha \in J(S)$ then $Ker\alpha$ is essential in M and so $Soc(M) \subseteq Ker\alpha \subseteq M$ Now, $Ker\alpha \subseteq_e M$ and $Soc(M) \subseteq M \Rightarrow Soc(M) \cap Ker\alpha \subseteq_e Soc(M)$, which by [3, Proposition 5.16(2)] implies $Ker\alpha \subseteq_e Soc(M) \Rightarrow Ker\alpha \subseteq Soc(M)$ $\Rightarrow \alpha \in Hom(Soc(M), M) \Rightarrow J(S) \subseteq Hom(Soc(M), M)$.

(c) Let $\alpha \in S$ be such that $\alpha \notin J(S)$ then $Ker\alpha \cap K = 0$ and $K \neq 0$ for some $K \subseteq M$. If $\nu : K \to M$ is the inclusion map, then $(\alpha o \nu)$ is an essential monomorphism from K to M, since $\alpha|_K$ is an essential monomorphism. By essential pseudo injectivity of M, there exists $\psi \in End(M)$ such that $\nu = \psi o \alpha o \nu$. We have $\alpha o \nu(K) = \alpha o \psi o \alpha o \nu(K)$ which implies that $(\alpha - \alpha o \psi o \alpha)\nu(K) = 0$ $\Rightarrow (\alpha - \alpha o \psi o \alpha)K = 0 \Rightarrow K \subseteq Ker(\alpha - \alpha o \psi o \alpha) \subseteq M$. As K is essential in M and K is a nonzero submodule of M, it implies that K is essential in $Ker(\alpha - \alpha o \psi o \alpha)$ and $Ker(\alpha - \alpha o \psi o \alpha)$ is essential in M. This shows that $(\alpha - \alpha o \psi o \alpha) \in J(S)$ and hence $S/J(S)$ is Von-Neumann regular ring.

□

Notation: If N is a direct summand of M it will be denoted by $N \subseteq^{\oplus} M$.

Proposition 3.4. *Let M be any pseudo injective module and $End(M)$ be the endomorphism ring of M. If $ker\alpha \subseteq^{\oplus} M$ for every $\alpha \in End(M)$ then $\alpha(M) \subseteq^{\oplus} M$.*

Proof. Follows from [5, Proposition 9]. □

Proposition 3.5. *Let M be any pseudo injective module and $End(M)$ denotes the endomorphism ring of M. Then if $ker\alpha \subseteq^{\oplus} M$ for every $\alpha \in End(M)$ then $End(M)$ is regular.*

Proof. Follows from [5, Proposition 11]. □

Corollary 5.1: Endomorphism ring of a completely reducible pseudo injective module is regular.

Proposition 3.6. *Let M be an essentially pseudo injective uniform module. If $S = End(M)$ then $Soc(M)$ is essential in M iff $J(S) = Hom(Soc(M), M)$.*

Proof. Let $J(S) = Hom(Soc(M), M)$ and $\alpha \in J(S)$ then by Proposition 3(a), $Ker\alpha$ is essential in M. As $\alpha \in J(S)$, $\alpha \in Hom(Soc(M), M)$ which implies that $Ker\alpha \subseteq Soc(M) \subseteq M \Rightarrow Soc(M)$ is essential in M.
Conversly, let $Soc(M)$ be essential in M. Let $\alpha \in J(S)$ then by 3(b) $J(S) \subseteq Hom(Soc(M), M)$.
Now, let $\alpha \in Hom(Soc(M), M)$ then $Ker\alpha \subseteq Soc(M)$. As $Soc(M) \subseteq_e M$ and $Ker\alpha \subseteq M \Rightarrow Ker\alpha \cap Soc(M) \subseteq_e Ker\alpha$ and by [3, Proposition 5.16(2)] we get $Soc(M) \subseteq_e Ker\alpha \Rightarrow Soc(M) \subseteq Ker\alpha \subseteq M \Rightarrow Ker\alpha \subseteq_e M$, which in turn implies that $\alpha \in J(S)$ and hence

$$Hom(Soc(M), M) \subseteq J(S) \quad \Rightarrow J(S) = Hom(Soc(M), M).$$

□

Proposition 3.7. [6, Proposition 3.3(1)] *If $Soc(M) \subseteq_e M$, then $\Delta = l_S(Soc(M))$, where $\Delta = \{\alpha \in S | ker\alpha$ is essential in $M\}$ and $l_S(Soc(M))$ is the annihilator of $Soc(M)$ in S.*

Proposition 3.8. *If M is an essentially pseudo injective uniform module and if $Soc(M) \subseteq_e M$ then $J(S) = l_S(Soc(M))$.*

Proof. By Proposition 3(a) we have $J(S) = \Delta$ and so by Proposition 7 we get, $J(S) = l_S(Soc(M))$ □

Proposition 3.9. *Let M be a duo, essentially pseudo injective module. Let $S = End(M)$ and $T = \{\alpha \in S | ker\alpha$ is essential in $M\}$. Then:*
(a) *for every $f \in T$, $Kerf$ is an essentially pseudo stable submodule of M.*
(b) *if M is extending then every submodule of M is essentially pseudo stable submodule of any direct summand of M.*

Proof. **(a)** Let $f \in T$ then $Kerf$ is essential in M. Let $g : A \to Kerf$ be an essential monomorphism, $\psi : A \to Kerf$ be a monomorphism and $\nu : Kerf \to M$ be the inclusion map. Then clearly $Im(\nu o g) + Im(\nu o \psi) \subseteq Kerf$. By essential pseudo injectivity of M there exists $h \in End(M)$
such that $h o \nu o g = \nu o \psi$. Since M is duo and $Kerf \subseteq M$, $h(Kerf) \subseteq Kerf$, proving that $Kerf$ is an essentially pseudo stable submodule of M.

(b) Let A be any submodule of M. As M is extending there exists a direct summand N of M such that A is essential in N. Since direct summand of essentially pseudo injective module is essentially pseudo injective implies that N is essentially pseudo injective. Rest of the proof follows from (a). □

For a uniform module M we give below a characterization as to when M is essentially pseudo injective in terms of its injective hull.

Proposition 3.10. *Let M be a uniform module. Then M is essentially pseudo injective iff for every $f \in End(E(M))$, $f(M) \subseteq M$,where $E(M)$ is the injective hull of M.*

Proof. Let M be essentially pseudo injective, let $f \in End(E(M))$ and $N = \{m \in M | f(m) \in M\}$. Then N is an essential submodule of M. Let $i : N \to M$ be inclusion map. As M is essentially pseudo injective, the map $f|_N$ can be extended to a map $g : M \to M$ such that $g|_N = f|_N$. Claim

that $M \cap (g-f)(M) = 0$. If not then there exists $0 \neq m \in M$ such that $m = (g-f)(m^{'})$ for some $m^{'} \in M$. This implies that $f(m^{'}) = g(m^{'}) - m \in M$. So, $m^{'} \in N$. Hence $m = g(m^{'}) - f(m^{'}) = 0$. We have M is essential submodule of $E(M)$. So, $(g-f)(M) = 0 \Rightarrow f(M) \subseteq M$.
The converse follows from [5, Proposition 17] with suitable changes. □

Proposition 3.11. *Let M be an essentially pseudo injective uniform, duo module. Let $S = End(M)$. For every $f \in J(S)$, $Ker f$ is an essentially pseudo stable submodule of M.*

Proof. By Proposition 3(a) we have $T = J(S)$. Rest of the proof follows from Proposition 9(a). □

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