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Generalized Fibonacci Sequences

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Abstract

The Fibonacci sequence is famous for possessing wonderful and amazing properties. In this paper, we introduce generalized Fibonacci sequences and related identities consisting even and odd terms. Also we present connection formulas for generalized Fibonacci sequences, Jacobsthal sequence and Jacobsthal-Lucas sequence.

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1 Introduction

It is well-known that the Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence and Jacobsthal-Lucas sequence are most prominent examples of recursive sequences.

The Fibonacci sequence [8] is defined by the recurrence relation

$$F_k = F_{k-1} + F_{k-2}\ ,\ k \geq 2 \quad \text{with} \quad F_0 = 0,\ F_1 = 1 \tag{1.1}$$

The Lucas sequence [8] is defined by the recurrence relation

$$L_k = L_{k-1} + L_{k-2}\ ,\ k \geq 2 \quad \text{with} \quad L_0 = 2,\ L_1 = 1 \tag{1.2}$$

The Jacobsthal sequence [2] is defined by the recurrence relation

$$J_k = J_{k-1} + 2J_{k-2}\ ,\ k \geq 2 \quad \text{with} \quad J_0 = 0,\ J_1 = 1 \tag{1.3}$$

The Jacobsthal-Lucas sequence [2] is defined by the recurrence relation

$$j_k = j_{k-1} + 2j_{k-2}\ ,\ k \geq 2 \quad \text{with} \quad j_0 = 2,\ j_1 = 1 \tag{1.4}$$

B. Singh, O. Sikhwal and S. Bhatnagar [5] defined Fibonacci-Like Sequence

$$S_k = S_{k-1} + S_{k-2}\ ,\ k \geq 2 \quad \text{with} \quad S_0 = 2,\ S_1 = 2 \tag{1.5}$$

The second order recurrence sequence has been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation.

Kalman and Mena [6] generalize the Fibonacci sequence by

$$F_n = aF_{n-1} + bF_{n-2}\ ,\ n \geq 2 \quad \text{with} \quad F_0 = 0,\ F_1 = 1 \tag{1.6}$$

Horadam [3] defined generalized Fibonacci sequence $\{H_n\}$ by

$$H_n = H_{n-1} + H_{n-2},\ n \geq 3 \quad \text{with} \quad H_1 = p,\ H_2 = p + q \tag{1.7}$$

where p and q are arbitrary integers.

In this paper, we introduce generalized Fibonacci sequence and present identities consisting even and odd terms. Further we defined connection formulas for generalized Fibonacci sequence, Jacobsthal sequence and Jacobsthal-Lucas sequence.

2 Generalized Fibonacci Sequence

We define generalized Fibonacci sequence as

$$F_k = pF_{k-1} + qF_{k-2}, \; k \geq 2 \quad \text{with} \quad F_0 = a, \; F_1 = b \; , \tag{2.1}$$

where p, q, a and b are positive integers

For different values of p, q, a and b many sequences can be determined.

We focus two cases of sequences $\{V_k\}_{k\geq 0}$ and $\{U_k\}_{k\geq 0}$ which generated in (2.1).

If $p = 1$, $q = a = b = 2$, we get

$$V_k = V_{k-1} + 2V_{k-2}, \quad \text{for } k \geq 2 \quad \text{with } V_0 = 2, V_1 = 2 \tag{2.2}$$

The first few terms of $\{V_k\}_{k\geq 0}$ are 2, 2, 6, 10, 22, 42 and so on. Its Generating function is defined by

$$V_k = \frac{2}{1 - x - 2x^2} \tag{2.3}$$

Its Binet's formula is defined by

$$V_k = 2\,\frac{\Re_1^{k+1} - \Re_2^{k+1}}{\Re_1 - \Re_2} \tag{2.4}$$

where $\Re_1$ and $\Re_2$ are the roots of the characteristic equation $2x^2 + x - 1 = 0$.

If $p = 1$, $q = a = 2$, $b = 0$ we get

$$U_k = U_{k-1} + 2U_{k-2} \quad \textit{for} \quad k \geq 2 \quad \textit{with} \quad U_0 = 2, U_1 = 0 \tag{2.5}$$

The first few terms of $\{U_k\}_{k\geq 0}$ are 2, 0, 4, 4, 12, 20 and so on. Its Generating function is defined by

$$U_k = \frac{2(1-x)}{1 - x - 2x^2} \tag{2.6}$$

Its Binet's formula is defined by

$$U_k = 4\,\frac{\Re_1^{k-1} - \Re_2^{k-1}}{\Re_1 - \Re_2} \tag{2.7}$$

where $\Re_1$ and $\Re_2$ are the roots of the characteristic equation $2x^2 + x - 1 = 0$.

3 Identities of Generalized Fibonacci Sequence

Now we present identities consisting even and odd terms.

Theorem 3.1. *If* $\{V_k\}_{k\geq 0}$ *and* $\{U_k\}_{k\geq 0}$ *are the generalized Fibonacci sequences, then*

$$V_k U_k = \frac{8}{9} j_{2k} - (-2)^{k-1} \frac{8}{3} = \begin{cases} \frac{8}{9} j_{2k} + 2^{k-1} \frac{8}{3}, & k \text{ even} \\ \frac{8}{9} j_{2k} - 2^{k-1} \frac{8}{3}, & k \text{ odd} \end{cases}$$

Proof. By Binet's formulas (2.4) and (2.7), we have

$$\begin{aligned} V_k U_k &= \frac{8}{9}\left\{(\Re_1^{k+1} - \Re_2^{k+1})(\Re_1^{k-1} - \Re_2^{k-1})\right\} \\ &= \frac{8}{9}\left\{(\Re_1^{2k} + \Re_2^{2k}) - \Re_1^k \Re_2^k (\Re_1 \Re_2^{-1} - \Re_2 \Re_1^{-1})\right\} \\ &= \frac{8}{9}\left\{ j_{2k} - (-2)^k \left(\frac{\Re_1}{\Re_2} - \frac{\Re_2}{\Re_1}\right)\right\} \\ &= \frac{8}{9}\left\{ j_{2k} - (-2)^{k-1} (\Re_1^2 - \Re_2^2)\right\} \\ &= \frac{8}{9} j_{2k} - \frac{8}{3}(-2)^{k-1} \end{aligned}$$

$$V_k U_k = \frac{8}{9} j_{2k} - (-2)^{k-1} \frac{8}{3} = \begin{cases} \frac{8}{9} j_{2k} + 2^{k-1} \frac{8}{3}, & k \text{ even} \\ \frac{8}{9} j_{2k} - 2^{k-1} \frac{8}{3}, & k \text{ odd} \end{cases}$$

□

Theorem 3.2. *If* $\{V_k\}_{k\geq 0}$ *is the generalized Fibonacci sequence, then*

(*i*) $V_{k-1}^2 + V_k^2 = \frac{4^{k+1}}{3} + \frac{8}{9}\left\{ j_{2k} + (-2)^k \right\}$

$$(ii)\;\; V_{k+1}^2 - V_{k-1}^2 = \begin{cases} \frac{5}{3}(4)^{k+1} - \frac{1}{3}(2)^{k+3}\;, & k\ even \\ \frac{5}{3}(4)^{k+1} + \frac{1}{3}(2)^{k+3}\;, & k\ odd \end{cases}$$

Proof. **(i):** By Binet's formula (2.4), we have

$$V_{k-1}^2 + V_k^2 = \frac{4}{9}\left\{(\Re_1^k - \Re_2^k)^2 + (\Re_1^{k+1} - \Re_2^{k+1})^2\right\}$$

$$= \frac{4}{9}\left\{\Re_1^{2k} + \Re_1^{2k+2} + \Re_2^k + \Re_2^{2k+2} - 2(\Re_1\Re_2)^k(\Re_1\Re_2 + 1)\right\}$$

$$= \frac{4}{9}\left\{3\Re_1^{2k} + 2(\Re_1^{2k} + \Re_2^{2k}) + 2(-2)^k\right\}$$

$$= \frac{4}{9}\left\{3\Re_1^{2k} + 2\,j_{2k} + 2(-2)^k\right\}$$

$$V_{k-1}^2 + V_k^2 = \frac{4^{k+1}}{3} + \frac{8}{9}\left\{j_{2k} + (-2)^k\right\}$$

(ii): By Binet's formula (2.4), we have

$$V_{k+1}^2 - V_{k-1}^2 = \frac{4}{9}\left\{(\Re_1^{k+2} - \Re_2^{k+2})^2 + (\Re_1^k - \Re_2^k)^2\right\}$$

$$= \frac{4}{9}\left[\Re_1^{2k}(\Re_1^4 - 1) + \Re_2^{2k}(\Re_2^4 - 1) - 2(\Re_1\Re_2)^k\left\{(\Re_1\Re_2)^k - 1\right\}\right]$$

$$V_{k+1}^2 - V_{k-1}^2 = \frac{4}{9}\left\{15\Re_1^{2k} - 6(-2)^k\right\}$$

$$= \frac{5}{3}(4)^{k+1} - \frac{(-1)^k}{3}(2)^{k+3}$$

$$V_{k+1}^2 - V_{k-1}^2 = \begin{cases} \frac{5}{3}(4)^{k+1} - \frac{1}{3}(2)^{k+3}\;, & k\ even \\ \frac{5}{3}(4)^{k+1} + \frac{1}{3}(2)^{k+3}\;, & k\ odd \end{cases}$$

□

Following theorems can be solved by Binet's formula (2.4) and (2.7)

Theorem 3.3. *If* $\{U_k\}_{k\geq 0}$ *is the generalized Fibonacci sequence, then*

$$(i)\quad U_{k+1}^2 + U_{k+2}^2 = \frac{4^{k+2}}{3} + \frac{32}{9}\left\{ j_{2k} + (-2)^k \right\}$$

$$(ii)\quad U_{k+3}^2 - U_{k+1}^2 = \begin{cases} \frac{5}{3}(4)^{k+2} - \frac{1}{3}(2)^{k+5}, & k \text{ even} \\ \frac{5}{3}(4)^{k+2} + \frac{1}{3}(2)^{k+5}, & k \text{ odd} \end{cases}$$

Theorem 3.4. *Prove that*

$$V_0 + V_3 + V_6 + \ldots + V_{3k-3} = \sum_{n=1}^{k} V_{3n-3} = \begin{cases} \frac{4}{7}\, J_{3k}, & k \text{ even} \\ \frac{2}{7}(2J_{3k} + 1), & k \text{ odd} \end{cases}$$

Corollary 3.5. $$V_0 + V_3 + V_6 + \ldots + V_{3k-3} = \begin{cases} \frac{4}{21}(8^k - 1), & k \text{ even} \\ \frac{2}{21}(2^{3k+1} + 5), & k \text{ odd} \end{cases}$$

Theorem 3.6. *Prove that*

$$U_2 + U_5 + U_8 + \ldots + U_{3k-1} = \begin{cases} \frac{8}{7} J_{3k}, & k \text{ even} \\ \frac{2}{7}(4J_{3k} + 1), & k \text{ odd} \end{cases}$$

Corollary 3.7. $U_2 + U_5 + U_8 + ... + U_{3k-1} = \begin{cases} \frac{8}{21}(8^k - 1), & k \ even \\ \frac{8^{k+1} + 20}{21}, & k \ odd \end{cases}$

Theorem 3.8. *Prove that*

$$V_1 + V_4 + V_7 + ... + V_{3k-2} = \sum_{n=1}^{k} V_{3n-2} = \frac{1}{3} j_{3k} + \frac{1}{21}(8)^k - \frac{5}{7}$$

Corollary 3.9. $V_1 + V_4 + V_7 + ... + V_{3k-2} = \begin{cases} \frac{8}{21}(8^k - 1), & k \ even \\ \frac{(8^{k+1} - 20)}{21}, & k \ odd \end{cases}$

Theorem 3.10. *Prove that*

$$U_3 + U_6 + U_9 + ... + U_{3k} = \sum_{n=1}^{k} U_{3k} = \frac{2}{3} j_{3k} + \frac{2}{21}(8)^k - \frac{10}{7}$$

Corollary 3.11. $U_3 + U_6 + U_9 + ... + U_{3k} = \begin{cases} \frac{16}{21}(8^k - 1), & k \ even \\ \frac{2}{21}(8^{k+1} - 22), & k \ odd \end{cases}$

Theorem 3.12. *Prove that*

$$U_1 + U_4 + U_7 + ... + U_{3k-2} = \begin{cases} \frac{4}{21}(j_{3k} - 2), & k \ even \\ \frac{4}{21}(j_{3k} - 7), & k \ odd \end{cases}$$

Corollary 3.13. $U_1 + U_4 + U_7 + ... + U_{3k-2} = \begin{cases} \frac{4}{21}(8^k - 1) & , k \ even \\ \frac{4}{21}(8^k - 8) & , k \ odd \end{cases}$

4 Connection Formulas

Finally we present connection formulas for generalized Fibonacci Sequences, Jacobsthal sequence and Jacobsthal-Lucas sequence.

Theorem 4.1. *Prove that*

(i) $2V_{k+1} - V_k = 2\, j_{k+1}$

(ii) $V_k + 4V_{k-1} = 2\, j_{k+1}$

Proof. **(i):** For $k = 0$,

$$2V_{0+1} - V_0 = 2\times 2 - 2 = 2 = 2\times 1 = 2j_1,$$

which is true for $k = 0$.

For $k = 1$,

$$2V_{1+1} - V_1 = 2\times 6 - 2 = 10 = 2\times 5 = 2j_2,$$

which is also true for $k = 1$.

If result is true for $k = n$, then $2V_{n+1} - V_n = 2j_{n+1}$. Now

$$2V_{(n+1)+1} - V_{(n+1)} = 2V_{n+2} - V_{n+1}$$

$$= 2j_{n+1} + 2(2j_n) \qquad \text{(By hypothesis)}$$

$$= 2(j_{n+1} + 2j_n) = 2j_{n+2}$$

Therefore, $2V_{(n+1)+1} - V_{(n+1)} = 2j_{n+2}$, which is true for $k = n + 1$.

This completes the proof.

(ii): By Binet's formula (2.4), we have

$$V_k + 4V_{k-1} = 2\,\frac{\Re_1^{k+1} - \Re_2^{k+1}}{\Re_1 - \Re_2} + 8\,\frac{\Re_1^{k} - \Re_2^{k}}{\Re_1 - \Re_2} = 2\,j_{k+1}$$

Following theorems can be solved with the help of Binet's formulas and as well as induction method. □

Theorem 4.2. *Prove that*

$(i)\quad U_{k+1} + 4U_k = 4\,j_k$

$(ii)\quad 2U_{k+2} - U_{k+1} = 4\,j_k$

Theorem 4.3. *Prove that*

$$(i)\qquad j_{k+1} + 4\,j_k = \begin{cases} \frac{9}{2}V_k, & k \geq 0 \\ \frac{9}{4}U_{k+2}, & k \geq 0 \end{cases}$$

$$(ii)\qquad 2\,j_{k+1} - j_k = \begin{cases} \frac{9}{2}V_{k-1}, & k \geq 1 \\ \frac{9}{4}U_{k+2}, & k \geq 0 \end{cases}$$

5 Conclusion

In this paper we have stated and derived many identities of generalized Fibonacci sequences consisting even and odd terms through Binet's formulas. Finally we presented some connection formulas and defined through induction method.

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