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Stability of solitary waves for symmetric coupled Klein-Gordon equations in 2-dimensional

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Abstract

In this paper, the authors discuss the existence and stability of solitary waves for the symmetric coupled Klein-Gordon equations in R^2. The existence is obtained by considering a minimization problem using the concentration compactness principle and the stability is proved by stability theory.

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Keywords: Klein-Gordon equations, Solitary waves, Existence, Stability

1 Introduction

Stability of solitary waves describes the phenomenon that two solitary waves still maintain their energy and velocity in respectively after they encounter. Now, the virtue is concerned by more and more investigators. [1-4] state the

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existence or stability of solitary waves for Schrödinger equation, Kadomtsev-Petvicshvili equation, Klein-Gordon equation and their coupled equations. Where [4] describe the interaction of a nucleon field with a meson field, whereas we use the following Klein-Gordon equations to interpret the motion between the meson.

$$\begin{cases} u_{tt} - \Delta u + u &= |v|^r |u|^{r-2} u, \\ v_{tt} - \Delta v + v &= |u|^r |v|^{r-2} v, \quad (x,y) \in R^2 \end{cases} \tag{1.1}$$

where (u, v) is a real function of $(x, y, t) \in R^2$. The periodic solutions, periodic traveling wave solutions, exact solutions and stability of the standing waves of the similar coupled Klein-Gordon equations in [5-9]. (1.1) describes the interact of the same masses charged meson in an electromagnetic field, the motion via a nonlinear coupling.

This paper is organized as follows: In Section 2 , we discuss the existence by considering a minimization problem using the concentration compactness principle; In Section 3 , based on the results obtained in section 2, we state the stability by *stability theory*, [10-11].

Here we set: $H^1 = H^1(R^2),\ L^2 = L^2(R^2),\ Y(R^2) = H^1(R^2) \times H^1(R^2)$.

The solitary waves of (1.1) are defined in the following form

$$\begin{cases} u(x,y,t) = \varphi_c(x - ct, y), \\ v(x,y,t) = \psi_c(x - ct, y). \end{cases} \tag{1.2}$$

Then

$$\begin{cases} c^2 \varphi_{xx} - \Delta \varphi_c + \varphi_c = |\psi_c|^r |\varphi_c|^{r-2} \varphi_c, \\ c^2 \psi_{xx} - \Delta \psi_c + \psi_c = |\varphi_c|^r |\psi_c|^{r-2} \psi_c. \end{cases} \tag{1.3}$$

2 Existence Of Solitary Waves

Theorem 2.1. *(1.3) admits one nontrivial solution in* $H^1(R^2) \times H^1(R^2)$.

It is clear that (1.3) is the critical point equations of the functional

$$T(\varphi, \psi) = \frac{1}{2} \int |\nabla \varphi|^2 - c^2 |\varphi_x|^2 + |\varphi_c|^2 + |\nabla \psi|^2 - c^2 |\psi_x|^2 + |\psi_c|^2 - \frac{2}{r} |\varphi \psi|^r$$

When $0 < c < 1$ consider the following minimization problem

$$I_\lambda = M_c = \inf_{(\varphi,\psi)\in\Sigma_\lambda} I_c(\varphi,\psi) \tag{2.1}$$

$$\Sigma_\lambda = \left\{(\varphi,\psi) \in H^1 \times H^1 : \varphi,\psi \not\equiv 0, K(\varphi,\psi) = \lambda\right\} \tag{2.2}$$

Where

$$I_c(\varphi,\psi) = \frac{1}{2}\int |\nabla\varphi|^2 - c^2|\varphi_x|^2 + |\varphi_c|^2 + |\nabla\psi|^2 - c^2|\psi_x|^2 + |\psi_c|^2$$

$$K(\varphi,\psi) = \int |\varphi\psi|^r$$

If (φ,ψ) is a minimizer of (2.1), then there exists a *Lagrange* multiplier θ such that

$$\begin{cases} c^2\varphi_{xx} - \Delta\varphi_c + \varphi_c - \theta|\psi_c|^r|\varphi_c|^{r-2}\varphi_c = 0, \\ c^2\psi_{xx} - \Delta\psi_c + \psi_c - \theta|\varphi_c|^r|\psi_c|^{r-2}\psi_c = 0. \end{cases}$$

Where $\varphi_1 = \theta^{\frac{1}{2(r-1)}}\varphi,\ \psi_1 = \theta^{\frac{1}{2(r-1)}}\psi$ is solution of (1.3).

We prove the Theorem 2.1 by the following several lemmas.

Lemma 2.2. *$I_\lambda > 0$, for any $\lambda > 0$.*

Proof. When $u \in H^1(R^2)$, then $\int |u_x|^q dxdy \le \int |\nabla u|^q dxdy, \forall q < \infty$. And since $\varphi \in H^1(R^2),\ \psi \in H^1(R^2)$, we get

$$\int |\varphi|^r|\psi|^r dxdy \le \frac{1}{2}\int (|\varphi|^{2r} + |\psi|^{2r})dxdy$$

Hence $\lambda = \int |\varphi|^r|\psi|^r dxdy \le A_1 I_\lambda^r$. Therefore there is a some $r > 0$, for any $\lambda > 0$, we have $I_\lambda > 0$. □

Lemma 2.3. *$\Sigma_\lambda \neq \varnothing$, for any $\lambda > 0$.*

Lemma 2.4. *(Strict Sub-additivity) We have $I_\lambda < I_{\lambda-\alpha} + I_\alpha$, for any $\alpha \in (0,\lambda)$.*

Proof. Let $\varphi_\lambda = \lambda^{\frac{1}{2r}}\varphi, \psi_\lambda = \lambda^{\frac{1}{2r}}\psi$. Then $(\varphi,\psi)\in\Sigma_1 \Leftrightarrow (\varphi_\lambda,\psi_\lambda)\in\Sigma_\lambda$ and $I_\lambda = \lambda^{\frac{1}{r}} I_1$. □

Proof of Theorem 2.1 (Concentration Compactness Principle): Let (φ_k,ψ_k) be the minimizing sequence of problem I_λ for $\forall\lambda > 0$ and set

$$\rho_k = \frac{1}{2}\left(|\nabla\varphi_k|^2 + |\varphi_k|^2 - c^2|\varphi_{kx}|^2 + |\nabla\psi_k|^2 + |\psi_k|^2 - c^2|\psi_{kx}|^2\right)$$

We have

$$\begin{aligned}\int \rho_k &= \frac{1}{2}\int \left(|\nabla\varphi_k|^2 + |\varphi_k|^2 - c^2|\varphi_{kx}|^2 + |\nabla\psi_k|^2 + |\psi_k|^2 - c^2|\psi_{kx}|^2\right)\\ &= I_c(\varphi_k,\psi_k) \to I_\lambda\end{aligned}$$

(i) If $\lim\limits_{k\to\infty}\sup\limits_{X\in R^2}\int\limits_{X+B_k}\rho_k dxdy = 0$, for any $R<\infty$,

Where $B_R = B_R(0)$ is a circle with radius R centered at 0 and $X=(x,y)\in R^2$. And since

$$\int\limits_{R^2}|\varphi\psi|^r dxdy \le A_1\left[\left(\sup_{X\in R^2}\int\limits_{X+B_1}|\varphi|^2\right)^r + \left(\sup_{X\in R^2}\int\limits_{X+B_1}|\psi|^2\right)^r\right]$$

Therefore $|\varphi_k|^r|\psi_k|^r \to 0$, strongly in L^r.

Which contradicts the fact that $(\varphi_k,\psi_k)\in\Sigma_\lambda$ and $\lambda>0$.

(ii) We may assume

$$\lim_{t\to\infty} N(t) = \gamma \in (0, I_\lambda) \tag{2.3}$$

Where

$$N(t) = \lim_{k\to\infty}\sup_{X_0\in R^2}\int\limits_{X_0+B_t}\rho_k dxdy \tag{2.4}$$

$X_0=(x_0,y_0)\in R^2$ and (φ_k,ψ_k) is the minimizing sequence.

We need the following several lemmas to prove the consequence.

Lemma 2.5. *Assume (2.3) holds. Then for any $\varepsilon>0$, there exists $\delta(\varepsilon)\to 0$ (as $\varepsilon\to 0$) such that we can find φ_k^1, φ_k^2, ψ_k^1, ψ_k^2 satisfying the following relations*

$$\left\|\varphi_k - (\varphi_k^1+\varphi_k^2)\right\|_{H^1} + \left\|\psi_k - (\psi_k^1+\psi_k^2)\right\|_{H^1} \le \delta(\varepsilon) \tag{2.5}$$

$$\left|I_c(\varphi_k^1,\psi_k^1)-\gamma\right|\le\delta(\varepsilon) \tag{2.6}$$

$$\left|I_c(\varphi_k^1,\psi_k^1)-(I_\lambda-\gamma)\right|\le\delta(\varepsilon) \tag{2.7}$$

$$\left|\int(|\varphi_k^1|^r\psi_k^1|^r+|\varphi_k^2|^r\psi_k^2|^r-|\varphi_k|^r\psi_k|^r)\right|<\delta(\varepsilon) \tag{2.8}$$

$$dist(supp\varphi_k^1,supp\varphi_k^2)\to\infty, dist(supp\psi_k^1,supp\psi_k^2)\to\infty \tag{2.9}$$

We may assume

$$K(\varphi_k^1,\psi_k^1)\to\lambda_1(\varepsilon),K(\varphi_k^2,\psi_k^2)\to\lambda_2(\varepsilon)\ as\ k\to\infty$$

Then

$$|\lambda-(\lambda_1(\varepsilon)+\lambda_2(\varepsilon))|\le\delta(\varepsilon)\to 0\ as\ \varepsilon\to 0$$

We distinguish the following two cases:

Case 1 When $\lambda_1(\varepsilon)\to 0$ as $\varepsilon\to 0$. Choosing ε small enough ,we have

$$K(\varphi_k^2,\psi_k^2)>0$$

Set

$$\left(\tilde{\varphi_k^2},\tilde{\psi_k^2}\right)=\left(\left(\frac{\lambda_2(\varepsilon)}{K(\varphi_k^2,\psi_k^2)}\right)^{\frac{1}{2r}}\varphi_k^2,(\frac{\lambda_2(\varepsilon)}{K(\varphi_k^2,\psi_k^2)}\right)^{\frac{1}{2r}}\psi_k^2\right)$$

And since $\dfrac{\lambda_2(\varepsilon)}{K(\varphi_k^2,\psi_k^2)}\to 1$ as $\varepsilon\to 0$.
We get

$$I_{\lambda_2(\varepsilon)}\le\limsup_{k\to\infty}I_c(\varphi_k^2,\psi_k^2)\le I_\lambda-\gamma+\delta(\varepsilon)$$

Which contradicts the fact that $\lambda_2(\varepsilon)\to\lambda\ as\ \varepsilon\to 0$.

Case 2 $\lim_{\varepsilon\to 0}|\lambda_1(\varepsilon)|>0$ and $\lim_{\varepsilon\to 0}|\lambda_2(\varepsilon)|>0$, similar to the Case 1, we obtain

$$\begin{aligned}I_{|\lambda_1(\varepsilon)|}+I_{|\lambda_2(\varepsilon)|}&\le\liminf_{k\to\infty}(I_c(\varphi_k^1,\psi_k^1)+I_c(\varphi_k^2,\psi_k^2))\\&\le I_\lambda+2\delta(\varepsilon)\end{aligned}$$

Then $I_s+I_{\lambda-s}\le I_\lambda$, for a some $s\in(0,\lambda)$ contradicts the strict sub-additivity.

(iii) The only possibility is that ρ_k is tight, there exists a sequence $X_k\in R^2$ such that for all $\varepsilon>0$, there is a finite number $R>0$ and $k_0>0$ such that

$$\frac{1}{2}\int\limits_{X_k+B_R}|\nabla\varphi|^2-c^2|\varphi_x|^2+|\varphi_c|^2+|\nabla\psi|^2-c^2|\psi_x|^2+|\psi_c|^2\ge I_\lambda-\varepsilon \tag{2.10}$$

Since (φ_k, ψ_k) is bounded in $H^1 \times H^1$, we may assume that $(\varphi_k(\cdot - X_k), \psi_k(\cdot - X_k))$ converges weakly in $H^1 \times H^1$ to (φ, ψ).
We prove that

$$\varphi(\cdot - X_k) \to \varphi, \ \textit{strongly in } L^q \tag{2.11}$$

$$\psi(\cdot - X_k) \to \psi, \ \textit{strongly in } L^q \ , \ \forall 2 \leq q < \infty \tag{2.12}$$

Indeed, from (2.10) it follows that for all $\forall k \geq k_0$,

$$\int\limits_{X_k + B_R} |\varphi_k|^2 \geq \int\limits_{R^2} |\varphi_k|^2 - 2\varepsilon$$

and hence

$$\int\limits_{R^2} |\varphi_k|^2 \leq \liminf_{k \to \infty} \int\limits_{X_k + B_R} |\varphi_k|^2 + 2\varepsilon$$

On the other hand, since $H^1(R^2) \subseteq L^2(R^2)$ is compact, we get $\varphi_k \to \varphi$ strongly in $L^2(R^2)$, therefore $\varphi_k(\cdot - X_k) \to \varphi$ strongly in $L^2(R^2)$. From interpolation and $H^1 \subset L^q$ $(as\ 2 \leq q < \infty)$, we see that (2.11) holds.
(2.12) can be proved in the similar manner.
Therefore

$$K(\varphi_k, \psi_k) \to K(\varphi, \psi) = \lambda$$

and I_λ is a minimizer of I_λ.

Proof of Lemma 2.5. Assume (2.3) holds.
We can find $R_0 > 0, R_k \geq R_0$, R_k with $R_k \nearrow \infty$ and $X_k \in R^2$ such that $\gamma - \varepsilon \leq \int\limits_{X + B_{R_0}} \rho_k dxdy \leq \gamma$ and $N_k(2R_k) \leq \gamma + \varepsilon$, for $k \geq k_0$, where

$$N_k(t) = \sup_{X_0 \in R^2} \int\limits_{X_0 + B_t} \rho_k dxdy.$$

Therefore

$$\int\limits_{R_0 \leq |X - X_k| \leq 2R_k} \rho_k dxdy \leq 2\varepsilon \tag{2.13}$$

Define φ_k^1, φ_k^2, ψ_k^1 *and* ψ_k^2 as follows:
Choose $\xi, \eta \in C_0^\infty(R^2)$, such that $0 \leq \xi, \eta \leq 1$ and $\xi \equiv 1$, on B_1, $supp\xi \subset B_2$;

$\eta \equiv 1$, on B_1^c ,$supp\eta \subset B_1^c$.

Let

$$\xi_k = \xi\left(\frac{\cdot - X_k}{R_1}\right),\ \eta_k = \eta\left(\frac{\cdot - X_k}{R_k}\right)$$

Set

$$\varphi_k^1 = \xi_k\varphi_k,\ \varphi_k^2 = \eta_k\varphi_k,\ \psi_k^1 = \xi_k\varphi_k,\ \psi_k^1 = \eta_k\varphi_k.$$

Now we prove (2.5)-(2.8)

$$\begin{aligned}
\|\varphi_k - (\varphi_k^1 + \varphi_k^2)\|_{L_2} &= \|\varphi_k - (\xi_k\varphi_k + \eta_k\varphi_k)\|_{L_2} \\
&\leq \Big(\int\limits_{R_0 \leq |X - X_k| \leq 2R_k} |\varphi_k|^2)^{\frac{1}{2}} + \|\xi_k\varphi_k + \eta_k\varphi_k\|_{L^2} \\
&\leq \sqrt{2\varepsilon} + \|\xi_k\varphi_k\|_{L^2} + \|\eta_k\varphi_k\|_{L^2}
\end{aligned}$$

However

$$\|\xi_k\varphi_k\|_{L^2} \leq \|\xi_k\|_{L^p}\|\varphi_k\|_{L^q}, \tag{2.14}$$

where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}.$$

By (2.13)

$$\|\xi_k\varphi_k\|_{L^2} \leq A\sqrt{\varepsilon}.$$

Similarly we can prove

$$\|\varphi_k\varphi_k\|_{L^2} \leq A\sqrt{\varepsilon}.$$

Then

$$\|\varphi_k - (\varphi_k^1 + \varphi_k^2)\|_{L_2} \leq A\sqrt{\varepsilon}.$$

By the same method as above, we obtain

$$\|\varphi_k - (\varphi_k^1 + \varphi_k^2)\|_{H_1} \leq A\sqrt{\varepsilon}, \quad \|\psi_k - (\psi_k^1 + \psi_k^2)\|_{H_1} \leq A\sqrt{\varepsilon},$$

therefore (2.5) holds.

(2.6)-(2.7) follows from the definition of $N(t)$, γ.

Since

$$\begin{aligned}\left|\int_{R^2}(|\varphi_k|^r|\psi_k|^r-|\varphi_k^1|^r|\psi_k^1|^r-|\varphi_k^2|^r|\psi_k^2|^r)\right| &= \left|\int_{R^2}|\varphi_k|^r|\psi_k|^r(1-\xi_k^{2r}-\eta_k^{2r})\right| \\ &\leq \int_{R_0\leq|X-X_k|\leq 2R_k}|\varphi_k|^r|\psi_k|^r \\ &\leq A(\int_{R_0\leq|X-X_k|\leq 2R_k}\rho_k)^r \\ &\leq \delta(\varepsilon)\end{aligned}$$

then (2.8) holds. □

3 Stability of Solitary Waves

Set

$$\vec{\varphi}(x,y)=(\varphi(x,y),\psi(x,y)),$$

where (φ,ψ) is solution of (1.3).

Theorem 3.1. *There exists $T_*=T_*(\|\vec{\varphi}_0\|)>0$ such that have only one solution of (1.1) and satisfy $\vec{\varphi}(0)=\vec{\varphi}_0$, for any $\vec{\varphi}(x,y)\in H^1(R^2)\times H^1(R^2)$.*

Definition 3.2. *Let (φ,ψ) be a solution of (1.3). If there is a some $\delta>0$, for all $\lambda>0$ such that for any $\vec{\varphi}(x,y,0)\in Y(R^2)$ with $\|\vec{\varphi}(x,y,0)-\vec{\varphi}\|_Y<\delta$ and $\vec{\varphi}(x,y,t)$ satisfies the following inequality is solution of (1.1)*

$$\sup_t\inf_\theta|\vec{\varphi}(x,y,t)-(\varphi(x-\theta,y),\psi(x-\theta,y))|_Y<\varepsilon,$$

Then $\vec{\varphi}(x,y,t)=(\varphi(x-ct,y),\psi(x-ct,y))$ is solution of (1.1) is orbital stability; Otherwise is orbital instability.

Note

$$S_c=\left\{(\varphi,\psi)\in Y|I_c(\varphi,\psi)=K(\varphi,\psi)=[M(c)]^{\frac{r}{r-1}}\right\},$$

where (φ,ψ) is solution of (1.3).

Choose $\vec{u}=(u,v)\in Y(R^2)$, set

$$E(\vec{u})=\frac{1}{2}\int_{R^2}|u_t|^2+|\nabla u|^2+|u|^2+|v_t|^2+|\nabla v|^2+|v|^2+\frac{2}{r}|uv|^r$$

$$Q(\vec{u})=-\int_{R^2}u_xu_t+v_xv_t \tag{3.1}$$

Consider $d(c)=E(\vec{\varphi})-cQ(\vec{\varphi}),\vec{\varphi}\in S_c$.

$$\begin{aligned} d(c) &= E(\vec{\varphi})-cQ(\vec{\varphi})\\ &= I_c(\vec{\varphi})-\frac{1}{r}K(\vec{\varphi})\\ &= \frac{r-1}{r}I_c(\vec{\varphi})\\ &= \frac{r-1}{r}K(\vec{\varphi}) \end{aligned} \tag{3.2}$$

Set

$$\begin{cases} \varphi(x,y) = (1-c^2)^{-\frac{1}{2}}G([2(1-c^2)]^{-\frac{1}{2}}x,y),\\ \psi(x,y) = (1-c^2)^{-\frac{1}{2}}W([2(1-c^2)]^{-\frac{1}{2}}x,y), \end{cases}$$

We get

$$M(c)=\sqrt{2}(1-c^2)^{-\frac{1}{2}}M(\frac{\sqrt{2}}{2}) \tag{3.3}$$

By (3.2)-(3.3)

$$d''(c)=\left[\frac{3r-2}{2(r-1)}c^2(1-c^2)^{-(\frac{r}{2(r-1)}+2)}+(1-c^2)^{-(\frac{r}{2(r-1)}+1)}\right]\left[\sqrt{2}M(\frac{\sqrt{2}}{2})\right]^{\frac{r}{r-1}}$$

Therefore $d''(c)>0$, for any $r>1$.

Lemma 3.3. *$d(c)$ is differentiable and strictly increasing function for $0<c<1$.*

Proof. By (3.2)-(3.3)

$$d(c)=\frac{r-1}{r}[M(c)]^{\frac{r}{r-1}}=\frac{r-1}{r}(1-c^2)^{-(\frac{r}{2(r-1)})}[\sqrt{2}M(\frac{\sqrt{2}}{2})]^{\frac{r}{r-1}},$$

$$d'(c)=c(1-c^2)^{-(\frac{r}{2(r-1)}+1)}[\sqrt{2}M(\frac{\sqrt{2}}{2})]^{\frac{r}{r-1}}>0.$$

□

Lemma 3.4. *Let* $d''(c) > 0$ *and* $0 < c < 1$, *there is a some* $\varepsilon > 0$, *such that* $c_1 > 0$ *and* $|c_1 - c|^2 < \varepsilon$, *then*

$$d(c_1) \geq d(c) + d'(c)(c_1 - c) + \frac{1}{4} d''(c)|c_1 - c|^2. \tag{3.4}$$

Proof. This follows by Taylor's expansion at $c_1 = c$. □

Now define

$$U_{c,\varepsilon} = \left\{ \vec{u} \in Y \middle| \inf_{\vec{\varphi} \in S_c} \|\vec{u} - \vec{\varphi}\|_Y < \varepsilon \right\}$$

Since $d(c)$ is differentiable and strictly increasing for $0 < c < 1$. By $\vec{u} \rightarrow \vec{\varphi}$, $\vec{\varphi} \in S_c$, then

$$c(\vec{u}) = d^{-1}\left(\frac{r-1}{r} K(\vec{u})\right), \tag{3.5}$$

is a C^1 map $c(\vec{u}) : U_{c,\varepsilon} \rightarrow R^+$ for small enough $\varepsilon > 0$ and $\vec{\varphi}_c \in S_c$.

Lemma 3.5. *Suppose* $d''(c) > 0$ *for* $0 < c < 1$, *then there exists* $\varepsilon > 0$ *such that for all* $\vec{u} \in U_{c,\varepsilon}$ *and* $\vec{\varphi}_c \in S_c$,

$$E(\vec{u}) - E(\vec{\varphi}_c) - c(\vec{u})(Q(\vec{u}) - Q(\vec{\varphi}_c)) \geq \frac{1}{4} d''(c)|c(\vec{u}) - c|^2, \tag{3.6}$$

where $c(\vec{u}) = d^{-1}\left(\frac{r-1}{r} K(\vec{u})\right)$, *for* $\vec{u} \in U_{c,\varepsilon}$.

Proof. It follows from (3.2), we have

$$E(\vec{u}) - c(\vec{u})Q(\vec{u}) = I_{c(\vec{u})}(\vec{u}) - \frac{1}{r} K(\vec{u}). \tag{3.7}$$

Since

$$\frac{r}{r-1} d(c(\vec{u})) = K(\vec{u}), \frac{r}{r-1} d(c(\vec{u})) = K(\vec{\varphi}_c(\vec{u})), \vec{\varphi}_c(\vec{u}) \in S_{c(\vec{u})}$$

Then

$$K(\vec{u}) = K(\vec{\varphi}_c(\vec{u}))$$

This implies that

$$I_{c(\vec{u})}(\vec{u}) \geq I_{c(\vec{u})}(\vec{\varphi}_c(\vec{u})). \tag{3.8}$$

And since $\vec{\varphi}_c(\vec{u})$ is a minimizer of $I_{c(\vec{u})}(\vec{u})$ subject to constraint and $c(\vec{u}) \in C^1$, then by (3.7) and Lemma 3.4 we have

$$\begin{aligned} E(\vec{u}) - c(\vec{u})Q(\vec{u}) &\geq I_{c(\vec{u})}(\vec{\varphi}_{c(\vec{u})}) - \frac{r-1}{r}K(\vec{\varphi}_{c(\vec{u})}) \\ &= d(c(\vec{u})) \\ &\geq d(c) + d'(c)(c(\vec{u}) - c) + \frac{1}{4}d''(c)|c(\vec{u}) - c|^2 \\ &\geq E(\vec{\varphi}_c) - c(\vec{u})Q(\vec{\varphi}_c) + \frac{1}{4}d''(c)|c(\vec{u}) - c|^2. \end{aligned} \tag{3.9}$$

Where $d'(c) = -Q(\vec{\varphi}_c)$. □

Now using the above lemmas we can prove the following theorem.

Theorem 3.6. *(Theorem of Stability) Let $r > 1$ and $0 < c < 1$, (φ, ψ) is solution of (1.3), then*

$$\vec{\varphi}(x, y, t) = (\varphi(x - ct, y), \psi(x - ct, y))$$

is a orbital stability solution of (1.1).

Proof. In fact, if it is instability. Then by the definition of stability, $\exists \delta > 0$ and initial data $\vec{U}_k(0) \in U_{t,\frac{1}{k}}$ such that

$$\sup_{t>0} \inf_{\vec{\varphi} \in S_c} \|\vec{u}_k(t) - \vec{\varphi}\|_Y \geq \delta \tag{3.10}$$

where $\vec{u}_k(t)$ is the solution of (1.1) with initial data $\vec{u}_k(0)$.

By continuity in t, we choose the first time t_k such that

$$\inf_{\vec{\varphi} \in S_c} \|\vec{u}_k(t_k) - \vec{\varphi}\|_Y = \delta \tag{3.11}$$

Since $E(\vec{u})$ and $Q(\vec{u})$ are conserved at t and continuous for $\vec{u}$, we find $\vec{\varphi}_k \in S_c$ such that

$$|E(\vec{u}_k(t_k)) - E(\vec{\varphi}_k)| = |E(\vec{u}_k(0)) - E(\vec{\varphi}_k)| \to 0, \ as \ k \to \infty$$

$$|Q(\vec{u}_k(t_k)) - Q(\vec{\varphi}_k)| = |Q(\vec{u}_k(0)) - Q(\vec{\varphi}_k)| \to 0, \ as \ k \to \infty \tag{3.12}$$

Choose δ small enough and by Lemma 3.4

$$E(\vec{u}_k(t_k)) - E(\vec{\varphi}_k) - c(\vec{u}_k(t_k)(Q(\vec{u}_k(t_k)) - Q(\vec{\varphi}_k))) \geq \frac{1}{4}d''(c)|c(\vec{u}_k(t_k)) - c|^2 \tag{3.13}$$

By (3.11)

$$\begin{aligned} \|\vec{u}_k(t_k)\|_Y &\leq \|\vec{\varphi}_k)\|_Y + 2\delta \\ &\leq (2 - \frac{2}{c^2})I_c(\vec{\varphi}_k) + 2\delta \\ &\leq C[M(c)]^{\frac{r}{r-1}} + 2\delta \\ &< \infty \end{aligned} \tag{3.14}$$

Since $c(\vec{u})$ is a continuous map, $c(\vec{u}_k(t_k))$ is uniformly bounded for k. By (3.13)

$$c(\vec{u}_k(t_k)) \to c, \quad k \to \infty \tag{3.15}$$

Hence

$$\lim_{k\to\infty} K(\vec{u}_k(t_k)) = \lim_{k\to\infty} \frac{r}{r-1} d(c(\vec{u}_k(t_k))) = \frac{r}{r-1} d(c) \tag{3.16}$$

On the other hand

$$\begin{aligned} I_c(\vec{u}_k(t_k)) &= E(\vec{u}_k(t_k)) - c(\vec{u}_k(t_k))Q(\vec{u}_k(t_k)) + \frac{1}{r}K(\vec{u}_k(t_k)) \\ &= d(c(\vec{u}_k(t_k))) - (c - c(\vec{u}_k(t_k)))Q(\vec{u}_k(t_k)) + \frac{1}{r}K(\vec{u}_k(t_k)) \end{aligned} \tag{3.17}$$

And since

$$Q(\vec{u}_k(t_k)) = Q(\vec{u}_k(0)) \leq ||\vec{u}_k(t_k)||_Y < \infty.$$

By (3.16)

$$I_c(\vec{u}_k(t_k)) \to d(c) + \frac{1}{r} \cdot \frac{r}{r-1} d(c) = \frac{r}{r-1} d(c), \ as\ k \to \infty \tag{3.18}$$

That is

$$I_c(\vec{u}_k(t_k)) \to I_c(\vec{\varphi}_c) = [M(c)]^{\frac{r}{r-1}} \tag{3.19}$$

Let

$$\begin{aligned} \vec{\zeta}_k(t_k) &= (K(\vec{u}_k(t_k)))^{-\frac{1}{r}} I_c(\vec{u}_k(t_k)) \\ &\to \frac{[M(c)]^{\frac{r}{r-1}}}{([M(c)]^{\frac{r}{r-1}})^{\frac{1}{r}}} \\ &= [M(c)]^{\frac{r}{r-1}} [M(c)]^{-\frac{r}{r-1}} \\ &= [M(c)] \end{aligned}$$

Hence $\vec{\zeta}_k(t_k)$ is a minimizing sequence, therefore exists $\vec{\varphi}_k \in S_c$ such that

$$\lim_{k\to\infty} \|\vec{\zeta}_k(t_k) - ([M(c)]^{-\frac{1}{2(r-1)}}\vec{\varphi}_k)\|_Y = 0 \tag{3.20}$$

where

$$K([M(c)]^{-\frac{1}{2(r-1)}}\vec{\varphi}_k) = 1.$$

$$\begin{aligned}
\lim_{k\to\infty} \|\vec{u}_k(t_k) - \vec{\varphi}_k\|_Y &= \lim_{k\to\infty}[(K(\vec{u}_k(t_k)))^{\frac{1}{2r}}\|(K(\vec{u}_k(t_k)))^{-\frac{1}{2r}}(\vec{u}_k(t_k) - \vec{\varphi}_k)\|_Y] \\
&\leq [M(c)]^{\frac{1}{2(r-1)}}[\lim_{k\to\infty}\|\vec{\zeta}_k(t_k) - ([M(c)]^{-\frac{1}{2(r-1)}}\vec{\varphi}_k)\|_Y] \\
&\quad + \lim_{k\to\infty}\left|([M(c)]^{-\frac{1}{2(r-1)}} - (K(\vec{u}_k(t_k)))^{-\frac{1}{2r}}\right| \|\vec{\varphi}_k\|_Y \\
&= 0
\end{aligned} \tag{3.21}$$

which contradicts with (3.11). □

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