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A Study of Pál-Type Interpolation

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Abstract

In this paper, we consider an interscaled set of nodes, which are the zeros of two different polynomials.Then we obtain the explicit forms of interpolatory polynomials and prove the convergence theorem such pál-type interpolation.

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1 Introduction

In 1975, L.G. Pal [4] introduced a new type of interpolation on the zeros of two different polynomials. He considered two system of real numbers $\{x_k\}_{k=0}^{n}$ and $\{x_k^*\}_{k=0}^{n-1}$, which are the zeros of $W_n(x)$ and $W_n'(x)$ respectively, then there exists a unique polynomial $P(x)$ of degree at most $2n-1$ satisfying the interpolation properties

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(1.1) $\begin{cases} P(x_k) = y_k, & (k = 1, 2, ...n) \\ P'(x_k^*) = y_k', & (k = 1, 2, ...n-1) \end{cases}$ and gave the explicit formula of this polynomial.

Later on many authors have dealt with the above method of interpolation on the various set of nodes *i.e.* on the real line or unit circle. In 2006, M. Lénárd [3] considered the weighted $(0, 2)$ pál-type interpolation problem on the zeros of Legendre polynomial $P_n(x)$ and gave the explicit formulae. In a paper V.Srivastava, N. Mathur, P Mathur [5] have considered a new kind of Pál-type interpolation. In other papers author $[1, 2]$ has also considered $(0, 1; 0)$ and $(0; 0, 1)$ interpolation on the unit circle.

In this paper, we consider two pairwise disjoint sets $\{x_k\}_{k=1}^n$ and $\{y_k\}_{k=1}^n$, which are the zeros of $P_n(x)$ and $\Pi_n(x)$ respectively with two additional conditions *i.e.* the function is also prescribed at ± 1. The Pál-type interpolation on these set of points means the determination of the polynomial, say $Q_n(x)$ of degree $\leq 2n+1$ satisfying the conditions:

(1.2) $\begin{cases} Q_n(x_k) = \alpha_k; & k = 0\,(1)\,n+1 \\ Q_n'(y_k) = \beta_k; & k = 1\,(1)\,n \end{cases}$

where α_k's $(k = 0\,(1)\,n+1)$ and β_k's ($k = 1\,(1)\,n$) are arbitrary real numbers.

In section 2 we give some preliminaries, in section 3 we give explicit representation and in sections 4 &5 estimates and convergence of interpolatory polynomials are respectively given.

2 Preliminaries

We shall use some well known facts about Legendre polynomials.

(2.1) $\Pi_n(x) = (1 - x^2)\, P_{n-1}'(x)$

(2.2) $\Pi_n'(x) = -n\,(n-1)\, P_{n-1}(x)$

(2.3) $(1 - x^2)\, P_n''(x) - 2x P_n'(x) + n\,(n+1)\, P_n(x) = 0.$

(2.4) $(1 - x^2)\, \Pi_n''(x) + n\,(n-1)\, \Pi_n(x) = 0$

(2.5) $H_k(x) = \int_{-1}^{x} x^n \Pi_n(x)\, dx$

(2.6) $\{x^{-n} P_n(x)\}_{x=y_j}' = 0,$ for $j = 1\,(1)\,n.$

We shall require the fundamental polynomials of Lagrange interpolation on zeros of $P_n(x)$ and $\Pi_n(x)$, $i.e$

(2.6) $$L_k(x) = \frac{P_n(x)}{(x-x_k)\,P_n'(x_k)} \quad \text{for } k = 1\,(1)\,n$$

(2.7) $$l_k(x) = \frac{\Pi_n(x)}{(x-y_k)\,\Pi_n'(y_k)} \quad \text{for } k = 1\,(1)\,n\ .$$

For $-1 \le x \le 1$.

(2.7) $$|P_n(x)| \le 1$$

(2.9) $$|\Pi_n(x)| \le \left(\frac{2n}{\pi}\right)^{\frac{1}{2}}$$

Let $x_k = \cos\theta_k \quad (k = 1\,(1)\,n)$ be the zeros of the n^{th} Legendre polynomial P_n, with $1 > x_1 > x_2 > .... > x_n > -1$, then by [6]

(2.10) $$\begin{cases} (1-x_k)^2 \ge k^2 n^{-2}, & k = 1, ..., \left[\frac{n}{2}\right] \\ (1-x_k)^2 \ge (n-k+1)^2\, n^{-2}, & k = \left[\frac{n}{2}\right]+1, ..., n \end{cases}$$

(2.11) $$\begin{cases} |P_n'(x_k)| \ge ck^{-\frac{3}{2}} n^2, & k = 1, ..., \left[\frac{n}{2}\right] \\ |P_n'(x_k)| \ge c(n-k+1)^{-\frac{3}{2}} n^2, & k = \left[\frac{n}{2}\right]+1, ..., n \end{cases}$$

Let $l_k(z)$ be defined in (2.7) , then

(2.12) $$\sum_{k=1}^{2n-1} |l_k(x)| \le c \log n.$$

For more details, one can see [6]

3 Explicit representation of polynomials

We shall write $Q_n(x)$ satisfying (1.2) as

(3.1) $$Q_n(x) = \sum_{k=0}^{n+1} \alpha_k A_k(x) + \sum_{k=1}^{n} \beta_k B_k(x)$$

where $A_k(x)$ and $B_k(x)$ are fundamental polynomials of first and second kind respectively each of degree $\leq 2n+1$, uniquely determined by the following conditions:

For $k = 0\,(1)\,n+1$

$$(3.2) \qquad \begin{cases} A_k(x_j) = \delta_{kj}, & j = 0\,(1)\,n+1, \\ A'_k(y_j) = 0, & j = 1\,(1)\,n. \end{cases}$$

for $k = 1\,(1)\,n$

$$(3.3) \qquad \begin{cases} B_k(x_j) = 0, & j = 0\,(1)\,n+1, \\ B'_k(y_j) = \delta_{kj}, & j = 1\,(1)\,n. \end{cases}$$

Theorem 3.1. *For* $k = 1\,(1)\,n$

$$(3.4) \qquad B_k(x) = x^{-n} P_n(x) \{a_k J_k(x) + b_k H_k(x)\}$$

where

$$(3.6) \qquad J_k(x) = \int_{-1}^{x} x^n l_k(x)\,dx$$

$$(3.7) \qquad a_k = \frac{1}{P_n(x_k)}$$

$$(3.8) \qquad b_k = -a_k \frac{\int_{-1}^{1} x^n l_k(x)\,dx}{\int_{-1}^{1} x^n \Pi_n(x)\,dx}.$$

Proof. Let

$$(3.9) \qquad B_k(x) = x^{-n} P_n(x)\, q(x),$$

where $q(x)$ is a polynomial of degree at most $2n+1$.

One can check that $B_k(x_j) = 0$, for $j = 1\,(1)\,n$.

Similarly from (3.9) ,using the 2^{nd} condition of (3.3) , *i.e.*

$$(3.10) \qquad B'_k(y_j) = \{x^{-n} P_n(x)\}_{x=y_j} q'(y_j) + \{x^{-n} P_n(x)\}'_{x=y_j} q(y_j) = \delta_{kj},$$

using (2.6) , we get

$$(3.11) \qquad y_j^{-n} P_n(y_j)\, q'(y_j) = \delta_{kj}$$

Hence we have

$$(3.12) \qquad q'(x) = x^n \{a_k l_k(x) + b_k \Pi_n(x)\}.$$

On integration , we get (3.4)

Again

$$B'_k(y_k) = a_k \delta_{kj}$$

we get (3.7) .For $x = 1$, we get (3.8) , which completes the proof. □

Theorem 3.2. *For* $k = 1\,(1)\,n$

$$A_k(x) = \frac{(1-x^2)\,\Pi_n(x)}{(1-x_k^2)\,\Pi_n(x_k)} L_k(x) + \tag{3.9}$$
$$+ \frac{x^{-n} P_n(x)}{(1-x_k^2)\,\Pi_n(x_k)\,P'_n(x_k)} \{S_k(x) + c_k H_k(x)\}$$

where

$$S_k(x) = -\int_{-1}^{x} x^n \frac{(1-x^2)\,\Pi'_n(x) + d_k \Pi_n(x)}{(x-x_k)} dx \tag{3.10}$$

with $d_k = -\dfrac{(1-x_k^2)\,\Pi'_n(x_k)}{\Pi_n(x_k)}$

$$c_k = -\frac{\int_{-1}^{1} x^n \frac{(1-x^2)\,\Pi'_n(x) + d_k \Pi_n(x)}{(x-x_k)} dx}{\int_{-1}^{1} x^n \Pi_n(x)\,dx}. \tag{3.11}$$

For $k = 0, n+1$

$$A_0(x) = \frac{x^{-n} P_n(x) \int_{-1}^{x} x^n \Pi_n(x)\,dx}{P_n(1) \int_{-1}^{1} x^n \Pi_n(x)\,dx}, \tag{3.12}$$

$$A_{n+1}(x) = \frac{x^{-n} P_n(x) \int_{x}^{1} x^n \Pi_n(x)\,dx}{P_n(1) \int_{-1}^{1} x^n \Pi_n(x)\,dx}. \tag{3.13}$$

Proof. Let $A_k(x) = \dfrac{(1-x^2)\,\Pi_n(x)}{(1-x_k^2)\,\Pi_n(x_k)} L_k(x) +$

$$+ \frac{x^{-n} P_n(x)}{(1-x_k^2)\,\Pi_n(x_k)\,P'_n(x_k)} \{S_k(x) + c_k H_k(x)\}$$

Obviously $A_k(x_j) = \delta_{kj}$, $j = 1\,(1)\,n$ and $A_k(1) = 0$ gives c_k. From the second condition of (3.2) , we get $S_k(x)$. Similarly one can find $A_k(x)$, for $k = 0, n+1$. □

4 Estimation of Fundamental Polynomials

In this section, we prove the following:

Lemma 4.1. *Let* $A_k(x)$ *be defined in theorem 2, then*

(4.1) $$\sum_{k=1}^{n} |A_k(x)| = O\left(n^{\frac{3}{2}} \log n\right), \quad \text{for } k = 1\,(1)\,n$$

(4.2) $$|A_k(x)| = O\left(n^{-\frac{3}{2}}\right), \ \text{for } k = 0, n+1 \ .$$

Lemma 4.2. *Let* $B_k(x)$ *be defined in theorem 1, then*

(4.1) $$\sum_{k=1}^{n} |B_k(x)| = O\left(n^{\frac{1}{2}} \log n\right), \quad \text{for } k = 1\,(1)\,n.$$

Lemmas 4.1 and 4.2 can be proved owing to conditions $(2.7) - (2.12)$ and the results in [6].

5 Convergence

In this section we prove the following.

Theorem 5.1. *If* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is continuously differentiable function , then*

(5.1) $$Q_n(x) = \sum_{k=0}^{n+1} f(x_k)\, A_k(x), \ \text{for } -1 \leq x \leq 1$$

satisfies the relation

(5.2) $$|Q_n(x) - f(x)| = O\left(n^{\frac{3}{2}} \omega_2\left(f, \tfrac{1}{n}\right) \log n\right).$$

Remark Let $f(x) \in C^r[-1,1]$ and f'$\in$ *Lip* $\alpha, \alpha > \frac{1}{2}$, then sequence $\{Q_n\}$ converges uniformly to $f(x)$, follows from (5.2) provided

(5.3) $$\omega_2\left(f, \tfrac{1}{n}\right) = O\left(n^{-1-\alpha}\right).$$

To prove Theorem 5.1, we shall need the following:

Let $f(x) \in C^r[-1,1]$.Then there exists a polynomial $F_n(x)$ of degree $\leq 2n-2$ satisfying inequality :

(5.4) $\quad |f(x) - F_n(x)| \leq c\omega_2\left(f, \frac{1}{n}\right).$

Proof. Theorem 5.1 Using $(5.1), (5.3) - (5.4)$ and Lemmas 4.1 and 4.2, we get (5.2). □

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