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Efficiency and Extensions in Infinite Dimensional Ordered Vector Spaces

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Abstract

This research paper is focused on the common concepts of the efficiency and set-valued map. After a short introduction, we propose some questions regarding the notion of efficiency and we emphasize the Pareto optimality as one of the first finite dimensional illustrative examples. We present the efficiency and the multifunctions in the infinite dimensional ordered vector spaces following also our recent results concerning the most general concept of approximate efficiency, as a natural generalization of the efficiency, with implications and applications in vector optimization and the new links between the approximate efficiency, the strong optimization - by the full nuclear cones - and Choquet's boundaries by an important coincidence result. In this way, the efficiency is strong related to the multifunctions and Potential theory through the agency of optimization and conversely. Significant examples of Isac's cones and several pertinent references conclude this study.

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1 Introduction

Throughout in our *Life*, *the Efficiency* was, it is and it will remain *essential* for *the Existence* and not only. Its mathematical models were unanimously accepted in various *knowledge fields*. *Also, we propose it as a New Frontier in Computational Science and Engineering - The CSE Program introduced in 1997 by Professor Rolf Jeltsch from Swiss Federal Institute of Technology Zürich - in the actual World context of priorities concerning the Alternative Energies, the Climate Exchange and the Education*. To it we dedicate our modest scientific contribution. The content of this research work is organized as follows: *Section 2* is dedicated to some useful matters on the efficiency. We briefly present in *Section 3 the Pareto Optimality* as one of the main starting points for the mathematical modelling of the *Economical Efficiency*. In *Section 4*, conceived as a concise Survey, we present *the most general notion* of *Approximate Efficiency* and its particular case of the usual *Efficiency*, with the immediate connections to *Multifunctions*, in *the Infinite Dimensional Ordered Vector Spaces*, completed by *the coincidence* between *Choquet's Boundaries and the Approximate Efficient Points Sets in Ordered Hausdorff Locally Convex Spaces*, this conclusion being based on *the first result established by us concerning such a property as this for Pareto type efficient points sets and the corresponding Choquet boundaries*. Our results represent *strong relationships* between *important Great Fields of Mathematics* and the *Human Knowledge*: *Vector and Strong Optimization*, *Set-valued Maps*, *the Axiomatic Theory of Potential together with* its *Applications* *and the Human Efficiency*. Finaly, we indicate the selected bibliography which refers only to the papers which were used. Sections 2, 3 and a part of Section 4 are presented following [74].

2 Some selected questions on the Efficiency

First of all, we present a short survey on the efficiency.The current language defines the efficiency as *"the ability to produce the desired effect in dealing with any problem"*. In the actual world characterized by: *globalization which has generated efficiency gains*, *liberalisation, individualization, informatization, informalization* ([78] and so on), the efficiency is perceived as follows: "working well, quickly and without waste". But, our life has its *Divine efficiency* and the projection in the reality deals with a lot of kind of descriptions: *information efficiency, energy efficiency, eco - efficiency* (see, as recent references, [42], [43], [50] and so on), home energy efficiency, water efficiency market enhancement programs, sports and efficiency, economic efficiency (agricultural and industrial efficiency, human efficiency in business, efficiency in financial and betting markets, efficiency in capital asset pricing models, etc.), efficiency in Mathematics (efficient algorithms in computational complexity, efficiency frontier in data envelopment analysis, statistics and so on, efficiency in multi – objective, stochastic and goal programming, efficiency in mathematical economics, etc.), efficiency in medicine, technical efficiency, etc., all of them being based on the fundamental question: *"whose valuations do we use, and how shall they be weighted ?"*[28]. Thus, the economic efficiency is characterized by the "*optimal*" relationships between the value of the ends (the physical outputs) and the value of the means (the physical inputs), both of them "measured" by the money. Hence, the monetary evaluations are essential for the economic efficiency which usually is associated with the sustainability defined as *"development that meets the needs* *of the present without compromising the ability of future generations to meet their own needs" to achieve a good society* ([11], p.43). However, the goals of efficiency and sustainability might not be enough to ensure a positive social development because the economic efficiency does not include any component of it (see, for example, the distribution of goods) ([54], p.13) and the sustainability was not very clear defined [77]. In terms of the technical efficiency, a machine is considered to be useful or *"more efficient than another" when it generates "more work output per unit of energy input"*. But, it is not possible to speak about complete efficiency in any process since we have not developed yet workable procedures concerning the corresponding evaluations (see, for instance, the urban automobile traffic which engages different people from many points of view; in general, the transport systems are highly complex with different goals often conflict). In fact, in all the processes, we try to approximate the real efficiency by some kinds of fuzzy or relative efficiency in order to obtain several controls on it, taking into account its complexity. For example, one of the main purposes of the data envelopment analysis is to *"to measure"* the relative efficiency and the comparison of decision making units by the estimation of *"the distance"* between the evaluated real units and the virtual units (see, for instance, [42]). Another strong argument and a significant example in this direction is represented by *the permanent measure and the continuous supervision of the interest rate risk to obtain the optimal efficiency balance in the management of banks' assets and liabilities* (see, for example, the pertinent asset liability management model together with the simulation analysis given in [43]). Generally speaking, from the decisional point of view, the main steps to obtain the efficiency are the following:

2.1. *Wording of the problems by the managerial staff in an adequate language for mathematicians and computer scientists because the "bilateral" dialogue is absolutely necessary to solve the programs through the agency of such as these cooperations.*

2.2. *The elaboration of the appropriate mathematical models and the* *corresponding numerical processings*.

2.3. *The selection , by a serious study, of the best multicriteria decisions.*

2.4. *The application of them.*

2.5. *The evaluation of the efficiency.*

We must remark that the multicriteria decision aid ([20], [78]) and the decision making under uncertainty viewed as an important area of decision making and recognized as *" the fact that in certain situations a person does not have the information which quantitatively and qualitatively is appropriate to describe, prescribe or predict determinastically and numerically a system, its behaviour or other characteristics",* that is,*" it relates to a state of the human mind"* characterized by the *"lack of complete knowledge about something"* [89] represents the real risk for the efficiency which, in our opinion, generates it. Following [89], we remember that the term of *"risk"* was initially applied for *"the situations in which the probabilities of outcomes were objectively known"*. In accordance with [22] and [79] now this concept *"means a possibility of something bad happening"* and the uncertainty *"is applied to the problems in which real alternatives with several possible outcomes exist"*. Another important matter is the efficacy defined as *"the quality of being efficacious"* in the sense of *"producing the desired effects or results"*. In our opinion, this means *to be efficient step by step*, that is, *a discrete efficiency* with *appropriate links* between *the stages*.

3 Pareto Optimality: an illustrative example of efficiency

Not even for the market economies there exists no an universal mathematical model. Pareto efficiency or Pareto optimality, the term being named for an Italian economist Vifredo Pareto [56], is a central theory in economics with broad applications in game theory, engineering and the social sciences. Pareto efficiency is important because it provides a weak but widely accepted standard for comparing the economic outcomes.It's a weak standard since there may be many efficient situations and Pareto's test doesn't tell us how to choose between them. Any policy or action that makes at least one person better off without hurting anyone is called *a Pareto improvement*. From the Mathematics point of view, the Pareto efficiency represents the actual finite dimensional part of the multiobjective programming in vector optimization. Thus, whenever a feasible deviation from a genuine solution S of an arbitrary multiobjective programme generates the improvement of at least one of the objectives while some other objectives degrade, any such a solution S as this is called efficient or nondominated. A system in economics and in politics is called Pareto efficient whenever "no individual can be made better off without another being made worse off" that is, a social state is economically efficient, or Pareto optimal, provided that "no person in society can become better off without anyone else becoming worse off". This characteristic of Pareto type efficiency has been pointed out in [1] - [3], [19], [25], [56], [82] and the others. In terms of the alternative allocations this means that given a set of alternative allocations and a set of individuals, any movement from one alternative allocation to another that can make at least one individual better off, without making any other individual worse off is called a Pareto improvement or a Pareto optimization. An allocation of resources is named *Pareto efficient* or *Pareto optimal* whenever *no further Pareto improvements can be made*. If the allocation is *strictly preferred* by one person and no *other allocation would be as good for everyone*, then it is called *strongly Pareto optimal*. A *weakly Pareto optimal allocation* is one where *any feasible reallocation would be strictly preferred by all agents* [89]. Consequently, *Pareto type efficiency* is *an important approximate criterion* for *evaluating the economic systems* and *the political policies*, with *minimum assumptions on the interpersonal comparability*. We said *"approximate criterion"* because it asks *"the ideal"* which may not reflect, for example, the workings of real economics, thanks to the folowing restrictive assumptions necessary for the existence of Pareto efficient outcomes: the markets exist for all possible goods, being perfectly competitive and the transaction costs are negligible. In the political policies, not every Pareto efficient outcome is regarded as desirable (see, for instance, the strategies based on the unilateral benefits). For these reasons, Pareto optimality was accepted with some or much uncertainty and controversy, but, by the Arrow's renowned impossibility theorem given in 1951 according to which *"no social preference ordering based on individual orderings only could satisfy a small set of very reasonable conditions"*- the Pareto criterion being one of them, it remains *"plausible and uncontroversial"* [78]. Usually, the concept of efficiency replaces the notion of optimality in multiple criteria optimization because *whenever the solutions of a multiple – objectives program exist in Pareto'sense, they cannot be improved following the ordering induced by the cone*.

4 Approximate Efficiency, Efficiency and Multifunctions in Infinite Dimensional Ordered Vector Spaces with Recent Connected Results

Seemingly, the concept of efficiency is equivalent to the optimality, as we can see from the next abstract construction. In reality, the optimality represents a particular case of the efficiency, that is, "the best approximation" of all the efficient points. Let X be a real or complex ordered vector space, let $\mathcal{K}$ be the class of all convex cones defined on X and let A be an arbitrary, non-empty subset of X. Following the next considerations, we consider that the set of all efficient points of A with respect to an arbitrary $K \in \mathcal{K}$ is in the following relation with the "vectorial" minimization or maximization:

$$Eff(A) \supseteq \bigcup_{K \in \mathcal{K}} MIN_K(A) \cup \bigcup_{K \in \mathcal{K}} MAX_K(A)$$

Clearly, any vector optimization program (which has its origin in the usual ordered Euclidean spaces programs thanks to Pareto optimality of vector-valued real functions) includes the corresponding strong optimization program and allows to be described as one of the next optimization problems. Moreover, it is possible to replace the convex cone K by any other convenient, non-empty subset of X:

$$(P_{T,K}): MIN_K f(T) \quad \text{or} \quad (P_{T,K}): MAX_K f(T)$$

where $K \in \mathcal{K}$, T is a given non-empty set and $f: T \to X$ is an appropriate application. If one denotes by $S_f(T,K)$ the corresponding set of solutions, then the announced equivalence seems to be justified by the following relation which justifies the optimization of the efficiency and the efficiency of the optimization:

$$\bigcup_{\emptyset \neq A \subseteq X} Eff(A) = \bigcup_{\substack{T \neq \emptyset \\ K \in \mathcal{K}, f: T \to X}} S_f(T,K)$$

but, in reality, only the next inclusion is valid :

$$\bigcup_{\emptyset \neq A \subseteq X} Eff(A) \supseteq \bigcup_{\substack{T \neq \emptyset \\ K \in \mathcal{K}, f: T \to X}} S_f(T,K)$$

thanks to the refinement of the notion of efficiency which can not be totally described for the moment by mathematical means, that is, the final agreement between “the dictionary’s concept of efficiency” and “the corresponding notion in mathematics” was not signed yet.

Now, let X be a non-empty set, let E be a vector space ordered by a convex, pointed cone K and let $f: X \to E$ be a function. We consider the next Vector Optimization Problem:

$$(P) \quad \begin{cases} \min f(x) \\ x \in X \end{cases},$$

To solve it means to identify all the efficient points $x_0 \in X$ in the sense that $f(X) \cap [f(x_0) - K] = \{f(x_0)\}$. In all these cases, x_0 is *an efficient solution* and $f(x_0)$ is called *a nondominated point* of the program (P) for Multicriteria

Optimization Programs in Finite Dimensional Vector Spaces. If one replaces K by K\{0} or int(K), then x_0 is called strictly(weakly) efficient for X, respectively, and $f(x_0)$ is named strictly (weakly) nondominated. Consequently, whenever E is an usual ordered Euclidean space, the above concepts for x_0 means *Pareto optimal*, *strictly Pareto optimal* and *weakly Pareto optimal solution*, respectively, with the corresponding nondominated concepts for *f* (see, for a recent instance, [21]). So, concerning the practical vector - optimization problems, it is important to know when the set of all efficient points is non-empty, to establish its main properties (existence, domination, connectedness, compactness, density in varied topologies, etc.) and to extend the concepts together with the results to multicriteria optimization in infinite dimensional ordered vector spaces. The proper efficiency introduced in [47] and developped in [5] – [7], [9], [10], [18], [24], [26], [27], [37], [40], [53], etc. appears as a refined case of the efficiency, that is, the set of all properly efficient solutions and the set of all positive proper efficient solutions of problem (P) are subsets of the set containing all the efficient points and the study of them has been proposed in order to eliminate some "undesirable" efficient solutions.

We recall that $x_o \in X$ is *a properly efficient solution* of (P) if it is *an efficient point* and $cl\left[cone(f(X)+K-\{f(x_o)\})\right] \cap K = \{0\}$; x_o is a positive proper efficient solution of (P) if there exists a linear continuous functional φ on E such that $\varphi(k) > 0$ for every $k \in K$ and $\varphi\left[f(x_o)\right] \leq \varphi\left[f(x)\right]$ for all $x \in X$. This section deals with a new generalization of the efficiency, named by us the approximate efficiency, in ordered Hausdorff locally convex spaces. All the elements concerning the ordered topological vector spaces used here are in accordance with [57].

Let E be a vector space ordered by a convex cone K, K_1 a non-void subset of K and A a non-empty subset of E. The following definition introduces a new concept of approximate efficiency which generalizes the well known notion of Pareto efficiency.

Definition 4.1.[73]. *We say that* $a_0 \in A$ *is a* $K_1 -$*Pareto (minimal) efficient point of* A, *in notation,* $a_0 \in eff\left(A,K,K_1\right)\left(or\ a_0 \in MIN_{K+K_1}\left(A\right)\right)$ *if it satisfies one of the following equivalent conditions:*

(i) $A\bigcap\left(a_0 - K - K_1\right) \subseteq a_0 + K + K_1$;

(ii) $\left(K + K_1\right)\bigcap\left(a_0 - A\right) \subseteq -K - K_1$;

In a similar manner one defines the Pareto (maximal) efficient points by replacing $K + K_1$ with $-\left(K + K_1\right)$. Clearly,

$A\bigcap\left(a_0 - K\right) \subseteq a_0 + K_1 \Rightarrow$ $A\bigcap\left(a_0 - K - K_1\right) \subseteq a_0 + K + K_1 \Rightarrow$ $A\bigcap\left(a_0 - K_1\right) \subseteq a_0 + K$,

which suggests other concepts for the approximate efficiency in ordered linear spaces.

Remark 4.1. The notion of *approximate efficiency* in the sense of the above definition is *the most general notion* of *approximate efficiency* introduced until now. We also remark that $a_0 \in eff\left(A,K,K_1\right)$ iff it is a *fixed point* for *the multifunction* $F : A \to A$ defined by

$$F\left(t\right) = \left\{a \in A : A\bigcap\left(a - K - K_1\right) \subseteq t + K + K_1\right\}.$$

Consequently, for the existence of the Pareto type efficient points we can apply appropriate fixed points theorems for multifunctions (see, for instance, [13], [55], [88] and any other proper scientific papers) and we need the next usual continuity properties of multifunctions. Namely, if X and Y are two topological spaces and $f : X \to 2^Y$ is a set-valued map, then: f is *upper semicontinuous (usc)* if for any $x \in X$ and any open set $D \supseteq f(x)$ we have $D \supseteq f(x_1)$ for all x_1 in some neighbourhood $V(x)$ of x. The multifunction f is *lower semicontinuous (lsc)* if for any $x \in X$ and every open set D with $f(x) \cap D \neq \varnothing$ it follows that

$f(x_1) \cap D \neq \varnothing$ for all x_1 from some neighbourhood $V(x)$ of x. The map f is *continuous* iff it is both usc and lsc. Whenever *the graph* of f defined by $gr(f) = \{(x,y) \in X \times Y : y \in f(x)\}$ is a *closed (open)* set of $X \times Y$ one says that f has a closed (open) graph. Any multifunction having a closed graph is also called *closed*. If $f(x)$ is a closed (compact) subset of Y for any $x \in X$, then f is *a closed-valued (compact-valued)* map. The set-valued map f is called *compact* if $im(f) = f(X) = \bigcup_{x \in X} f(x)$ is contained in a compact subset of Y. A topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish (see for example, the contractible spaces; in particular, the convex sets and the star-shaped sets are acyclic). The multifunction f is *acyclic* if it is usc and $f(x)$ is non-empty, compact and acyclic for all $x \in X$. The next theorem is useful to establish the existence of the (approximate) efficient points taking into account the above mentioned connection with the fixed points of multifunctions.

Theorem 4.1. [55]. *If A is any non-empty convex subset of an arbitrary Hausdorff separated locally convex space and $F : A \to 2^A$ a compact acyclic multifunction, then F has a fixed point, that is, there exists $a_0 \in A$ such that $a_0 \in F(a_0)$.*

Remark 4.2. In [52] it was proved that whenever $K_1 \subset K \setminus \{0\}$, the existence of this new type of efficient points for bounded from below sets characterizes the semi -Archimedian ordered vector spaces and the regular ordered locally convex spaces.

Remark 4.3. When K is pointed, that is, $K \cap (-K) = \{0\}$, then $a_0 \in eff(A, K, K_1)$ means that $A \cap (a_0 - K - K_1) = \varnothing$ or, equivalently, $(K + K_1) \cap (a_0 - A) = \varnothing$ for $0 \notin K_1$ and $A \cap (a_0 - K - K_1) = \{a_0\}$, respectively, if $0 \in K_1$. Whenever $K_1 = \{0\}$, from Definition 1 one obtains the usual concept of *efficient* (Pareto minimal, optimal or admissible) point :

$$a_0 \in eff(A,K) \left(or\ a_0 \in MIN_K(A) \right)$$

if it fulfils *(i), (ii)* or any of the next equivalent properties:

(iii) $(A+K) \cap (a_0 - K) \subseteq a_0 + K$;

(iv) $K \cap (a_0 - A - K) \subseteq -K$

These relations show that a_0 is *a* fixed point for at least one of the following multifunctions:

$$F_1 : A \to A,\ F_1(t) = \{\alpha \in A : A \cap (\alpha - K) \subseteq t + K\},$$

$$F_2 : A \to A,\ F_2(t) = \{\alpha \in A : A \cap (t - K) \subseteq \alpha + K\},$$

$$F_3 : A \to A,\ F_3(t) = \{\alpha \in A : (A + K) \cap (\alpha - K) \subseteq t + K\},$$

$$F_4 : A \to A,\ F_4(t) = \{\alpha \in A : (A + K) \cap (t - K) \subseteq \alpha + K\},$$

that is, $a_0 \in F_i(a_0)$ for some $i = \overline{1,4}$. If, in addition, K is pointed, then $a_0 \in A$ is an efficient point of A with respect to K if and only if one of the following equivalent relations holds:

(v) $A \cap (a_0 - K) = \{a_0\}$;

(vi) $A \cap (a_0 - K \setminus \{0\}) = \varnothing$;

(vii) $K \cap (a_0 - A) = \{0\}$;

(viii) $(K \setminus \{0\}) \cap (a_0 - A) = \varnothing$;

(ix) $(A+K) \cap (a_0 - K \setminus \{0\}) = \varnothing$.

and we notice that $eff(A,K) = \bigcap_{\{0\} \neq K_2 \subseteq K} eff(A,K,K_2)$. Moreover, $a_0 \in eff(A,K)$ iff it is *a critical (equilibrium) point* ([33], [34], [69], [72]) for the *generalized dynamical system* $\Gamma : A \to 2^A$ defined by $\Gamma(a) = A \cap (a - K),\ a \in A$. Thus,

$eff(A,K)$ describes the moments of equilibrium for Γ and *the ideal equilibria* are contained in this set. In particular, this kind of critical points generates the special class of critical boundaries for dynamical systems represented by Pareto type boundaries.Taking $K_1 = \{\varepsilon\}$ ($\varepsilon \in K \setminus \{0\}$), it follows that $a_0 \in eff(A,K,K_1)$ iff $A \cap (a_0 - \varepsilon - K) = \varnothing$. In all these cases, the set $eff(A,K,K_1)$ is denoted by $\varepsilon - eff(A,K)$ and it is obvious that $eff(A,K) = \bigcap_{\varepsilon \in K \setminus \{0\}} [\varepsilon - eff(A,K)]$. Concerning existence results on the efficient points and significant properties for the efficient points sets we refer the reader to [12], [33] – [39], [48], [49], [52], [53], [61] – [72], [81], [85], [86] for a survey. The following theorem offers the first immediate connection between the strong optimization and this kind of approximate efficiency, in the environment of the ordered vector spaces.

Theorem 4.2. [41]. *If we denote by* $S(A,K,K_1) = \{a_1 \in A : A \subseteq a_1 + K + K_1\}$ *and* $S(A,K,K_1) \neq \varnothing$, *then* $S(A,K,K_1) = eff(A,K,K_1)$.

Remark 4.4. We shall denote by $S(A,K)$ the set $S(A,K,\{0\})$. If $S(A,K,K_1) \neq \varnothing$, then $K + K_1 = K$, hence $eff(A,K,K_1) = eff(A,K)$. Indeed, let $a \in S(A,K,K_1)$. Then, $a \in a + K + K_1$ which implies that $0 \in K + K_1$. Therefore, $K \subseteq K_1 + K + K = K_1 + K \subseteq K$.

The above theorem shows that, for any non-empty subset of an arbitrary vector space, the set of all strong minimal elements with respect to any convex cone through the agency of every non-noid subset of it coincides with the corresponding set of the efficient points, whenever there exists at least a strong minimal element. Obviously, the result remains valid for the strong maximal elements and the corresponding efficient points, respectively.

Using this conclusion and the abstract construction given in [59] and [66] for the splines in the H-locally convex spaces introduced in [75] as separated locally convex spaces $(X, P = \{p_\alpha : \alpha \in I\})$ with any semi- norm p_α ($\alpha \in I$)satisfying the parallelogram law:

$$p_\alpha^2(x+y) + p_\alpha^2(x-y) = 2[p_\alpha^2(x) + p_\alpha^2(y)]$$

whenever $x, y \in X$, linear topological spaces also studied in [45], it follows that the only best simultaneous and vectorial approximation for each element in the direct sum of any (closed) linear subspace and its orthogonal, with respect to any linear (continuous) operator between two arbitrary H-locally convex spaces, is its spline function. We also note that it is possible to have $S(A, K, K_1) = \varnothing$ and $eff(A, K, K_1) = A$. Thus, for example, if one considers $X = R^n (n \in N, n \geq 2)$ endowed with the separated H - locally convex topology generated by the semi - norms $p_i : X \to R_+$, $p_i(x) = |x_i|$, $\forall$ $x = (x_i) \in X$, $i = \overline{1,n}$, $K = R_+^n$, $K_1 = \{(0,...,0)\}$ and for each real number c we define $A_c = \left\{(x_i) \in X : \sum_{i=1}^{n} x_i = c\right\}$, then it is clear that $S(A_c, K, K_1)$ is empty and $eff(A_c, K, K_1) = A_c$.

At the same time, in the usual real linear space of all sequences, ordered by the convex cone

$$K = \left\{(x_n) : n \in N^*, n \geq 2, \ x_n \geq 0, \ \forall n \geq 2\right\},$$

for $A = \left\{(x_{n\alpha}) : n \in N^*, \ n \geq 2, \ \alpha > 0\right\}$

with

$$x_{n\alpha} = (n-1)^{-\alpha} - n^{-\alpha}, \ n \in N^*, \ n \geq 2, \ \alpha > 0 \quad \text{and} \quad K_1 = \{(0, 0, ...)\}$$

we have

$$Eff(A, K, K_1) = S(A, K, K_1) = \varnothing.$$

In all our further considerations we suppose that X is *a Hausdorff Locally Convex Space* having *the topology* induced by a family $P = \{p_\alpha : \alpha \in I\}$ of *semi-norms*, *ordered* by *a convex cone* K and *its topological dual space* X^*. In this framework, the next theorem contains a significant criterion for the existence of the approximate efficient points, in particular, for the usual efficient points, taking into account that *the dual cone* of K is defined by $K^* = \{x^* \in X^* : x^*(x) \geq 0,\ \forall x \in K\}$ and *its attached polar cone* is $K^0 = -K^*$. The cone K is called *Isac's cone* (*supernormal or nuclear in* [33], [34]) if for every seminorm $p_\alpha \in P$ there exists $f_\alpha \in X^*$ such that $p_\alpha(k) \leq f_\alpha(k)$ for all $k \in K$ .Moreover, if $\varphi : P \to K^*$ is a function, then the convex cone $K_\varphi = \{x \in X : p_\alpha(x) \leq \varphi(p_\alpha)(x), \forall p_\alpha \in P\}$ is the *full nuclear cone* associated to K, P and φ [39]. A characterization of the supernormality by the full nuclearity is gven in the next Remark 7.

Theorem 4.3. [41]. *If A is any non-empty subset of X and* K_1 *is every non-void subset of K, then* $a_0 \in \mathit{eff}(A, K, K_1)$ *whenever for each* $p_\alpha \in P$ *and* $\eta \in (0,1)$ *there exists* x^* *in the polar cone* K^0 *of* K *such that* $p_\alpha(a_0 - a) \leq x^*(a_0 - a) + \eta, \quad \forall a \in A$.

Remark 4.5. The above theorem represents an immediate extension of Proposition 1.2 in [76]. Generally, the converse of this theorem is not valid at least in partially ordered separated locally convex spaces as we can see from the example considered in Remark 4.4. Indeed, if one assumes the contrary in the corresponding mathematical background then, taking $\eta = \frac{1}{4}$, it follows that for each $\lambda_0 \in [0,1]$, there exists $c_1, c_2 \leq 0$ such that

$$|\lambda_0 - \lambda| \leq (c_1 - c_2)(\lambda_0 - \lambda) + \frac{1}{4}, \forall \lambda \in [0,1].$$

Taking $\lambda_0 = \frac{1}{4}$ one obtains $|1 - 4\lambda| \leq (c_1 - c_2)(1 - 4\lambda) + 1, \forall \lambda \in [0,1]$ which for

$\lambda = 0$ implies that $c_2 \le c_1$ and for $\lambda = \frac{1}{2}$ leads to $c_1 \le c_2$, that is, $|1-4\lambda| \le 1, \forall \lambda \in [0,1]$, a contradiction.

Remark 4.6. If $a_0 \in A$ and for every $p_\alpha \in P$, $\eta \in (0,1)$ there exists $x^* \in K^0$ such that $p_\alpha(a_0\text{-}a) \le x^*(a_0\text{-}a)+\eta$, $\forall a \in A$, then $K \cap (a_0 - A) = \{0\}$ even if K is not pointed. Indeed, if $x \in K \cap (a_0 - A) = \{0\}$, then $a_0 - x \in A$ and for each $p_\alpha \in P$ and $\eta \in (0,1)$ there exists $x^* \in K^0$ with

$$p_\alpha(x) = p_\alpha(a_0\text{-}(a_0\text{-}x)) \le x^*(x)+\eta \le \eta .$$

Because η is arbitrarily chosen in *(0,1)*, we obtain $p_\alpha(x)=0$ and since X is separated it follows that $x=0$. If $0 \in K+K_1$, then $K+K_1=K$ and $0 \notin K+K_1$ implies that $(K+K_1) \cap (a_0 - A)=\varnothing$. Consequently, $a_0 \in eff(A,K,K_1)$ in both cases and in this way we indicated also another proof of the theorem.The beginning and the considerations in Section 4 of [39] suggested us to consider for each function $\varphi : P \to K^* \setminus \{0\}$ *the full nuclear cone*

$$K_\varphi = \{x \in X : p_\alpha(x) \le \varphi(p_\alpha)(x), \forall p_\alpha \in P\}$$

in order to give the next generalization of Theorem 7 indicated in [38].

Theorem 4.4. [41]. *If* $0 \in K_1$ *and there exists* $\varphi : P \to K^* \setminus \{0\}$ *with* $K \subseteq K_\varphi$, *then*

$$eff(A,K,K_1) = \bigcup_{\substack{a \in A \\ \varphi \in P \to K^* \setminus \{0\}}} S\left(A \cap (a - K - K_1), K_\varphi\right)$$

for any non-empty subset K_1 *of K.*

Remark 4.7. If $0 \notin K_1$, then $a_0 \in \text{eff}(A,K,K_1)$ implies that $A \cap (a_0 - K\text{-}K_1)=\varnothing$. Therefore, it is not possible to have $a_0 \in S(\varnothing, K_\varphi)$.In case of $0 \in K_1$, then $\text{eff}(A,K,K_1)=\text{eff}(A,K)$ and $a_0 \in \text{eff}(A,K)$ iff $A \cap (a_0 - K)=\{a_0\}$, so in the right member of the first proved inclusion it can be selected any convex cone, not necessary K_φ.The hypothesis $K \subseteq K_\varphi$ imposed upon the convex cone K is automatically satisfied whenever K is an *Isac's* (nuclear or supernormal) cone ([33] – [37], [39]) and it was used only to prove the inclusion

$$eff\left(A,K,K_1\right) \subseteq \bigcup_{a\in A,\varphi:P\to K^*\setminus\{0\}} S\left(A\bigcap\left(a-K-K_1\right),K_\varphi\right).$$

Moreover, K is an Isac's cone if and only if there exists $\varphi : P \to K^* \setminus \{0\}$ such that $K \subseteq K_\varphi$. Indeed, Lemma 5 of [39] ensures the necessity of the above inclusion condition. Conversely, since for every seminorm $p_\alpha \in P$ there exists $\varphi(p_\alpha) \in K^* \setminus \{0\}$ and for any $x \in K \subseteq K_\varphi$ it follows that $p_\alpha(x) \le \varphi(p_\alpha)(x)$, we conclude the nuclearity of K . When K is an arbitrary pointed convex cone, A is a non-empty subset of X and $a_0 \in eff\left(A,K\right)$, then, by virtue of (v) in Remark 3, we have $A\bigcap\left(a_0-K\right)=\left\{a_0\right\}$, that is, $A\bigcap\left(a_0-K\right)-a_0=\left\{0\right\}\subset K_\varphi$. Hence, $a_0 \in S\left(A\bigcap\left(a_0-K\right),K_\varphi\right)$ for every mapping $\varphi: P\to K^*\setminus\{0\}$ and the next corollary is valid.

Corollary 4.1. *For every non-empty subset A of any Hausdorff locally convex space ordered by an arbitrary, pointed convex cone K with its dual cone* K^* *we have*

$$eff\left(A,K\right)=\bigcup_{\substack{a\in A\\ \varphi:P\to K^*\setminus\{0\}}} S\left(A\bigcap\left(a-K\right),K_\varphi\right)$$

Remark 4.8. The hypothesis of Theorem 4.4 together with Lemma 3 of [39] involves K to be pointed. Consequently, $0 \in K_1$ iff $0 \in K+K_1$. If $a_0 \in S(A\bigcap(a\text{-}K\text{-}K_1),K_\varphi)$ for some $\varphi: P\to K^*$ and $a\in A$ with $a_0=a-k-k_1$, $k\in K$, $k_1\in K_1$, then $K\bigcap(a_0-A)=\{0\}$ because $A\bigcap(a-K-K_1)\subseteq a_0+K_\varphi$ in any such a case as this. Indeed, let $x\in K\bigcap(a_0-A)$ be an arbitrary element.

Then, $a_0 - x \in A$ and $a_0 - x = a-k-k_1 - x \in a-K-K_1$.Therefore, $a_0 - x \in a_0 + K_\varphi$, that is, $x \in K_\varphi$. For every $p_\alpha \in P$ we have $p_\alpha(-x) \leq \varphi(p_\alpha)(-x) = -\varphi(p_\alpha)(x) \leq 0$. Since p_α was arbitrary chosen in P and X is a Hausdorff locally convex space, it follows that $x = 0$.

Remark 4.9. Clearly, the announced theorem represents a significant result concerning the possibilities of scalarization for the study of some Pareto efficiency programs in separated locally convex spaces, as we can see also in the final comments of [39] for the particular cases of Hausdorff locally convex spaces ordered by closed, pointed and normal cones.

Remark 4.10. As an open problem, it is interesting to replace K_1 with any non-empty subset of an ordered linear space *X*, under proper hypotheses.

Remark 4.11. It is well known that the Choquet boundary represents a basic concept in the axiomatic theory of potential and its applications and the efficiency is a fundamental notion in vector optimization. The main aim of the last part in this section is to indicate the recent generalization of our coincidence result established in [12] between the set of all Pareto type minimum points of any non-empty, compact set in an ordered Hausdorff locally convex space and the Choquet boundary of the same set with respect to the convex cone of all real, increasing and continuous functions defined on the set, using our new concept of approximate efficiency. Following this line, firstly the Choquet boundary concept is revised in an original manner.

Let us consider an arbitrary Hausdorff locally space (E,τ), where τ denotes its topology and let K be any closed, convex, pointed cone in E. The usual order relation $\leq_K$ associated with *K* is defined by $x \leq_K y$ ($x, y \in E$) if there exists $k \in K$ with $y = x + k$. Clearly, this order relation on E is closed, that is, the set G_K given by $G_K = \{(x,y) \in E \times E : x \leq_K y\}$ is a closed subset of $E \times E$

endowed with the induced product topology. If S is any convex cone satisfying the properties:

a) $\forall x \in X,\ \exists s \in S,\ \ s > 0$, and $s(x) < +\infty$;

b) S linearly separates $X_1 = \{x \in X:\ \exists s \in S \ \ \text{with}\, s(x) < 0\}$, that is, for every $x, y \in X_1,\ x \neq y$, there exists $s,t \in S$ with real values in x and y such that $s(x)t(y) \neq s(y)t(x)$, then, on the set $M_+(X)$ of all positive Radon measures defined on X, one associates the following natural pre-order relation: if $\mu, \upsilon \in M_+(X)$, then $\mu \leq_S \upsilon$ means that $\mu(s) \leq \upsilon(s)$ for all $s \in S$.

Let S_1 be the convex cone of all lower semicontinuous and bounded from below real functions s on X having the next property: if $x \in X$ and $\mu \leq_S \varepsilon_x$, where $\varepsilon_x(f) = f(x)$ for every real continuous function f on X denotes the Dirac measure, implies that $\mu(s) \leq s(x)$. Any non-empty subset $T \subseteq X$ will be called S - *boundary* if, whenever $s \in S_1$ and its restriction on T denoted by $s_{/T}$ is positive, it follows that $s \geq 0$. The smallest, closed S - boundary is usually called *the Silov boundary of X with respect to S.*

A closed set $A \subseteq X$ is called S - *absorbent* if $x \in A$ and $\mu \leq_S \varepsilon_x$ implies that $\mu(X \setminus A) = 0$. The set $\partial_S X = \{x \in X_1 : \{x\} \text{ is } S - \text{absorbent}\}$ is named *the Choquet boundary of X with respect to S* and clearly its closure coincides with *the Silov boundary* of X with respect to S. The trace on $\partial_S X$ of the topology on X in which the closed sets coincide with X or with any of the S-absorbent subsets of X contained in X_1 is usually called *the Choquet topology of* $\partial_S X$.

Definition 4.2. [73]. *A real function* $f : X \to R$ *is called* $(K + K_1)$ - *increasing if* $f(x_1) \geq f(x_2)$ *whenever* $x_1, x_2 \in X$ *and* $x_1 \in x_2 + K_1 + K$.

It is obvious that every real increasing function defined on any linear space ordered by an arbitrary convex cone K is $K+K_1$ - increasing, for each non-empty subset K_1 of K .

Now, we present the coincidence of the approximate efficient points sets and the Choquet boundaries, which generalizes the main results given in [12] and [63], respectively, and can not be obtained as a consequence of the Axiomatic Potential Theory.

Theorem 4.5. [74]. *If A is any non-void, compact subset of X and*

(i) K is an arbitrary, closed, convex, pointed cone in X;

(ii) K_1 is a non-empty subset of K such that $K+K_1$ is closed with respect to the Hausdorff separated locally convex topology on X.

Then, $eff(A,K,K_1)$ coincides with the Choquet boundary of A with respect the convex cone S_1 of all $K+K_1$ - increasing real continuous functions on A. Consequently, the set $eff(A,K,K_1)$ endowed with the corresponding trace topology is a Baire space and, if (A,τ_A) is metrizable, then $eff(A,K,K_1)$ is a G_δ - subset of X.

Corollary 4.2.

(i)

$$eff(A,K,K_1)=\{a\in A: f(a)=\sup\{f(a'): a'\in A\cap(a-K-K_1)\} \text{ for all } f\in C(A)\};$$

(ii) $eff(A,K,K_1)$ *and* $eff(A,K,K_1)\cap\{a\in A: s(a)\le 0\}$ $(s\in S)$ *are compact sets with respect to Choquet's topology;*

(iii) $eff(A,K,K_1)$ *is a compact subset of A.*

(iv) under the above hypotheses (i) and (ii) in Theorem 5, for any non-empty, compact set A in X such that (A,τ_A) is metrizable and every non-void G_δ - subset T of A there exists a convex cone S of $K+K_1$ - increasing real continuous functions on A which contains the constants and separates the points of A such that T coincides with $eff(A,K,K_1)$.

Corollary 4.3. *Under the hypotheses of the above theorem we proved that* $eff\left(A,K,K_1\right)=\left\{a\in A: f\left(a\right)=\overline{f}\left(a\right), \forall f\in C\left(A\right)\right\}=\partial_{S_1}A=\left\{a\in A: a\in F(a)\right\}$ *where* $F: A\to A$ *is the multifunction defined by*

$$F\left(t\right)=\left\{a\in A: A\bigcap\left(a-K-K_1\right)\subseteq t+K+K_1\right\}, \forall t\in A..$$

Remark 4.12. In general, $eff\left(A,K,K_1\right)$ coincides with the Choquet boundary of A only with respect to the convex cone of all real, continuous and $K+K_1$-increasing functions on *A*. Thus, for example, if *A* is a non-empty, compact and convex subset of *X,* then the Choquet boundary of *A* with respect to the convex cone of all real, continuous and concave functions on *A* coincides with the set of all extreme points for *A*. But, it is easy to see that, even in finite dimensional cases, an extreme point for a compact convex set is not necessary an efficient point and conversely.

Remark 4.13. As we have already specified before Theorem 4.1, there exists more general conditions than compactness imposed upon a non-empty set *A* in a separated locally convex space ordered by a convex cone *K* ensuring that $eff\left(A,K\right)\neq\varnothing$. Perhaps our coincidence result suggests a natural extension of the Choquet boundary at least in these cases. Anyhow, Theorem 4.5 represents an important link between vector optimization and potential theory and a new way for the study of the properties of efficient points sets and the Choquet boundaries. Indeed, one of the main question in potential theory is to find the Choquet boundaries. This fact is relatively easy for particular cases but, in general, it is an unsolved problem. Since in a lot of cases the efficient points sets contain dense subsets which can be identified by adequate numerical optimization methods, it is possible to determine the corresponding Choquet boundaries in all these situations. In this direction of study, an important role is attributed to the density properties of the efficient points sets with respect to varied topologies. Consequently, our coincidence result has its practical consequences at first for the axiomatic theory of potential and its applications. At the same time, by the above coincidence result, the Choquet boundaries offer important properties for the efficient points sets. In this way, the above coincidence result establishes a strong relationship between the approximate (in particular, strong) solutions for vector optimization programs in separated, ordered topological vector spaces and Choquet's boundaries of non-empty compact sets. Similar to the Choquet integral considered as an important risk measure [79], Choquet's boundary represents a very significant efficiency mathematical model. Nevertheless, even if these both concepts belong to Choquet, they are completely different at least because Choquet's boundary is defined as a non-empty set with respect to a convex cone and Choquet's integral is considered a measure. Thus, any possible connection between Choquet's boundary and Choquet's integral represents a genuine new open problem.

5 Isac's (nuclear or supernormal) cones

Throughout the research works devoted to nuclear (supernormal) cones professor Isac considered any locally convex space in the sense of the next definition.

Definition 5.1. (Treves, F., 1967). *A locally convex space is any couple* $(X, Spec(X))$ *which is composed of a real linear space* X *and a family* $Spec(X)$ *of seminoms on* X *such that:*

(i) $\chi p \in Spec(X), \quad \forall \chi \in R_+, \quad p \in Spec(X)$;

(ii) *if* $p \in Spec(X)$ *and* q *is an arbitrary seminorm on* X *such that* $q \le p$, *then* $q \in Spec(X)$;

(iii) $\sup(p_1, p_2) \in Spec(X), \forall p_1, p_2 \in Spec(X)$, *where*

$$\sup(p_1, p_2)(x) = \sup(p_1(x), p_2(x)), \forall x \in X .$$

It is well known [84] that whenever such a family as this $Spec(X)$ is given on a real vector space X, there exists a locally convex topology τ on X such that (X,τ) is a topological linear space and a seminorm p on X is τ - continous iff $p \in Spec(X)$. A a non-empty subset B of $Spec(X)$ is a *base* for it if for every $p \in Spec(X)$ there exist $\chi \rhd 0$ and $q \in \mathrm{B}$ such that $p \le \chi q$ and (X,τ) is a Hausdorff locally convex space iff $Spec(X)$ has a base B, named *Hausdorff base* , with the property that $\{x \in X : p(x) = 0, \forall p \in \mathrm{B}\} = \{\theta\}$, where θ is the null vector in X. In this research paper we will suppose that the space (X,τ) sometimes denoted by X is a Hausdorff locally convex space. Every non-empty subset K of X satisfying the following properties: $K + K \subseteq K$ and $\chi K \subseteq K, \forall \chi \in R_+$ is named *convex cone*. If, in addition, $K \bigcap K = \{\theta\}$, then K is called *pointed*. Clearly, any pointed convex cone K in X generates an ordering on X defined by $x \le y (x, y \in X)$ iff $y - x \in K$. If X^* is the dual of X, then the *dual cone* of K is defined by $K^* = \left\{x^* \in X^* : x^*(x) \ge 0, \forall x \in K\right\}$ and its corresponding *polar* is $K^0 = -K$. We recall that a pointed convex cone $K \subset (X, Spec(X))$ is *normal* with respect to the topology defined by $Spec(X)$ if it fulfils one of the next equivalent assertions:

(i) there exists at a base Ω *of neighborhoods for the origin* θ *in* X *such that* $V = (V + K) \cap (V - K), \forall V \in \Omega$*;*

(ii) there exists a base B *of* $Spec(X)$ *with*

$$p(x) \le p(y), \ \ \forall x, y \in K, \ \ x \le y, \ \ \forall p \in B;$$

(iii) for any two nets $\{x_i\}_{i\in I}, \{y_i\}_{i\in I} \subset K$ *with* $\theta \le x_i \le y_i, \forall i \in I$ *and* $\lim y_i = \theta$ *it follows that* $\lim x_i = \theta$*. In particular, a convex cone* K *is normal in a normed linear space* $(E, \|.\|)$ *iff there exists* $t \in (0,\infty)$ *such that* $x, y \in E$ *and* $y - x \in K$ *implies that* $\|x\| \le t\|y\|$.

It is well known that the concept of *normal cone* is the most important notion in the theory and applications of convex cones in topological ordered vector spaces. Thus, for example, *for every separated locally convex space* $(X, Spec(X))$ *and any closed normal cone* $K \subset (X, Spec(X))$ *we have* $X^* = K^* - K^*$ (see, for instance, [29], [32]).

Each pointed convex cone $K \subset (X, Spec(X))$ for which there exists a non-empty, convex bounded set $B \subset X$ such that $0 \notin \overline{B}$ and $K = \bigcup_{\chi \geq 0} \chi B$ is called *well-based*.

A cone $K \subset (X, Spec(X))$ is well-based iff there exists a base $B = \{p_i\}_{i \in I}$ of $Spec(X)$ and a linear continuous functional $f \in K^*$ such that for every $p_i \in B$ there exists $c_i > 0$ with $c_i p_i(x) \leq f(x), \forall x \in K$ ([29], [30]). Clearly, every well-based cone is a normal cone, but, in general, the converse is not true, as we can see in the examples below, starting from the next basic notion.

Definition 5.2. (Isac, G., 1981, 1983) *In a Hausdorff locally convex space* $(X, Spec(X))$ *a pointed convex cone* $K \subset X$ *is nuclear (supernormal) with respect to the topolgy induced by* $Spec(X)$ *if there exists a base* $B = \{p_i\}_{i \in I}$ *of* $Spec(X)$ *such that for every* $p_i \in B$ *there exists* $f_i \in X^*$ *with* $p_i(x) \leq f_i(x), \forall x \in K$.

Remark 5.1. For the first time, we called any such as this cone "Isac's cone" in [74], with the permission of the regretted professor George Isac, taking into account that the above definition of locally convex spaces is equivalent with the following: Let X be a real or complex linear space and $P = \{p_\alpha : \alpha \in A\}$ a family of seminorms defined on X. For every $x \in X, \varepsilon > 0$ and $n \in N^*$ let

$$V(x; p_1, p_2, ..., p_n; \varepsilon) = \{y \in X : p_\alpha(y - x) < \varepsilon, \ \ \forall \alpha = \overline{1, n}\},$$

then the family

$$\varsigma_0(x) = \left\{V(x; p_1, p_2, ..., p_n; \varepsilon) : \ n \in N^*, \ \ p_\alpha \in P, \ \ \alpha = \overline{1, n}, \ \ \varepsilon > 0\}\right\}$$

has the properties :

(V$_1$) $x \in V, \forall V \in \varsigma_0(x)$;

(V$_2$) $\forall V_1, V_2 \in \varsigma_0(x)$, $\exists V_3 \in \varsigma_0(x) : V_3 \subseteq V_1 \cap V_2$;

(V$_3$) $\forall V \in \varsigma_0(x)$, $\exists U \in \varsigma_0(x)$, $U \subseteq V$ such that $\forall y \in U$, $\exists W \in \varsigma_0(y)$ *with* $W \subseteq V$.

Therefore, ς_0(x) is *a base of neighborhoods* for x and taking $\varsigma(x) = \{V \subseteq X : \exists U \in \varsigma_0(x)$ cu $U \subseteq V\}$, the set $\tau = \{D \subseteq X : D \in \varsigma(x)$, $\forall x \in D\} \cup \{\varnothing\}$ is the locally convex topology generated by the family P. Obviously, the usual operations which induce the structure of linear space on X are continuous with respect to this topology. The corresponding topological space (X, τ) is a Hausdorff locally convex space iff the family P is sufficient, that is, $\forall\ x_0 \in X \setminus \{\theta\}, \exists p_\alpha \in P$ with $p_\alpha(x_0) \neq 0$. In this context, a convex cone $K \subset X$ is an Isac's cone iff $\forall p_\alpha \in P, \exists f_\alpha \in X^* : p_\alpha(x) \le f_\alpha(x), \forall x \in K$. The best special, refined and non-trivial Isac's cones class associated to normal cones in Hausdorff locally convex spaces was introduced and studied in [39] as *the full nuclear cones* family defined as follows: if $(X, Spec(X))$ is an arbitrary locally convex space $B \subset Spec(X)$ is a Hausdorff base of $Spec(X)$ and $K \subset X$ is a normal cone, then for any mapping $\varphi : B \to K^* \setminus \{0\}$ one says that the set

$$K_\varphi = \{x \in X : p(x) \le \varphi(p)(x), \forall p \in B\}$$

is *a full nuclear cone* associated to K whenever $K_\varphi \neq \{\theta\}$. Taking into account that in a real normed linear space $(E, \|.\|)$ a non-empty set $T \subset E$ is called *a Bishop-Phelps cone* if there exists y^* in the usual dual space E^* of E and $\alpha \in (0,1]$ such that $T = \{y \in E : \alpha\|y\| \le y^*(y)\}$ and the applications of such as these cones in Nonlinear Analysis and in Pareto type optimization for vector-valued mappings, we conclude that, for Haudorff locally convex spaces, the full nuclear cones are similar and generalizations of Bishop-Phelps cones in normed vector spaces.

In the next considerations we offer significant examples and adequate remarks on the supernormal cones. The existence of the efficient points and important properties of the efficient points sets are ensured in separated locally convex spaces ordered by (weak) supernormal cones named by us “Isac’s cones”, through the agency of the (weak) completeness instead of compactness (the reader is referred to, Isac, G., 1981, 1983, 1985, 1994, 1998, Isac, G., Postolică, V., 1993, Postolică, V., 1993, 1994, 1995, 1997, 1999, 2001, 2002, 2009, Truong, X. D. H., 1994 and so on).

Theorem 5.1. *(Bahya, A. O., 1989) A convex and normal cone* K *in a Hausdorff locally convex space is supernormal if and only if every net of* K *weakly convergent to zero converges to zero in the locally convex topology.*

Let us consider some pertinent examples

1. Any convex, closed and pointed cone in an arbitrary usual Euclidean space R^k with k in N* is supernormal.
2. In every locally convex space any well-based convex cone is an Isac’s cone.
3. A convex cone is an Isac’s cone in a normed linear space if and only if it is well-based.
4. Let $n \in N^*$ be arbitrary fixed and let Y be the space of all real symmetric (n, n) matrices ordered by the pointed, convex cone C = $\{A \in Y : x^T Ax \geq 0, \forall x \in R^n\}$. Then, Y is a real Hilbert space with respect to the scalar product defined by $<A, B>= =trace (A. B)$ for *all* $A, B \in Y$ and C is well-based by $B=\{A \in \mathbf{C}: <A, I>=1\}$ where I denotes the identity matrix.
5. Every pointed, locally or weakly locally compact convex cone in any Hausdorff locally convex space is an Isac’s cone.
6. A convex cone is an Isac’s cone in a nuclear space (Pietch, A., 1972) if and only if it is a normal cone.
7. In any Hausdorff locally convex space a convex cone is an weakly Isac’s

cone if and only if it is weakly normal.

8. In $L^p([a, b])$, $(p\geq),1$ the convex cone $K_p=\{x\in L^p([a, b]):x(t)\geq 0$ almost everywhere$\}$ is an Isac's cone if and only if p=1, being well based in this case by the set $B=\{x\in K_1: \int_a^b x(t)dt = 1\}$. Indeed, if p>1, then the sequence (x_n) defined by

$$x_n(t) = \begin{cases} n^{1/p}, & a \leq t \leq a+(b-a)/2n \\ 0, & a+(b-a)/2n < t \leq b \end{cases} \quad n \in N$$

converges to 0 in the weak topology but not in the usual norm topology. Therefore, by virtue of Theorem 5.1, K_p is not an Isac's cone. Generally, for every p>1, K_p has a base $B=\{x\in K_p: \int_a^b x(t)dt = 1\}$ which is unbounded and any cone generated by a closed and bounded set $B_t=\{x\in B: \int_a^b |x(t)|^p dt \leq t\}$ with $t\geq 0$ is certainly an Isac's cone.

A similar result holds for $L^p(R)$. Thus, if we consider a countable family (A_n) of disjoint sets which covers R such that (A_n) = 1 for all n in N, where μ is the Lebesgue measure, then the sequence (y_n) given by $y_n(t)$ = 1 if $t\in A_n$ and $y_n(t)$ = 0 for $t\in R-A_n$ converges weakly to zero while it is not convergent to zero in the norm topology. Taking into account the above theorem, it follows that the usual positive cone in $L^p(R)$ is not an Isac's cone if p>1, that is, it is not well-based in all these cases. However, these cones are normal for every $p\geq 1$. The same conclusion concerning the non-supernormality is valid for the positive orthant of the usual Orlicz spaces.

9. In l^p $(p\geq 1)$ equipped with the usual norm $\|\cdot\|_p$ the positive cone

$C_p= \{(x_n)\in l^p: x_n \in 0$ *for all* $n\in N\}$ is also *normal* with respect to *the norm topology*, but it is not an Isac's cone excepting the case p = 1. Indeed, for every p>1, the sequence (e_n) having 1 at the nth coordinate and zeros elsewhere converges to zero in the weak topology, but not in the norm topology and by virtue of Theorem 5.1 it follows that C_p is not an Isac's cone. For p = 1, C_p is well-based by the set B = $\{x\in C_1: \|x\|_1 = 1\}$ and Proposition 5 (Isac, G., 1983) ensures that it is an Isac's cone. If we consider in this case the locally convex topology in l^1 defined by the seminorms $p_n((x_k)) = \sum_{k=0}^{n} |x_k|$ for every (x_k) in l^1 and $n \in N$, which is weaker than its usual weak topology, then the usual positive cone remains an Isac's cone with respect to this topology (now it is normal in a nuclear space and one applies Proposition 6 of (Isac, G., 1983) but it is not well based. Taking into account the concept of H-locally convex space introduced by Precupanu, T. in 1969 and defined as any Hausdorff locally convex space with the seminorms satisfying the parallelogram law and the property that every nuclear space is also a *H*-locally convex space with respect to an equivalent system of seminorms (Pietch, A., 1972), the above example shows that in a *H-locally convex* space a proper convex cone may be an Isac's cone without to be well-based. Moreover, if we consider in l^2 the *H*-locally convex topology induced by the seminorms

$$\tilde{p}_n((x_k)) = \left(\sum_{i \geq n} |x_i|^2 \right)^{1/2}, \; n \in N, \; (x_k) \in \ell^2,$$

then the convex cone $C_2 = \{(x_k) \in l^2 : x_k \in 0 \text{ for all } k \in N\}$ is normal in the *H*-locally *convex space* $(l^2, \{\tilde{p}_n\}_{n \in N})$, but it is not a supernormal cone because the same sequence (e_k) is weakly convergent to zero while $(\bar{p}_n(e_k))$ is convergent to 1 for each $n \in N$ and one applies again Theorem 5.1. Another interesting example of normal cone in a *H*-locally convex space which is not supernormal is the usual positive cone in the space $L^2_{loc}(R)$ of all functions from *R* to C which are square integrable over any finite interval of *R*, endowed with the system of seminorms $\{\bar{p}_n : n \in N\}$ defined by $\bar{p}_n(x) = \left(\int_{-n}^{n} |x(t)|^2 \, dt^{1/2} \right)$ for every *x in* $L^2_{loc}(R)$. In this case, the sequence (x_k) given by:

$$x_k(t) = \begin{cases} 0, & t \in (-\infty, 0) \cup (1/k, +\infty) \\ \sqrt{k}, & t \in [0, 1/k] \end{cases}$$

converges weakly to zero, but it is not convergent in the H-locally convex topology. The results follows by Theorem 5.1. It is clear that every weak topology is a H-locally convex topology and, in these cases, the supernormality of convex cones coincides with the normality thanks to the Corollary of Proposition 2 in (Isac, G., 1983).

10. In the space $C([a,b])$ of all continuous, real valued functions defined on every non-trivial, compact interval $[a,b]$ equipped with the usual supremum norm the convex cone $K = \{x \in C([a, b]): x \text{ is concave}, x(a)=x(b) = 0$ and $x(t) \geq 0$ for all $t \in [a,b]\}$ is supernormal, being well based by the set $\{x \in K: x(t_0) = 1\}$ for some arbitrary $t_0 \in [a,b]$. The hypothesis that all $x \in K$ are concave is essential for the supernormality.

11. The convex cone of all nonnegative sequences in the space of all absolutely convergent sequences is the dual of the usual positive cone in the space of all convergent sequences. Consequently, it has a weak star compact base and hence it is a weak star supernormal cone.

12. In l^{∞} or in c_0 equipped with the supremum norm, the convex cone consisting of all sequences having all partial sums non-negative is not normal, hence it is not supernormal.

13. In every Hausdorff locally convex space any normal cone is supernormal with respect to the weak topology.

14. In every locally convex lattice which is a (L)-space the ordering cone is supernormal (see also the Example 7 given by Isac, G. in 1994).

15. If we consider the space of all locally integrable functions on a locally compact space Y with respect to a Radon measure μ endowed with the topology induced by the family of seminorms $\{p_A\}$ where $p_A(f) = \int_A |f(x)| d\mu$ for every non-empty and compact subset A of Y and every locally integrable function f, then the convex cone $K = \{f : f(x) \geq 0, x \in Y\}$ is supernormal.

16. If Z is any locally convex lattice ordered by an arbitrary convex cone K

and Z^* is its topological dual ordered by the corresponding dual cone K^*, then the cone K is supernormal with respect to the locally convex topology defined on Z by the neighbourhood base at the origin $\{[-f, f]^\circ\}_{f\in\tilde{K}}$

17. In every regular vector space (E, K) (that is, the order dual E^* separates the points of E) with the property that $E = K - K$ the convex cone K is supernormal with respect to the topology defined in the preceding example.

18. Any semicomplete cone in a Hausdorff locally convex space is supernormal (for this concept see the Example 11 of Isac, G., 1994).

Remark 5.2. Clearly, if a convex cone K is supernormal in a normed space, then K admits a strictly positive, linear and continuous functional, that is, there exists a linear, continuous functional f such that $f(k)>0$ *for all* $k\in K\setminus\{0\}$. Generally, the converse is not true even in a Banach space as we can see in the following examples:

19. If one considers in the usual space l^p $(1\leq p\leq\infty)$ the convex cone $K=\ell^p_+=\{x=(x_i)\in\ell^p : x_i\geq 0 \text{ for every } i\in \mathrm{N}\}$ of infinite vectors with non-negative components, then the functional φ defined by $\varphi(k)=\sum_{i=1}^{\infty}k_i$ for any $k=(k_i)\in l^p$ is linear, continuous and strictly positive. But, as we have seen in the above considerations (Example 9), this cone is supernormal if and only if $p=1$.

20. Let K be the usual positive cone

$$L^p{}_+=\{x\in L^p([a,b]) : x(t)\geq 0 \text{ almost everywhere}\}$$

in $L^p([a,b])$, $(1\leq p\leq\infty)$. Then, the linear and continuous functional ψ on $L^p([a,b])$ given by $\psi(x)=\int_a^b x(t)dt$ for every $x\in L^p([a, b])$ is strictly positive on K while K is supernormal (see the above Example 8) if and only if $p = 1$.

Therefore, $l^1{}_+$ and $L^1{}_+$ are supernormal cones with empty topological interiors and for every $p \in (1,+\infty)$ it follows that l^{p}_{+} and L^{p}_{+} are normal cones with empty interiors which are not supernormal. Hence, these convex cones are not well based. A very simple example of supernormal cones having non-empty topological interior is R_+^n ($n \in N^*$).

Remark 5.3. In the order complete vector lattice $B([a,b])$ of all bounded, real valued functions on a compact non-singleton interval $[a,b]$ endowed with its usual norm the standard positive cone $K = \{u \in B([a,b]) : \ u(t) \tilde{\geq} \ \text{for all } t \in [a,b]\}$ is normal but it has not a base, that is, it is not supernormal. However, this cone has non-empty interior. If we consider the linear space l^1 endowed with the separated locally convex topology generated by the famil $\{p_n : n \in \mathrm{N}\}$ of seminorms *defined by* $\mathrm{p_n(x)} = \sum_{k=0}^{n} |\mathrm{x_k}|$ for every $x = (x_k) \in l^1$, then the convex cone $K = \{x = (x_k) \in l^1 : x_k \geq 0 \ \text{whenever } k \in \mathrm{N}\}$ is supernormal but it is not well based.

Remark 5.4. The natural context of supernormality (nuclearity) for convex cones is any separated locally convex space. Isac, G. introduced the concept of “nuclear cone” in 1981, published it in 1983 and he showed that in a normed space a convex cone is nuclear if and only if it is well based or equivalently iff it is “with plastering”, the last concept being defined by Krasnoselski, M. A. in fifties (see ,for example , Krasnoselski, M. A., 1964 and so on). Such a convex cone was initially called “nuclear cone” by Isac, G. (1981) because in every nuclear space (Pietch, A., 1972) any normal cone is a nuclear cone in Isac’s sense (Proposition 6 of Isac, G., 1983). Afterwards, since the nuclear cone introduced by Isac appears as a reinforcement of the normal cone, it was called supernormal. The class of supernormal cones in Hausdorff locally convex spaces was initially imposed by the theory and the applications of the efficient (Pareto minimum type) points (especially existence conditions based on completeness instead of compactness were decisive together with the main properties of the efficient points sets), the study of critical points for dynamical systems and conical support points and their importance was very well illustrated by important results, examples and comments in the specified references and in other connected papers. It is also very significant to mention again that the concept of supernormality introduced by Isac, G. (1981) is not a simple generalization of the corresponding notion defined in normed linear spaces by Krasnoselski, M. A. and his colleagues in the fifties. Thus, for example, Isac's supernormality attached to the convex cones has his sense in every Hausdorff locally convex space identically with the well known Grothendieck's nuclearity. By analogy with the fact that a normed space is nuclear in Grothendieck's sense if and only if it is isomorphe with an usual Euclidean space, a convex cone is supernormal in a normed space if and only if it is well based, that is, it is generated by a convex bounded set which does not contain the origin in its closure.Beside Pareto type optimization, we also mention Isac's significant contributions, through the agency of supernormal cones, to the convex cones in product linear spaces and Ekeland's variational type principles (Isac.G.,2003;Isac, G.,Tammer, Chr.,2003). Therefore, the more appropriate background for Isac's cones is any separated locally convex space.

6 Useful splines for the best approximation and optimization in H-locally convex spaces

We conclude this research report with some topics on the best approximation (simultaneous and vectorial) and the optimization in *H-locally* convex spaces. So, it is known that the concept of *H*-locally convex space was introduced and studied for the first time by Precupanu, T. (1969) and defined as any Hausdorff locally convex space with the seminorms satisfying the parallelogram law. At the same time, we introduced the notion of spline function in *H-locally* convex space

(Postolică, V., 1981) and we established the basic properties of approximation and optimal interpolation for these splines. Our splines are natural extensions in *H*-locally convex spaces of the usual abstract splines which appear in any Hilbert space like the minimizing elements for a seminorm subject to the restrictions given by a set of linear continuous functionals.

Let $(X, \mathcal{P} = \{p_\alpha : \alpha \in I\})$ be a *H*-locally convex space with each seminorm p_α being induced by a scalar semiproduct $(.,.)_\alpha$ $(\alpha \in I)$ and M a closed linear subspace of X for which there exist a *H-locally* convex space $(Y, \mathcal{Q} = \{ q_\alpha : \alpha \in I\})$ with each seminorm $q_\alpha \in \mathcal{Q}$ generated by a scalar semiproduct $<.,.>_\alpha (\alpha \in I)$ and a linear (continuous) operator $U : X \to Y$ such that

$$M = \{x \in X: \quad (x,y)_\alpha = <Ux, Uy>_\alpha, \quad \forall \ \alpha \in I\}.$$

The space of spline functions with respect to U was defined by Postolică, V.(1981) as the U-orthogonal of M, that is,

$$M^\perp = \{x \in X: \quad <Ux, U\zeta>_\alpha = 0, \quad \forall \ \zeta \in M, \ \alpha \in I\}.$$

Clearly, $M^\perp$ is the orthogonal of M in the *H-locally* convex sense.

Let $x_0 \in X$ and G a non-empty subset of X.

Definition 6.1. (Postolică V.,1993) $g_0 \in G$ *is said to be a best simultaneous approximation for* x_0 *by the elements of* G *with respect the family* $\mathcal{P}$ *(abbreviated* g_0 *is a* $\mathcal{P}$ *-b.s.a. of* x_0*) when*

$$p_\alpha (x_0 - g_0) \ \leq \ p_\alpha(x_0 - g) \quad \text{for all} \quad g \in G \quad \text{and} \quad p_\alpha \in \mathcal{P}.$$

If, in addition, each element $x \in X$ *possesses at least one* $\mathcal{P}$ *- b.s.a. in* G*, then the set* G *is called* $\mathcal{P}$ *- simultaneous proximinal.*

Definition 6.2. (Postolică V., 1993) $g_0 \in G$ *is said to be a best vectorial approximation of* x_0 *by* G *with respect to* $\mathcal{P}$ *(abbreviated* g_0 *is a* $\mathcal{P}$ *- b.v.a. of* x_0*) if*

$$(p_\alpha (x_0 - g_0)) \ \in \mathrm{MIN}_K (\{(p_\alpha(x_0 - g)) : g \in G\}) \text{ where } K = R^I_+ .$$

The set G *is called* $\mathcal{P}$ *- vectorial proximinal whenever each element* $x \in X$

possesses at least one P *- b.v.a. in* G.

Let us consider the direct sum $X'=M\oplus M^{\perp}$ and for every $x\in X'$, we denote its projection onto $M^{\perp}$ *by* s_x. Then, taking into account the Theorem 4 obtained by Postolică, V. in 1981, it follows that this spline is a best simultaneous *U*-approximation of x with respect to $M^{\perp}$ since it satisfies all the next conditions : $p_\alpha(x - s_x) \leq p_\alpha(x - y)\ \ \forall\ y\in M^{\perp}, p_\alpha\in P$.

Moreover, following the definition of the approximate efficiency, the results given in Chapter 3 of (Isac, G., Postolică, V., 1993) and the conclusions obtained by (Postolică, V.,1981,1993, 1998), we have:

Theorem 6.1.

(i) for every $x\in X'$ *the only elements of best simultaneous and vectorial approximation with respect to any family of seminorms which generates the* H*-locally convex topology on* X *by the linear subspace of splines are the spline functions* s_x. *Moreover, if* M *and* $M^{\perp}$ *supply an orthogonal decomposiion for* X, *that is* $X=M\oplus M^{\perp}$, *then* $M^{\perp}$ *is simultaneous and vectorial proximinal;*

(ii) if $K=R^{I}_{+}$, *then for each* $s\in M^{\perp}$, *every* $\sigma\in M^{\perp}$ *is the only solution of following optimization problem* $MIN_K(\{(q_\alpha(U(\eta\text{-}s))): \eta\in X'$ *and* $\eta-\sigma\in M\})$;

(iii) for every $x\in X'$ *its spline function* s_x *is the only solution for the next vectorial optimization problems:*

$MIN_K(\{(q_\alpha(U(\eta\text{-}x))): \eta\in M^{\perp}\})$, $MIN_K(\{(p_\alpha(x\text{-}y)): y\in M^{\perp}\})$, $MIN_K(\{(q_\alpha(Uy)): y\text{-}x\in M\})$.

Finally, let us consider two numerical examples in which, following Postolică,V.,(1981), Isac, G., Postolică, V.,(1993) and Postolică, V., (1998), we specify the expressions of splines and M together with $M^{\perp}$ realizes orthogonal decompositions.

Example 6.1. Let

$X = H^m(R) = \{f \in C^{m-1}(R) : f^{(m-1)}$ is locally absolutely continuous and $f^{(m)} \in L^2_{loc}(R)\}$ $m \geq 1$ endowed with the H-locally convex topology generated by the scalar semiproducts

$$(x,y)_k = \sum_{h=0}^{m-1} [x^{(h)}(k)\, y^{(h)}(k) + x^{(h)}(-k)\, y^{(h)}(-k)] + \int_{-k}^{k} x^{(m)}(t)\, y^{(m)}(t)dt, \quad k = 0,1,2,\ldots$$

and $Y = L^2_{loc}(R)$ with the H-locally convex topology induced by the scalar semiproducts $<x,y>_k = \int_{-k}^{k} x(t)\, y(t)dt, \quad k = 0,1,2,\ldots$

If $U : X \to Y$ is the derivation operator of order m, then

$$\mathrm{M} = \{x \in H^m(R) : \quad x^{(h)}(\nu) = 0, \quad \forall\, h = \overline{0, m-1}, \ \nu \in Z\}$$

and

$$\mathrm{M}^{\perp} = \{s \in H^m(R) : \int_{-k}^{k} s^{(m)}(t)\, x^{(m)}(t)dt \ , \ \forall x \in \mathrm{M}, \ k = 0,1,2\ldots\}$$

We proved in (Postolică, V., 1981) that

$\mathrm{M}^{\perp} = \{s \in H^m(R) : s_{/(\nu,\ \nu+1)}$ is a polynomial function of degree $2m-1$ at most$\}$

and if $y = (y_\nu)$, $y' = (y'_\nu)$, $y'' = (y''_\nu)$, $y^{(m-1)} = (y^{(m-1)}_\nu)$ are m sequences of real numbers, then there exists an unique spline $S \in \mathrm{M}^{\perp}$ satisfying the following conditions of interpolation: $S^{(h)}(\nu) = y^{(h)}(\nu)$ *whenever* $h = \overline{0,\ m-1}$ *and* $\nu \in Z$. Moreover, we observed in the paragraph 3 of (Isac, G., Postolică, V., 1993) that any spline function S such as this is defined by

$$S(x) = p(x) + \sum_{h=0}^{m-1} c_1^{(h)}(x-1)_+^{2m-1} + \sum_{h=0}^{m-1} c_2^{(h)}(x-2)_+^{2m-1} + \ldots + \sum_{h=0}^{m-1} c_0^{(h)}(-x)_+^{2m-1} + \ldots$$

where $u_+ = (|u| + u)/2$ for every real number u, p is a polynomial function of degree $2m$-1 at most perfectly determined by the conditions $p^{(h)}(0) = y_0^{(h)}$ *and* $p^{(h)}(1) = y_1^{(h)}$ for all $h = \overline{0, m-1}$ and the coefficients $c_\nu^{(h)}$ ($h = \overline{0, m-1}$, $\nu \in Z$) are successively given by the general interpolation.

Therefore, for every function $f \in H^m(R)$, there exists an unique function denoted by $S_f \in \mathrm{M}^{\perp}$ such that $S_f^{(h)}(\nu) = f^{(h)}(\nu)$, $\forall\ h = \overline{0, m-1}$ and $\nu \in Z$. Hence, in this case, M and $\mathrm{M}^{\perp}$ give an orthogonal decomposition for the space $H^m(R)$.

Example 6.2. Let

$X = F_m = \{f \in C^{m-1}(R) : f^{(m-1)}$ is locally absolutely continuous and $f^{(m)} \in L^2(R)\}$

endowed with the *H*-locally convex topology induced by the scalar semiproducts $(x,y)_\nu = x(\nu)y(\nu) + \int_R x^{(m)}(t)\, y^{(m)}(t)dt$, $\nu \in Z$, $Y = L^2(R)$ with the topology generated by the inner product $(x,y)_\nu = \ = \int_R x(t)\, y(t)dt$, $\nu \in Z$ and $U : X \to Y$ be the derivation operator of order *m*. Then,

$$\mathrm{M} = \{x \in F_m :\ \ x(\nu) = 0, \quad \text{for all } \ \nu \in Z\}$$

and

$$\mathrm{M}^{\perp} = \{s \in F_m : \int_R x^{(m)}(t)\, y^{(m)}(t)dt = 0 \ \text{ for every } \ x \in \mathrm{M}\}.$$

In a similar manner as in Example 6.1 it may be proved that $\mathrm{M}^{\perp}$ coincides with the class of all piecewise polynomial functions of order 2*m* (degree 2*m*-1 at most) having their knots at the integer points. Moreover, for every function f in F_m there exists an unique spline function $S_f \in \mathrm{M}^{\perp}$, which interpolates f on the set Z of all integer numbers, that is, S_f satisfies the equalities $S_f(\nu) = f(\nu)$ for every $\nu \in Z$, being defined by

$$S_f(x) = p(x) + a_1(x-1)_+^{2m-1} + a_2(x-2)_+^{2m-1} + \ldots + a_0(-x)_+^{2m-1} + a_{-1}(-x-1)_+^{2m-1} + \ldots$$

where u_+ has the same signification as in Example 6.1, the coefficients a_ν ($\nu \in Z$) are successively and completely determined by the interpolation conditions

$S_f(\nu) = f(\nu)$, $\nu \in Z - \{0,1\}$ and p is a polynomial function satisfying the conditions $p(0) = f(0)$ and $p(1) = f(1)$. The uniqueness of S_f is ensured in Theorem 6.2 given by (Postolică, V., 1981).

Thus, M and $M^{\perp}$ give an orthogonal decomposition of the space F_m and, as in the preceding example, $M^{\perp}$ is simultaneous and vectorial proximinal with respect to the family of seminorms generated by the above scalar semiproducts.

Remark 6.1. Our examples show that the abstract construction of splines can be used to solve also several frequent problems of interpolation and approximation, having the possibility to choose the spaces and the scalar semiproducts. It is obvious that for a given (closed) linear subspace of a *H*-locally convex space *X* such a *H*-locally convex space *Y* (respectively, a linear (continuous) operator $U : X \to Y$ would not exist. Otherwise, the problem of best vectorial approximation by the corresponding orthogonal space of any (closed) linear subspace M for the elements in the direct sum $M \oplus M^{\perp}$ might be always reduced to the best simultaneous approximation. But, in general, such a possibility doesn't exist. Even in a *H*-locally convex space it is possible that there exist best vectorial approximations and the set of all best simultaneous approximations to be empty for some element of the space. We confine ourselves to mention the following simple example.

Example 6.3. Let $X = R^N$ endowed with the topology generated by the family $P = \{p_i : i \in N\}$ of seminorms defined by $p_i(x) = |x_i|$ $(i \in N)$ for every $x = (x_i) \in X$ and $G = \{(x_i) \in X : x_i \geq 0$ whenever $i \in N$ and $\sum_{i \in N} x_i = 1\}$

X is a P -simultaneous strictly convex (Isac, G., Postolică, V., 1993) *H*-locally convex space. Nevertheless, every element of *G* is P -b.v.a. for the origin while its corresponding set of the best simultaneous approximations with respect to P is empty.

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