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Solutions of Higher-Degree Reduced Polynomial Equations by Using Unipodal Numbers

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Abstract

In this paper, motivated by some earlier work [6], we formulated a general way of solving a certain type of higher-degree reduced polynomial equations having only real solutions. In order to do it, we discussed about hyperbolic numbers (also known as unipodal numbers) and their remarkable properties. We have utilized these properties to formulate the procedure, and finally, we gave an example of how to solve a heptagonal reduced polynomial equation to demonstrate the applicability of the new method.

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1 Introduction

Mathematicians [2, 7] over the ages tried to find general rules for solving a higher-degree polynomial equation without much success. Niels Henrik Abel [1] and Evariste Galois [4] proved that there are no formulae for solving general polynomial equations of degree 5 or higher. The present author solved pentagonal equations under certain conditions [6]. Although, our goal was to find solutions of general higher-degree polynomial equations, we are to be happy with a partial success. Here we are able to show how to solve higher-degree reduced polynomial equations with the aid of unipodal numbers. We have described the unipodal numbers and their properties in Section 2. In Section 3, we defined what we called higher-degree reduced polynomial equations and developed the method of solving them. We also solved a reduced heptagonal equation as an example.

2 The Unipodal Number System

A unipodal number [5] w in the *standard basis* $\{1, u\}$ has the form $w = x + uy,$ where $u^2 = 1,$ however, $u \neq \pm 1,$ and $x,\ y$ are complex numbers. The basis $\{u_+, u_-\}$ defined by

$$u_+ = \frac{1}{2}(1+u) \quad \text{and} \quad u_- = \frac{1}{2}(1-u), \tag{1}$$

and satisfying the relations $u_+ + u_- = 1$ and $u_+ - u_- = u,$ is known as the *idempotent basis*, because it has the properties $u_+{}^2 = u_+$ and $u_-{}^2 = u_-.$ The idempotent basis $\{u_+, u_-\}$ has the mutually annihilating property $u_+ u_- = 0.$ Using the idempotent basis, we can write

$$w = w(u_+ + u_-) = w_+ u_+ + w_- u_-, \text{ where } w_+ = x + y \text{ and } w_- = x - y. \tag{2}$$

The coordinates of the standard basis can be recovered by

$$x = \frac{1}{2}(w_+ + w_-), \qquad y = \frac{1}{2}(w_+ - w_-). \tag{3}$$

It is noted that the idempotent basis makes calculations simple and the Binomial Theorem under this basis becomes extremely simple as we can see below:

$$w^n = (w_+u_+ + w_-u_-)^n = (w_+)^n u_+{}^n + (w_-)^n u_-{}^n = (w_+)^n u_+ + (w_-)^n u_-, \tag{4}$$

since $u_+u_- = 0$.

The relation defined in (4) is valid for any real *n*. Because of (4) we can extend the definitions of all the elementary functions in the complex plane to the elementary functions in the unipodal plane. If $f(w)$ is such a function for $w = w_+u_+ + w_-u_-$, we define

$$f(w) \equiv f(w_+)u_+ + f(w_-)u_-, \tag{5}$$

provided that $f(w_+)$ and $f(w_-)$ are defined. The basic unipodal equation $w^n = r$ can easily be solved using the idempotent basis, with the help of equation (4). Writing $w = w_+u_+ + w_-u_-$ and $r = r_+u_+ + r_-u_-$, we have

$$w^n = w_+{}^n u_+ + w_-{}^n u_- = r = r_+u_+ + r_-u_-. \tag{6}$$

Hence $w_+{}^n = r_+$ and $w_-{}^n = r_-$. It follows that $w_+ = |r_+|^{\frac{1}{n}} \alpha^j$ and $w_- = |r_-|^{\frac{1}{n}} \alpha^k$ for some integers $0 \le j, k \le n-1$, where α is primitive *n*th root of unity [3].

This proves the following theorem.

Theorem 2.1 For every positive integer *n*, the unipodal equation $w^n = r$ has n^2 solutions

$$w = \alpha^j r_+^{\frac{1}{n}} u_+ + \alpha^k r_-^{\frac{1}{n}} u_-, \tag{7}$$

for $j, k = 0, 1, 2, \ldots, n-1$, where $\alpha = \exp(2\pi i / n)$.

The number of roots to the equation $w^n = r$ can be reduced by adding some constraints. The following corollary follows from the Theorem 2.1, by having the constraint $w_+ w_- = \rho \neq 0,$ that is equivalent to $w_- = \rho / w_+$.

Corollary 2.1 The unipodal equation $w^n = r$, subject to the constraint $w_+ w_- = (x+y)(x-y) = x^2 - y^2 = \rho \neq 0$, has the *n* solutions given by

$$w = \alpha^j r_+^{\frac{1}{n}} u_+ + \frac{\rho}{\alpha^j r_+^{\frac{1}{n}}} u_-, \tag{8}$$

for $j = 0, 1, 2, \ldots, n-1$, where $\alpha = \exp(2\pi i / n)$, and $r_+^{\frac{1}{n}}$ denotes any *n*th root of the complex number r_+.

3 Reduced Polynomial Equations

Let n be an odd positive integer. Then by *reduced n-degree polynomial equation* we mean an equation of the form

$$x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \ldots + a_3 x^3 + a_1 x + a_0 = 0, \tag{9}$$

where $a_{n-2}, a_{n-4}, \ldots, a_3, a_1, a_0$ are rational numbers.

Theorem 3.1 The reduced *n*-degree polynomial equation

$$x^n + n\, a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \ldots + a_3 x^3 + a_1 x + a_0 = 0 \tag{10}$$

has the solutions, for $j = 0, 1, 2, \ldots, n-1,$

$$x = \frac{1}{2}\left(\alpha^j \sqrt[n]{s+t} + \frac{\rho}{\alpha^j \sqrt[n]{s+t}} \right), \tag{11}$$

where $\alpha = \exp(2\pi i / n)$ is a primitive *n*th root of unity,

$$\rho = -4a_{n-2}, \quad s = -\, 2^{n-1} a_0, \quad \text{and} \quad t = \sqrt{s^2 - \rho^n}.$$

Proof. The unipodal equation $w^n = r,$ where $r = s + ut,$ is equivalent in the standard basis to

$$(x+u\,y)^n = s+u\,t,$$

$$\Rightarrow\left[x^n+\frac{n(n-1)}{2!}x^{n-2}y^2+\frac{n(n-1)(n-2)(n-3)}{4!}x^{n-4}y^4+\cdots+n\,x\,y^{n-1}\right]+$$

$$\left[y^n+\frac{n(n-1)}{2!}y^{n-2}x^2+\frac{n(n-1)(n-2)(n-3)}{4!}y^{n-4}x^4+\cdots+n\,y\,x^{n-1}\right]u = s+ut. \tag{12}$$

Equating the complex scalar parts, we have

$$\left[x^n+\frac{n(n-1)}{2!}x^{n-2}y^2+\frac{n(n-1)(n-2)(n-3)}{4!}x^{n-4}y^4+\cdots+n\,x\,y^{n-1}\right]=s. \tag{13}$$

Now substituting $x^2-y^2=\rho$ in (13) we can write

$$\begin{aligned}&x^n+\frac{n(n-1)}{2!}x^{n-2}\left(x^2-\rho\right)+\frac{n(n-1)(n-2)(n-3)}{4!}x^{n-4}\left(x^2-\rho\right)^2+\cdots+\\&+n\,x\left(x^2-\rho\right)^{\frac{n-1}{2}}-s=0.\end{aligned} \tag{14}$$

Then the coefficient of x^n in (14) is given by

$$1+\frac{n(n-1)}{2!}+\frac{n(n-1)(n-2)(n-3)}{4!}+\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6!}+\cdots+n$$
$$=\frac{2^n}{2}=2^{n-1},$$

since for an odd n, the sum of the coefficients of the odd powers of x is equal to that of the even powers of x and the sum of the binomial coefficients is equal to 2^n.

The coefficient of x^{n-2} in (14), after simplification, could be written as

$$-\rho n\left[\begin{array}{l}\frac{n-1}{2!}+2\cdot\frac{(n-1)(n-2)(n-3)}{4!}+3\cdot\frac{(n-1)\cdots(n-5)}{6!}+\frac{(n-1)\cdots(n-7)}{8!}+\\ \cdots+k\cdot\frac{(n-1)(n-2)\cdots(n-2k+1)}{(2k)!}\end{array}\right] \tag{15}$$
$$=-\rho n\sum_{k=1}^{(n-1)/2}k\cdot\frac{(n-1)(n-2)\cdots(n-2k+1)}{(2k)!},$$

and by mathematical induction, we can show that

$$\sum_{k=1}^{(n-1)/2} k \cdot \frac{(n-1)(n-2)\cdots(n-2k+1)}{(2k)!} = 2^{n-3}. \tag{16}$$

Hence the coefficient of x^{n-2} is equal to $-2^{n-3}\rho n.$

Then dividing each term of the equation (14) (after expansion and simplification) by the coefficient of x^n and comparing the coefficients x^{n-2} in (10) and (14), we have

$$a_{n-2} = -\frac{2^{n-3}}{2^{n-1}}\rho \Rightarrow \rho = -4a_{n-2}. \tag{17}$$

Similarly, equating the constant terms of (10) and (14), we can show that

$$a_0 = -\frac{s}{2^{n-1}} \Rightarrow s = -2^{n-1}a_0. \tag{18}$$

In this way, every coefficient in (10) could be written in terms of ρ. The constraint $w_+w_- = \rho$ further implies that

$$\rho^n = (w_+w_-)^n = w_+{}^n w_-{}^n = r_+r_- = s^2 - t^2 \Rightarrow t = \sqrt{s^2 - \rho^n}. \tag{19}$$

From the equation (8) we have

$$\begin{aligned} w &= \alpha^j r_+^{\frac{1}{n}} u_+ + \frac{\rho}{\alpha^j r_+^{\frac{1}{n}}} u_- \\ \Rightarrow x + uy &= \alpha^j r_+^{\frac{1}{n}} \cdot \frac{1}{2}(1+u) + \frac{\rho}{\alpha^j r_+^{\frac{1}{n}}} \cdot \frac{1}{2}(1-u) \\ &= \frac{1}{2}\left[\alpha^j r_+^{\frac{1}{n}} + \frac{\rho}{\alpha^j r_+^{\frac{1}{n}}}\right] + \frac{1}{2}\left[\alpha^j r_+^{\frac{1}{n}} - \frac{\rho}{\alpha^j r_+^{\frac{1}{n}}}\right]u \end{aligned} \tag{20}$$

We have used the equation (1) in the second line of (20). Equating the complex scalar parts, we have

$$x = \frac{1}{2}\left[\alpha^j r_+^{\frac{1}{n}} + \frac{\rho}{\alpha^j r_+^{\frac{1}{n}}}\right] = \frac{1}{2}\left[\alpha^j \sqrt[n]{s+t} + \frac{\rho}{\alpha^j \sqrt[n]{s+t}}\right]. \tag{21}$$

We have substituted $r_+ = s + t,$ in the above equation to get the desired solutions and this completes the proof of the Theorem 3.1. □

Here we give an example of solving a reduced equation for $n = 7$. Cases for $n = 3$ and $n = 5$ have been worked out respectively in [6] and [8].

Example 3.1 Find the solutions of the reduced heptagonal equation

$$x^7 + 7\left(-\frac{1}{2}\right)x^5 + \frac{7}{2}x^3 - \frac{7}{8}x + \frac{8}{64} = 0.$$

Solution. Here $a_5 = -\frac{1}{2},\ a_0 = \frac{8}{64}$ so $\rho = 2,\ s = -8$.

Then $s + t = -8 + 8i = 8(-1+i) = 2^{7/2}\exp\left(\frac{3\pi}{4}i\right)$ and $\sqrt[7]{s+t} = \sqrt{2}\exp\left(\frac{3\pi}{28}i\right)$.

Thus

$$x = \frac{1}{2}\left[\sqrt{2}\exp\left(\frac{3\pi}{28}i\right)\alpha^k + \frac{2}{\sqrt{2}\exp\left(\frac{3\pi}{28}i\right)\alpha^k}\right]$$

$$= \frac{1}{2}\left[\sqrt{2}\exp\left(\frac{3\pi}{28}i\right)\exp\left(\frac{2k\pi}{7}i\right) + \frac{2}{\sqrt{2}\exp\left(\frac{3\pi}{28}i\right)\exp\left(\frac{2k\pi}{7}i\right)}\right]$$

$$= \sqrt{2}\cos\left[\left(\frac{3}{4} + 2k\right)\pi/7\right] \quad \text{for} \quad k = 0, 1, 2, \ldots, 6. \tag{22}$$

Equation (22) indicates that solutions are real. This means that this method is able to solve higher-degree reduced equations that have only real solutions, it cannot handle any reduced equations that have some complex solutions too.

I. For $k = 0$,

$$x = \sqrt{2}\cos\left(\frac{3\pi}{28}\right) = 1.334839806$$

II. For $k = 1$,

$$x = \sqrt{2}\cos\left[\left(\frac{3}{4}+2\right)\pi/7\right] = \sqrt{2}\cos\left(\frac{11}{28}\pi\right) = 0.467085129$$

III. For $k = 2$,

$$x = \sqrt{2}\cos\left[\left(\frac{3}{4}+4\right)\pi/7\right] = \sqrt{2}\cos\left(\frac{19}{28}\pi\right) = -0.75240$$

IV. For $k = 3$,

$$x = \sqrt{2}\cos\left[\left(\frac{3}{4}+6\right)\pi/7\right] = \sqrt{2}\cos\left(\frac{27}{28}\pi\right) = -1.40532184$$

V. For $k = 4$,

$$x = \sqrt{2}\cos\left[\left(\frac{3}{4}+8\right)\pi/7\right] = \sqrt{2}\cos\left(\frac{5}{4}\pi\right) = -1.00$$

VI. For $k = 5$,

$$x = \sqrt{2}\cos\left[\left(\frac{3}{4}+10\right)\pi/7\right] = \sqrt{2}\cos\left(\frac{43}{28}\pi\right) = 0.1583416806$$

VII. For $k = 6$,

$$x = \sqrt{2}\cos\left[\left(\frac{3}{4}+12\right)\pi/7\right] = \sqrt{2}\cos\left(\frac{51}{28}\pi\right) = 1.197448846$$

4 Conclusion

Following the above method, we can solve any reduced polynomial equations of degree *n*, where *n* is an odd positive integer.

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