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# Real hypersurfaces in complex two-plane Grassmannians whose Jacobi operators corresponding to $\mathcal{D}^\perp$ -directions are $\mathcal{D}$ -parallel

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## Abstract

We prove the non-existence of Hopf real hypersurfaces in complex two-plane Grassmannians whose Jacobi operators corresponding to the directions in the distribution  $\mathcal{D}^\perp$  are  $\mathcal{D}$ -parallel if they satisfy a further condition.

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## 1 Introduction

The geometry of real hypersurfaces in complex space forms or in quaternionic space forms is one of interesting parts in the field of differential geometry.

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Until now there have been many characterizations for homogeneous hypersurfaces of type  $(A_1)$ ,  $(A_2)$ ,  $(B)$ ,  $(C)$ ,  $(D)$  and  $(E)$  in complex projective space  $\mathbb{C}P^m$ , of type  $(A_1), (A_2)$  and  $(B)$  in quaternionic projective space  $\mathbb{H}P^m$ , or of type  $(A)$  and  $(B)$  in complex two-plane Grassmannians  $G_2(\mathbb{C}^{m+2})$ . Each corresponding geometric features are classified and investigated by Kimura [5], Martinez and Pérez [8], Berndt and Suh [3] and [4], respectively.

Let  $(\bar{M}, \bar{g})$  be a Riemannian manifold. A vector field  $U$  along a geodesic  $\gamma$  in  $\bar{M}$  is said to be a *Jacobi field* if it satisfies a differential equation

$$\bar{\nabla}_{\dot{\gamma}(t)}^2 U + \bar{R}(U(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0,$$

where  $\bar{\nabla}_{\dot{\gamma}(t)}$  and  $\bar{R}$  respectively denote the covariant derivative of the vector field  $U$  along the curve  $\gamma(t)$  in  $\bar{M}$  and the curvature tensor of the Riemannian manifold  $(\bar{M}, \bar{g})$ . Then this equation is called the *Jacobi equation*.

The Jacobi operator  $\bar{R}_X$  for any tangent vector field  $X$  at  $x \in \bar{M}$ , is defined by

$$(\bar{R}_X(Y))(x) = (\bar{R}(Y, X)X)(x)$$

for any  $Y \in T_x \bar{M}$ , and it becomes a self adjoint endomorphism of the tangent bundle  $T\bar{M}$  of  $\bar{M}$ . That is, the Jacobi operator satisfies  $\bar{R}_X \in \text{End}(T_x \bar{M})$  and is symmetric in the sense of  $\bar{g}(\bar{R}_X(Y), Z) = \bar{g}(\bar{R}_X(Z), Y)$  for any vector fields  $Y$  and  $Z$  on  $\bar{M}$ .

Now let us consider real hypersurfaces in complex two-plane Grassmannians  $G_2(\mathbb{C}^{m+2})$  which consist of all complex 2-dimensional linear subspaces in  $\mathbb{C}^{m+2}$ . The complex two-plane Grassmannians  $G_2(\mathbb{C}^{m+2})$  have a remarkable geometric structure. It is known to be the unique compact irreducible Riemannian symmetric spaces equipped with both a Kähler structure  $J$  and a quaternionic Kähler structure  $\mathfrak{J}$  (see Berndt and Suh [3]).

In [2] the authors prove that any tube  $M$  around a complex submanifold in complex projective space  $\mathbb{C}P^m$  is characterized by the invariancy of  $A\xi = \alpha\xi$ , where the Reeb vector  $\xi$  is defined by  $\xi = -JN$  for a Kähler structure  $J$  and a unit normal  $N$  to real hypersurface  $M$  in  $\mathbb{C}P^m$ .

Moreover, the corresponding geometrical feature for hypersurfaces in  $\mathbb{H}P^m$  is the invariance of the distribution  $\mathcal{D}^\perp = \text{Span} \{\xi_1, \xi_2, \xi_3\}$  by the shape operator, where  $\xi_i = -J_i N$ ,  $J_i \in \mathfrak{J}$ .

In fact every tube around a quaternionic submanifold of  $\mathbb{H}P^m$  admits a geometrical structure of this type (see Alekseevskii [1]). From such a view

point, we considered two natural geometric conditions for real hypersurfaces in  $G_2(\mathbb{C}^{m+2})$  that  $[\xi] = \text{Span} \{\xi\}$  and  $\mathcal{D}^\perp = \text{Span} \{\xi_1, \xi_2, \xi_3\}$  are invariant under the shape operator. By using such conditions and the result in Alekseevskii [1], Berndt and Suh [3] have proved the following.

**Theorem A.** *Let  $M$  be a connected real hypersurface in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ . Then both  $[\xi]$  and  $\mathcal{D}^\perp$  are invariant under the shape operator of  $M$  if and only if*

*[(A)] $M$  is an open part of a tube around a totally geodesic  $G_2(\mathbb{C}^{m+1})$  in  $G_2(\mathbb{C}^{m+2})$ , or  $m$  is even, say  $m = 2n$ , and  $M$  is an open part of a tube around a totally geodesic  $\mathbb{H}P^n$  in  $G_2(\mathbb{C}^{m+2})$ .*

If the *Reeb* vector field  $\xi$  of a real hypersurface  $M$  in  $G_2(\mathbb{C}^{m+2})$  is invariant by the shape operator,  $M$  is said to be a *Hopf hypersurface*. In such a case the integral curves of the *Reeb* vector field  $\xi$  are geodesics (see Berndt and Suh [4]). Moreover, the flow generated by the integral curves of the structure vector field  $\xi$  for Hopf hypersurfaces in  $G_2(\mathbb{C}^{m+2})$  is said to be *geodesic Reeb flow*. Moreover, if the corresponding principal curvature  $\alpha$  is non-vanishing we say  $M$  is with non-vanishing *geodesic Reeb flow*.

In this paper we will consider the Jacobi operators associated to a basis of the distribution  $\mathcal{D}^\perp$ ,  $R_{\xi_i}$ ,  $i = 1, 2, 3$ . In [7] we have proved the non-existence of real hypersurfaces in complex two-plane Grassmannians whose Jacobi operators  $R_{\xi_i}$ ,  $i = 1, 2, 3$  are of Codazzi type if the distribution  $\mathcal{D}$  or the  $\mathcal{D}^\perp$ -component of the *Reeb* vector field is invariant by the shape operator. As a consequence we obtained the non-existence of real hypersurfaces for which such Jacobi operators are parallel with the same further condition. Now we will deal with a weaker condition: The  $\mathcal{D}$ -paralellism of the Jacobi operators  $R_{\xi_i}$ ,  $i = 1, 2, 3$ . Thus we will prove the following.

**Theorem 1.1.** *There do not exist any connected Hopf real hypersurface in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ , such that  $\nabla_X R_{\xi_i} = 0$ ,  $i = 1, 2, 3$ , for any  $X \in \mathcal{D}$  if the distribution  $\mathcal{D}$  or the  $\mathcal{D}^\perp$ -component of the *Reeb* vector field is invariant by the shape operator.*

## 2 Preliminaries

For the study of Riemannian geometry of  $G_2(\mathbb{C}^{m+2})$  see [3]. All the notations we will use since now are the ones in [3] and [4]. For computational reasons we will suppose that the metric  $g$  of  $G_2(\mathbb{C}^{m+2})$  is normalized for the maximal sectional curvature of  $(G_2(\mathbb{C}^{m+2}), g)$  to be eight. Then the Riemannian curvature tensor  $\bar{R}$  of  $G_2(\mathbb{C}^{m+2})$  is locally given by

$$\begin{aligned} \bar{R}(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ \\ &\quad + \sum_{\nu=1}^3 \{g(J_\nu Y, Z)J_\nu X - g(J_\nu X, Z)J_\nu Y - 2g(J_\nu X, Y)J_\nu Z\} \\ &\quad + \sum_{\nu=1}^3 \{g(J_\nu JY, Z)J_\nu JX - g(J_\nu JX, Z)J_\nu JY\}, \end{aligned} \quad (1)$$

where  $J_1, J_2, J_3$  is any canonical local basis of  $\mathfrak{J}$ .

Let  $M$  be a real hypersurface of  $G_2(\mathbb{C}^{m+2})$ , that is, a submanifold of  $G_2(\mathbb{C}^{m+2})$  with real codimension one. The induced Riemannian metric on  $M$  will also be denoted by  $g$ , and  $\nabla$  denotes the Riemannian connection of  $(M, g)$ . Let  $N$  be a local unit normal field of  $M$  and  $A$  the shape operator of  $M$  with respect to  $N$ . The Kähler structure  $J$  of  $G_2(\mathbb{C}^{m+2})$  induces on  $M$  an almost contact metric structure  $(\phi, \xi, \eta, g)$ . Furthermore, let  $J_1, J_2, J_3$  be a canonical local basis of  $\mathfrak{J}$ . Then each  $J_\nu$  induces an almost contact metric structure  $(\phi_\nu, \xi_\nu, \eta_\nu, g)$  on  $M$ . Using the above expression for the curvature tensor  $\bar{R}$ , the Gauss and Codazzi equations are respectively given by

$$\begin{aligned} R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ &\quad + \sum_{\nu=1}^3 \{g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z\} \\ &\quad + \sum_{\nu=1}^3 \{g(\phi_\nu \phi Y, Z)\phi_\nu \phi X - g(\phi_\nu \phi X, Z)\phi_\nu \phi Y\} \\ &\quad - \sum_{\nu=1}^3 \{\eta(Y)\eta_\nu(Z)\phi_\nu \phi X - \eta(X)\eta_\nu(Z)\phi_\nu \phi Y\} \\ &\quad - \sum_{\nu=1}^3 \{\eta(X)g(\phi_\nu \phi Y, Z) - \eta(Y)g(\phi_\nu \phi X, Z)\}\xi_\nu \\ &\quad + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

and

$$\begin{aligned}
(\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\
&+ \sum_{\nu=1}^3 \{ \eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu \} \\
&+ \sum_{\nu=1}^3 \{ \eta_\nu(\phi X)\phi_\nu \phi Y - \eta_\nu(\phi Y)\phi_\nu \phi X \} \\
&+ \sum_{\nu=1}^3 \{ \eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X) \} \xi_\nu,
\end{aligned}$$

where  $R$  denotes the curvature tensor of  $M$  in  $G_2(\mathbb{C}^{m+2})$ .

To be used in the sequel we mention the following Propositions due to Berndt and Suh, [4].

**Proposition 2.1.** *Let  $M$  be a connected real hypersurface of  $G_2(\mathbb{C}^{m+2})$ . Suppose that  $AD \subset \mathcal{D}$ ,  $A\xi = \alpha\xi$  and  $\xi$  is tangent to  $\mathcal{D}^\perp$ . Let  $J_1 \in \mathfrak{J} = \text{Span}\{J_1, J_2, J_3\}$  be the almost Hermitian structure such that  $JN = J_1N$ . Then  $M$  has three (if  $r = \pi/2\sqrt{8}$ ) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8} \cot(\sqrt{8}r), \quad \beta = \sqrt{2} \cot(\sqrt{2}r), \quad \lambda = -\sqrt{2} \tan(\sqrt{2}r), \quad \mu = 0$$

with some  $r \in (0, \pi/\sqrt{8})$ . The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu)$$

and as the corresponding eigenspaces we have

$$\begin{aligned}
T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1, \\
T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3, \\
T_\lambda &= \{X \mid X \perp \mathbb{H}\xi, JX = J_1X\}, \\
T_\mu &= \{X \mid X \perp \mathbb{H}\xi, JX = -J_1X\},
\end{aligned}$$

where  $\mathbb{R}\xi$ ,  $\mathbb{C}\xi$  and  $\mathbb{H}\xi$  respectively denotes real, complex and quaternionic span of the structure vector  $\xi$  and  $\mathbb{C}^\perp\xi$  denotes the orthogonal complement of  $\mathbb{C}\xi$  in  $\mathbb{H}\xi$ .

**Proposition 2.2.** *Let  $M$  be a connected real hypersurface of  $G_2(\mathbb{C}^{m+2})$ . Suppose that  $AD \subset \mathcal{D}$ ,  $A\xi = \alpha\xi$  and  $\xi$  is tangent to  $\mathcal{D}$ . Then the quaternionic dimension  $m$  of  $G_2(\mathbb{C}^{m+2})$  is even, say  $m = 2n$ , and  $M$  has five distinct constant principal curvatures*

$$\alpha = -2 \tan(2r) , \beta = 2 \cot(2r) \quad \gamma = 0 , \lambda = \cot(r) , \mu = -\tan(r)$$

with some  $r \in (0, \pi/4)$ . The corresponding multiplicities are

$$m(\alpha) = 1 , m(\beta) = 3 = m(\gamma) , m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi , T_\beta = \mathfrak{J}J\xi , T_\gamma = \mathfrak{J}\xi , T_\lambda , T_\mu ,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp , \mathfrak{J}T_\lambda = T_\lambda , \mathfrak{J}T_\mu = T_\mu , JT_\lambda = T_\mu .$$

**Proposition 2.3.** *If  $M$  is a connected orientable real hypersurface in  $G_2(\mathbb{C}^{m+2})$  with geodesic Reeb flow, then*

$$\begin{aligned} & \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi X, Y) + 2g(\phi X, Y) \\ &= 2 \sum_{\nu=1}^3 \left( \eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi) \right. \\ & \quad \left. - 2\eta(X)\eta_\nu(\phi Y)\eta_\nu\xi + 2\eta(Y)\eta_\nu(\phi X)\eta_\nu(\xi) \right) \end{aligned}$$

for any  $X, Y \in TM$  where  $\alpha = g(A\xi, \xi)$ .

Recently, Lee and Suh, [6], have proved the following.

**Proposition 2.4.** *Let  $M$  be a connected orientable Hopf real hypersurface in  $G_2(\mathbb{C}^{m+2})$ ,  $m \geq 3$ . Then the Reeb vector  $\xi$  belongs to the distribution  $\mathcal{D}$  if and only if  $M$  is locally congruent to an open part of a tube around a totally geodesic  $\mathbb{H}P^n$  in  $G_2(\mathbb{C}^{m+2})$ , where  $m = 2n$ .*

### 3 Proof of Theorem 1.1

From the expression of the curvature tensor of  $G_2(\mathbb{C}^{m+2})$  we get

$$\begin{aligned}
R_{\xi_i}(X) &= X - \eta_i(X)\xi_i - 3g(\phi X, \xi_i)\phi\xi_i - 3\sum_{\nu=1}^3 g(\phi_\nu X, \xi_i)\phi_\nu\xi_i \\
&\quad + \sum_{\nu=1}^3 g(\phi_\nu\phi\xi_i, \xi_i)(\phi_\nu\phi X - \eta(X)\xi_\nu) + \sum_{\nu=1}^3 g(\phi_\nu\phi X, \xi_i)(\eta(\xi_i)\xi_\nu - \phi_\nu\phi\xi_i) \\
&\quad - \eta(\xi_i)\phi_i\phi X + \eta(X)\phi_i\phi\xi_i + g(A\xi_i, \xi_i)AX - g(AX, \xi_i)A\xi_i \tag{2}
\end{aligned}$$

for any tangent vector field  $X$ . From (2) we have

$$\begin{aligned}
(\nabla_X R_{\xi_i})Y &= -g(Y, \nabla_X \xi_i)\xi_i - \eta_i(Y)\nabla_X \xi_i - 3\left\{(\eta(Y)\eta_i(AX) \right. \\
&\quad \left. - \eta(\xi_i)g(AX, Y) + g(\phi Y, \nabla_X \xi_i))\phi\xi_i + \eta_i(\phi Y)(\eta(\xi_i)AX - \eta_i(AX)\xi \right. \\
&\quad \left. + \phi\nabla_X \xi_i)\right\} - 3\sum_{\nu=1}^3 \left\{(-q_{\nu+1}(X)\eta_i(\phi_{\nu+2}Y) \right. \\
&\quad \left. + q_{\nu+2}(X)\eta_i(\phi_{\nu+1}Y) + \eta_\nu(Y)\eta_i(AX) - \eta_\nu(\xi_i)g(AX, Y) \right. \\
&\quad \left. + g(\phi_\nu Y, \nabla_X \xi_i))\phi_\nu\xi_i + \eta_i(\phi_\nu Y)(-q_{\nu+1}(X)\phi_{\nu+2}\xi_i \right. \\
&\quad \left. + q_{\nu+2}(X)\phi_{\nu+1}\xi_i + \eta_\nu(\xi_i)AX - \eta_i(AX)\xi_\nu + \phi_\nu\nabla_X \xi_i)\right\} \\
&\quad + \sum_{\nu=1}^3 \left\{(-g(AX, \phi_\nu\xi_i)\eta(\xi_i) + \eta_i(AX)\eta(\phi_\nu\xi_i) \right. \\
&\quad \left. + g(\nabla_X \xi_i, \phi\phi_\nu\xi_i) - q_{\nu+1}(X)\eta_i(\phi\phi_{\nu+2}\xi_i) + q_{\nu+2}(X)\eta_i(\phi\phi_{\nu+1}\xi_i) \right. \\
&\quad \left. - \eta_\nu(\xi_i)g(AX, \phi\xi_i) + \eta_i(AX)\eta_\nu(\phi\xi_i) + g(\phi_\nu\phi\xi_i, \nabla_X \xi_i))\phi_\nu\phi Y \right. \\
&\quad \left. + \eta_i(\phi_\nu\phi\xi_i)(-q_{\nu+1}(X)\phi_{\nu+2}\phi Y + q_{\nu+2}(X)\phi_{\nu+1}\phi Y + \eta_\nu(\phi Y)AX \right. \\
&\quad \left. - g(AX, \phi Y)\xi_\nu + \eta(Y)\phi_\nu AX - g(AX, Y)\phi_\nu\xi)\right\} - \sum_{\nu=1}^3 \left\{(-\eta(\xi_i)g(AX, \phi_\nu\xi_i) \right. \\
&\quad \left. + \eta_i(AX)\eta_\nu(\phi\xi_i) + g(\nabla_X \xi_i, \phi\phi_\nu\xi_i) - q_{\nu+1}(X)\eta_i(\phi\phi_{\nu+2}\xi_i) \right. \\
&\quad \left. + q_{\nu+2}(X)\eta_i(\phi\phi_{\nu+1}\xi_i) - \eta_\nu(\xi_i)g(AX, \phi\xi_i) + \eta_i(AX)\eta_\nu(\phi\xi_i) \right. \\
&\quad \left. + g(\nabla_X \xi_i, \phi_\nu\phi\xi_i))\eta(Y)\xi_\nu + \eta_i(\phi\phi_\nu\xi_i)(g(Y, \phi AX)\xi_\nu + \eta(Y)\nabla_X \xi_\nu)\right\} \\
&\quad + \sum_{\nu=1}^3 \left\{(-q_{\nu+1}(X)g(Y, \phi\phi_{\nu+2}\xi_i) + q_{\nu+2}(X)g(Y, \phi\phi_{\nu+1}\xi_i) \right. \\
&\quad \left. + \eta_\nu(\phi Y)\eta_i(AX) - g(AX, \phi Y)\eta_\nu(\xi_i) + \eta(Y)\eta_i(\phi_\nu AX) + g(AX, Y)\eta_\nu(\phi\xi_i) \right.
\end{aligned}$$

$$\begin{aligned}
& +g(\phi_\nu\phi Y, \nabla_X\xi_i)\eta(\xi_i)\xi_\nu + g(Y, \phi\phi_\nu\xi_i)((\eta(\nabla_X\xi_i) \\
& +\eta_i(\phi AX))\xi_\nu + \eta(\xi_i)\nabla_X\xi_\nu) \Big\} - \sum_{\nu=1}^3 \left\{ (-q_{\nu+1}(X)\eta_i(\phi_{\nu+2}\phi Y) \right. \\
& +q_{\nu+2}(X)\eta_i(\phi_{\nu+1}\phi Y) + \eta_\nu(\phi Y)\eta_i(AX) - g(AX, \phi Y)\eta_\nu(\xi_i) + \eta(Y)\eta_i(\phi_\nu AX) \\
& +g(AX, Y)\eta(\phi_\nu\xi_i) + g(\phi_\nu\phi Y, \nabla_X\xi_i)\phi_\nu\phi\xi_i + \eta_i(\phi_\nu\phi Y)(-q_{\nu+1}(X)\phi_{\nu+2}\phi\xi_i \\
& +q_{\nu+2}(X)\phi_{\nu+1}\phi\xi_i + \eta_\nu(\phi\xi_i)AX - g(AX, \phi\xi_i)\xi_\nu + \eta(\xi_i)\phi_\nu AX \\
& \left. -\eta_i(AX)\phi_\nu\xi + \phi_\nu\phi\nabla_X\xi_i) \right\} - \left\{ (\eta(\nabla_X\xi_i) + \eta_i(\nabla_X\xi)) \right\} \phi_i\phi Y \\
& +\eta(\xi_i)(-q_{i+1}(X)\phi_{i+2}\phi Y + q_{i+2}(X)\phi_{i+1}\phi Y + \eta_i(\phi Y)AX - g(AX, \phi Y)\xi_i \\
& +\eta(Y)\phi_i AX - g(AX, Y)\phi_i\xi) \Big\} + g(Y, \nabla_X\xi)\phi_i\phi\xi_i + \eta(Y)(-q_{i+1}(X)\phi_{i+2}\phi\xi_i \\
& +q_{i+2}(X)\phi_{i+1}\phi\xi_i - g(AX, \phi\xi_i)\xi_i + \phi_i\nabla_X\xi_i + \eta(\xi_i)\phi_i AX - \eta_i(AX)\phi_i\xi \\
& +\phi_i\phi\nabla_X\xi_i) + (\eta_i(\nabla_X A\xi_i) + g(A\xi_i, \nabla_X\xi_i))AY + \eta_i(A\xi_i)(\nabla_X A)Y \\
& -\left\{ (\eta_i((\nabla_X A)Y) + g(AY, \nabla_X\xi_i))A\xi_i + \eta_i(AY)\nabla_X A\xi_i \right\} \tag{3}
\end{aligned}$$

for any  $X, Y$  tangent to  $M$ .

We will write  $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ , for a unit  $X_0 \in \mathcal{D}$ , where we suppose  $\eta(X_0)\eta(\xi_1) \neq 0$ . Then we have  $g(\phi_\nu\phi\xi_1, \xi_1) = 0$ ,  $\nu = 1, 2, 3$ . Notice this is true even if  $\xi \in \mathcal{D}$ . Thus, the covariant derivative of  $R_{\xi_1}$  is given by

$$\begin{aligned}
(\nabla_X R_{\xi_1})Y &= \nabla_X(R_{\xi_1}(Y)) - R_{\xi_1}(\nabla_X Y) \\
&= -g(Y, \nabla_X\xi_1)\xi_1 - \eta_1(Y)\nabla_X\xi_1 \\
&\quad -3\left\{ \eta(Y)g(AX, \xi_1) - g(AX, Y)\eta(\xi_1) + g(\phi Y, \nabla_X\xi_1) \right\} \phi\xi_1 \\
&\quad +g(\phi Y, \xi_1)(\eta(\xi_1)AX - g(AX, \xi_1)\xi + \phi\nabla_X\xi_1) \Big\} \\
&\quad -3\sum_{\nu=1}^3 \left\{ \left\{ -q_{\nu+1}(X)g(\phi_{\nu+2}Y, \xi_1) + q_{\nu+2}(X)g(\phi_{\nu+1}Y, \xi_1) \right. \right. \\
&\quad +\eta_\nu(Y)g(AX, \xi_1) - g(AX, Y)\eta_\nu(\xi_1) + g(\phi_\nu Y, \nabla_X\xi_1) \Big\} \phi_\nu\xi_1 \\
&\quad +g(\phi_\nu Y, \xi_1)\left\{ -q_{\nu+1}(X)\phi_{\nu+2}\xi_1 + q_{\nu+2}(X)\phi_{\nu+1}\xi_1 + \eta_\nu(\xi_1)AX \right. \\
&\quad \left. -g(AX, \xi_1)\xi_\nu + \phi_\nu\nabla_X\xi_1 \right\} + \sum_{\nu=1}^3 \left\{ \left\{ -q_{\nu+1}(X)g(Y, \phi\phi_{\nu+2}\xi_1) \right. \right. \\
&\quad +q_{\nu+2}(X)g(Y, \phi\phi_{\nu+1}\xi_1) + \eta_\nu(\phi Y)g(AX, \xi_1) - g(AX, \phi Y)\eta_\nu(\xi_1) \\
&\quad +\eta(Y)g(\phi_\nu AX, \xi_1) + g(AX, Y)\eta_\nu(\phi\xi_1) + g(\phi_\nu\phi Y, \nabla_X\xi_1) \Big\} \eta(\xi_1)\xi_\nu \\
&\quad \left. +g(Y, \phi\phi_\nu\xi_1)\left\{ g(\nabla_X\xi_1, \xi) + g(\xi_1, \phi AX) \right\} \xi_\nu + \eta(\xi_1)\nabla_X\xi_\nu \right\}
\end{aligned}$$



$$\begin{aligned}
& - \sum_{\nu=1}^3 \left\{ \{-q_{\nu+1}(X)g(\phi_{\nu+2}\phi Y, \xi_1) + q_{\nu+2}(X)g(\phi_{\nu+1}\phi Y, \xi_1) \right. \\
& + \eta_\nu(\phi Y)g(AX, \xi_1) - g(AX, \phi Y)\eta_\nu(\xi_1) + g(\phi_\nu AX, \xi_1)\eta(Y) \\
& + g(AX, Y)\eta(\phi_\nu \xi_1) + g(\phi_\nu \phi Y, \nabla_X \xi_1)\} \phi_\nu \phi \xi_1 \\
& + g(\phi_\nu \phi Y, \xi_1) \{-q_{\nu+1}(X)\phi_{\nu+2}\phi \xi_1 + q_{\nu+2}(X)\phi_{\nu+1}\phi \xi_1 + \eta_\nu(\phi \xi_1)AX \\
& - g(AX, \phi \xi_1)\xi_\nu + \eta(\xi_1)\phi_\nu AX - g(AX, \xi_1)\phi_\nu \xi + \phi_\nu \phi \nabla_X \xi_1\} \\
& - \left\{ \{g(\nabla_X \xi_1, \xi) + g(\xi_1, \nabla_X \xi)\} \phi_1 \phi Y + \eta(\xi_1) \{-q_2(X)\phi_3 \phi Y + q_3(X)\phi_2 \phi Y \right. \\
& + \eta_1(\phi Y)AX - g(AX, \phi Y)\xi_1 + \eta(Y)\phi_1 AX - g(AX, Y)\phi_1 \xi\} \\
& + g(Y, \nabla_X \xi)\phi_1 \phi \xi_1 + \eta(Y)(-q_2(X)\phi_3 \phi \xi_1 + q_3(X)\phi_2 \phi \xi_1 \\
& - g(AX, \phi \xi_1)\xi_1 + \phi_1 \nabla_X \xi_1 + \eta(\xi_1)\phi_1 AX - g(AX, \xi_1)\phi_1 \xi + \phi_1 \phi \nabla_X \xi_1\} \\
& + \{g(\nabla_X A\xi_1, \xi_1) + g(A\xi_1, \nabla_X \xi_1)\} AY + g(A\xi_1, \xi_1)(\nabla_X A)Y \\
& \left. - \left\{ \{g((\nabla_X A)Y, \xi_1) + g(AY, \nabla_X \xi_1)\} A\xi_1 + g(AY, \xi_1)\nabla_X A\xi_1 \right\} \right\} \quad (4)
\end{aligned}$$

for any  $X, Y$  tangent to  $M$ . From this expression we have:

**Lemma 3.1.** *Let  $M$  be a Hopf real hypersurface in  $G_2(\mathbb{C}^{m+2})$  such that  $\mathcal{D}$  or the  $\mathcal{D}^\perp$ -component of the Reeb vector field is  $A$ -invariant. If  $\nabla_X R_{\xi_i} = 0$ ,  $i = 1, 2, 3$ , for any  $X \in \mathcal{D}$ , then  $\xi \in \mathcal{D}$  or  $\xi \in \mathcal{D}^\perp$ .*

**Proof.** As we suppose  $A\xi = \alpha\xi$  and have written  $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$  with  $\eta(X_0)$  and  $\eta(\xi_1)$  nonnull, where  $X_0 \in \mathcal{D}$  is unit, we get from (4)

$$\begin{aligned}
0 & = g((\nabla_{X_0} R_{\xi_1})\xi_1, X_0) \\
& = -g(\nabla_{X_0} \xi_1, X_0) + \eta(\xi_1)g(\phi_1 \nabla_{X_0} \xi_1, X_0) + g(\phi_1 \phi \nabla_{X_0} \xi_1, X_0) - \alpha g(A\nabla_{X_0} \xi_1, X_0).
\end{aligned} \quad (5)$$

Now we have

$$g(\nabla_{X_0} \xi_1, X_0) = g(\phi_1 AX_0, X_0) = \alpha g(\phi_1 X_0, X_0) = 0.$$

We also have  $g(\phi_1 \nabla_{X_0} \xi_1, X_0) = -g(\nabla_{X_0} \xi_1, \phi_1 X_0) = -g(\phi_1 AX_0, \phi_1 X_0)$   
 $= -\alpha g(\phi_1 X_0, \phi_1 X_0) = -\alpha$ ,  $g(\phi_1 \phi \nabla_{X_0} \xi_1, X_0) = g(\nabla_{X_0} \xi_1, \phi \phi_1 X_0) = g(\phi_1 AX_0, \phi \phi_1 X_0)$   
 $= \alpha g(\phi_1 X_0, \phi \phi_1 X_0) = 0$ , and

$$g(A\nabla_{X_0} \xi_1, X_0) = \alpha g(\nabla_{X_0} \xi_1, X_0) = \alpha g(\phi_1 AX_0, X_0) = \alpha^2 g(\phi_1 X_0, X_0) = 0.$$

From (5) we get  $-\alpha\eta(\xi_1) = 0$ . If  $\alpha \neq 0$  in (3) we get  $\xi \in \mathcal{D}$ . If  $\alpha = 0$ , the result follows from [9].  $\square$

With the hypothesis in Lemma 3.1 we can prove.

**Lemma 3.2.** *If  $\xi \in \mathcal{D}^\perp$  then  $g(A\mathcal{D}, \mathcal{D}^\perp) = 0$ .*

**Proof.** In this case we can take  $\xi = \xi_1$ . Thus it is enough to prove that  $\eta_2(AX) = \eta_3(AX) = 0$ , for any  $X \in \mathcal{D}$ . For such an  $X$  we get  $0 = (\nabla_X R_{\xi_1})\xi_1 = (\nabla_X R_\xi)\xi = -\phi AX - \alpha A\phi AX - 2\eta_3(AX)\xi_2 + 2\eta_2(AX)\xi_3 - \phi_1 AX$ . If  $\alpha = 0$ , taking the scalar product of (3) and  $\xi_2$ , respectively  $\xi_3$ , we obtain the result. If  $\alpha \neq 0$ , taking the scalar product of (3) and  $\xi_2$  we get

$$\alpha g(A\phi AX, \xi_2) + 2\eta_3(AX) = 0. \quad (6)$$

On the other hand, by Proposition 2.3 we obtain

$$2g(A\phi AX, \xi_2) = \alpha g(A\phi X, \xi_2) + 2\eta_3(AX). \quad (7)$$

From (6) and (7) we have  $\alpha^2 g(A\phi X, \xi_2) + (\alpha^2 + 4)\eta_3(AX) = 0$ . Thus

$$g(A\phi X, \xi_2) = -\frac{\alpha^2 + 4}{\alpha^2} \eta_3(AX). \quad (8)$$

If we change  $X$  by  $\phi X$  in (8), it follows

$$-\alpha^2 \eta_2(AX) + (\alpha^2 + 4)\eta_3(A\phi X) = 0. \quad (9)$$

Now applying the same procedure to the scalar product of (3) and  $\xi_3$  we arrive to

$$g(A\phi X, \xi_3) = \frac{\alpha^2 + 4}{\alpha^2} \eta_2(AX) \quad (10)$$

and changing  $X$  by  $\phi X$  in (10) it gives

$$\alpha^2 \eta_3(AX) + (\alpha^2 + 4)\eta_2(A\phi X) = 0. \quad (11)$$

From (8) and (10) we obtain

$$\left(-\alpha^2 + \frac{(\alpha^2 + 4)^2}{\alpha^2}\right) \eta_2(AX) = 0, \quad (12)$$

and from (9) and (11) we have

$$\left(-\alpha^2 + \frac{(\alpha^2 + 4)^2}{\alpha^2}\right)\eta_3(AX) = 0. \quad (13)$$

From (12) and (13) we get  $\eta_2(AX) = \eta_3(AX) = 0$ , finishing the proof.  $\square$

From this Lemma and Proposition 2.4, in order to finish the proof of our Theorem, we only have to see if the real hypersurfaces of either type (A) or type (B) satisfy our condition.

In the case of a real hypersurface of type (A) we get from Proposition 2.1, considering  $\xi = \xi_1$  and taking  $X_i \in T_\lambda$ ,  $\phi X_i \in T_\lambda$ . Thus

$$0 = (\nabla_{X_i} R_\xi)\xi = -2\lambda\phi X_i - \alpha\lambda A\phi X_i = (-2\lambda - \alpha\lambda^2)\phi X_i. \quad (14)$$

Thus  $2\lambda + \alpha\lambda^2 = 0$ . This yields either  $\lambda = 0$  or  $2 + \alpha\lambda = 0$ . As from Proposition 2.1,  $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ , for some  $r \in (0, \frac{\pi}{\sqrt{8}})$ ,  $\lambda \neq 0$ . Thus  $2 + \alpha\lambda = 0$ . Then

$$\begin{aligned} 0 &= (\sqrt{8}\cot(\sqrt{8}r))(-\sqrt{2}\tan(\sqrt{2}r)) + 2 \\ &= \cot(\sqrt{8}r)\tan(\sqrt{2}r) - 2 = 2\tan^2(\sqrt{2}r). \end{aligned}$$

But for  $r \in (0, \pi/\sqrt{8})$ , then  $\tan(\sqrt{2}r) \neq 0$ , thus these real hypersurfaces do not satisfy our condition.

If we consider now a real hypersurface of type (B), from Proposition 2.2 we get

$$g((\nabla_\xi R_{\xi_1})\xi_1, \phi\xi_1) = -4\alpha. \quad (15)$$

If the real hypersurface satisfies our condition, from (15) we have  $\alpha = 0$ , but from Proposition 2.2,  $\alpha = -2\tan(2r)$  for some  $r \in (0, \pi/4)$ . This gives a contradiction, so real hypersurfaces of type (B) do not satisfy our condition. This finishes the proof.

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