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On a General Even Order Structure on a Differentiable Manifold

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Abstract

K. Yano defined and studied the structures defined by a tensorfield $f(\neq 0)$ of type (1, 1) satisfying $f^3 + f = 0$, $f^4 \pm f^2 = 0$ ([1], [3]). In this paper, we have considered the structure of order $2n$ defined by (1, 1) tensorfield f where n is a positive integer. Certain interesting results have been obtained. Local coordinate system is introduced in the manifold and it has been shown that there exist complementary distributions L^* and M^* and a positive definite Riemannian metric G such that they are orthogonal.

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1 General even order structure

Let M be an m-dimensional differentiable manifold of differentiability class C^∞. Suppose there exists on M, a tensor field $f(\neq 0)$ of type (1, 1) satisfying

$$f^{2n} + af^n + bI_{2n} = 0 \tag{1.1}$$

where n is a positive integer $(n > 1)$, a, b scalars not equal to zero and I_{2n} denotes the unit tensor field.

Then we say that the manifold M is equipped with general even order structure. We now prove the following theorem:

Theorem 1.1 *The general even order structure is not unique.*

Proof. Let μ be a non-singular real valued function and f' a tensorfield of type (1, 1) on M such that

$$\mu f' = f\mu \tag{1.2}$$

Then, by (1.2)

$$\mu f'^2 = f(\mu f') = f^2\mu \quad .$$

In a similar manner, we have

$$\mu f'^3 = f^3\mu, \dots, \mu f'^{2n} = f^{2n}\mu$$

Therefore

$$\begin{aligned} \mu(f')^{2n} + a\mu(f')^n + b\mu I_{2n} &= (f^{2n})\mu + a(f)^n\mu + bI_{2n}\mu \\ &= (f^{2n} + af^n + bI_{2n})\mu \\ \text{by (1.1)} \qquad &= 0 \end{aligned}$$

Since μ is non-singular we have

$$(f')^{2n} + a(f')^n + bI_{2n} = 0$$

Thus f' gives to M another general even order structure. Therefore, such structure is not unique. □

Theorem 1.2 *The rank of the general even order structure is equal to dimension of the manifold.*

Proof. Let M be of dimension m. If X be a vector field on M such that

$$f(X)=0 \Rightarrow f^2(X)=f^3(X)=...=f^n(X)=0.$$

Also $f^{2n}(X)=0$. Hence from (1.1) it follows that $X=0$.

Hence kernel of f contains only zero vector field. So if $\nu(f)$ be nullity of f, $\nu(f)=0$.

If $\rho(f)$ be rank of f, then from a well known theorem of Linear Algebra

$$\rho(f)+\nu(f)=\text{dimension of } M$$

As $\nu(f)=0$, therefore

$$\rho(f)=m$$

Hence we have the theorem. □

Theorem 1.3 *Let f and f' be two general even order structures on a differentiable manifold M such that the equation (1.2) holds. If V is an eigenvector of f' corresponding to some eigenvalue, μV is the eigenvector of f corresponding to same eigenvalue.*

Proof. As given, V is the eigenvector of f' for the eigenvalue λ. Then

$$f'V=\lambda V$$

Therefore

$$(\mu f')(V)=\mu(\lambda V)$$

or by (1.2)

$$f(\mu V)=\lambda(\mu V)$$

So μV is the eigenvector of f for the same eigenvalue λ. □

Theorem 1.4 *The dimension m of the manifold M equipped with general even order structure satisfying the equation (1.1) for $a^2<4b$ is even.*

Proof. Let V be eigenvector of f corresponding to eigenvalue λ. So

$$f(V) = \lambda V, \quad f^2(V) = \lambda^2 V, ..., \ f^n(V) = \lambda^n V, ...$$

Hence by virtue of the equation (1.1), it follows that

$$\lambda^{2n} + a\lambda^n + b = 0$$

which has solution of the form

$$\lambda^n = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

If $a^2 < 4b$, the values of λ^n are complex. Hence the eigenvalues of f are complex numbers. Since complex roots occur in pair, hence number of the eigenvalues must be even. Consequently dimension of M is even and $m = 2n$ as f has $2n$ non-zero distinct eigenvalues. □

2 Necessary and sufficient condition for existence of the general even order structure

For the manifold M equipped with general even order structure, the eigenvalues of f are given by

$$\lambda^n = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Taking $a^2 < 4b$ and $-\frac{a}{2} = \cos\theta$, $\frac{\sqrt{4b - a^2}}{2} = \sin\theta$. Then f has $2n$ eigenvalues given by

$$\lambda = e^{\pm \frac{i\theta}{n}}, \quad n = 1, 2, 3, ..., n$$

Let P_x, $x = 1, 2, \ldots, n$ be eigenvectors of f corresponding to eigenvalue $e^{\frac{i\theta}{x}}$ and Q_x, $x = 1,2,.....,n$ be eigenvectors for the eigenvalue $e^{-\frac{i\theta}{x}}$. Then $\{P_x\}$ and $\{Q_x\}$ are linearly independent sets.

For the set $\{P_x, Q_x\}$, suppose that

$$a^x P_x + b^x Q_x = 0, \quad x = 1, 2, \ldots, n \quad \text{and} \quad a^x, b^x \in R \tag{2.1}$$

Then operating the above equation (2.1) by f and taking into account that $\{P_x, Q_x\}$ are eigenvectors for eigenvalues $e^{\frac{i\theta}{x}}$ and $e^{-\frac{i\theta}{x}}$ of f, we get

$$a^x e^{\frac{i\theta}{x}} P_x + b^x e^{-\frac{i\theta}{x}} Q_x = 0 \tag{2.2}$$

In view of the equations (2.1) and (2.2), we get

$$b^x (1 - e^{-\frac{2i\theta}{x}}) Q_x = 0 \Rightarrow b^x = 0,\ x = 1, 2, ..., n$$

Consequently from (2.1), it follows that $a^x = 0$ as $\{P_x\}$ is linearly independent.

Thus the set $\{P_x, Q_x\}$ is linearly independent. Let us assume that $\pi_1, \pi_2, ..., \pi_n$ be tangent sub-bundles spanned by $P_1, P_2, ..., P_n$ respectively and $\tilde{\pi}_1, \tilde{\pi}_2, ..., \tilde{\pi}_n$ spanned by $Q_1, Q_2, ..., Q_n$ respectively.

Then

$$\pi_1 \cap \tilde{\pi}_1 = \phi, \pi_2 \cap \tilde{\pi}_2 = \phi, ..., \pi_n \cap \tilde{\pi}_n = \phi$$

and

$$\pi_1 \cup \pi_2 \cup ... \cup \pi_n \cup \tilde{\pi}_1 \cup \tilde{\pi}_2 \cup ... \cup \tilde{\pi}_n$$

is a tangent bundle of dimension $2n$.

Thus if the manifold M admits the general even order structure of rank $2n$, it possesses tangent subbundles $\pi_1, \pi_2, ..., \pi_n$ each of dimension unit and subbundles $\tilde{\pi}_1, \tilde{\pi}_2, ..., \tilde{\pi}_n$ conjugate to $\pi_1, \pi_2, ..., \pi_n$, respectively, such that

$\pi_1, \tilde{\pi}_1; \pi_2, \tilde{\pi}_2; ...; \pi_n, \tilde{\pi}_n$ are mutually disjoint and they span together a tangent bundle of dimension $2n$.

Suppose conversely that M admits the general even order structure of rank $2n$. Let $p^1, p^2, ..., p^n, q^1, q^2, ..., q^n$ be 1-forms dual to vector fields $P_1, P_2, ..., P_n, Q_1, Q_2, ..., Q_n$ respectively. So

$$p^1 \otimes P_1 + p^2 \otimes P_2 + ... + p^n \otimes P_n + q^1 \otimes Q_1 + q^2 \otimes Q_2 + ... + q^n \otimes Q_n = I_{2n}$$

or equivalently $p^x \otimes P_x + q^x \otimes Q_x = I_{2n}$, x takes the values $1,2,......,n$ and I_{2n} denotes the unit tensor field.

Let us now put

$$f = e^{-\frac{in\theta}{x}} p^x \otimes P_x + e^{\frac{in\theta}{x}} q^x \otimes Q_x$$

Then it is easy to show

$$f^{2n} = e^{\frac{in\theta}{x}} p^x \otimes P_x + e^{-\frac{in\theta}{x}} q^x \otimes Q_x$$

and

$$f^n = p^x \otimes P_x + q^x \otimes Q_x$$

Thus

$$f^{2n} + af^n = (a + e^{\frac{in\theta}{x}}) p^x \otimes P_x + (b + e^{-\frac{in\theta}{x}}) q^x \otimes Q_x \qquad (2.3)$$

It is possible to set

$$(a + e^{\frac{in\theta}{x}}) = (b + e^{-\frac{in\theta}{x}}) = -b$$

Hence the equation (2.3) takes the form

$$f^{2n} + af^n = -b\{p^x \otimes P_x + q^x \otimes Q_x\}$$

or

$$f^{2n} + af^n + bI_{2n} = 0$$

Thus the manifold M admits the general even order structure of rank $2n$. Thus we have.

Theorem 2.1 *In order that the differentiable manifold M admits the general even order structure of rank $2n$, it is necessary and sufficient that it possesses tangent subbundles $\pi_1, \pi_2, ..., \pi_n$ each of dimension unit and their respective complex conjugates $\tilde{\pi}_1, \tilde{\pi}_2, ..., \tilde{\pi}_n$ such that*

$$\pi_1 \cap \tilde{\pi}_1 = \phi,\ \pi_2 \cap \tilde{\pi}_2 = \phi,\ ...,\ \pi_n \cap \tilde{\pi}_n = \phi,$$

and they span together a tangent bundle of dimension $2n$.

3 General even order structure when $b = 0$

Suppose the manifold M admits the general even order structure for $b = 0$. Hence we have

$$f^{2n} + af^n = 0 \qquad (a \neq 0)$$

If we take the operators

$$l = -af^{-n} \text{ and } m = I + af^{-n} \tag{3.1}$$

Then it is easy to show

$$l^2 = l, \quad m^2 = m, \quad l + m = I, \quad lm = ml = 0.$$

Thus for general even order structure for $b = 0$, the operators l and m defined by (3.1) when applied to the tangent space of M at a point are complementary projection operators. Corresponding to projection operators l and m, we get complementary distributions L^* and M^* respectively. If rank of f is constant every where and equal to r, the dimensions of L^* and M^* are r and $(n-r)$ respectively.

Let us now introduce in the manifold M a local coordinate system and denote by

$$f_i^h;\ l_i^h;\ m_i^h$$

the local components of f, l and m respectively.

Let $u_a^h \, (a,b,c = 1,2,.......,r)$ be r mutually orthogonal unit vectors in L^* and $(2n-r)$ such vectors in M^* denoted by $u_B^h \, (B = 1,2,.......,2n-r)$. Thus we have

$$l_i^h u_b^i = u_b^h , \; l_i^h u_B^i = 0$$
$$m_i^h u_b^i = 0 , \; m_i^h u_B^i = u_B^h \tag{3.2}$$

If (v_i^a , v_i^A) be the matrix inverse to (u_b^h , u_B^h), then we can write

$$v_i^a u_b^i = \delta_b^a , v_i^a u_B^i = 0$$
$$v_i^A u_b^i = 0 , v_i^A u_B^i = \delta_B^A \tag{3.3}$$

δ_b^a denotes the Kroneker delta. Also

$$v_i^a u_a^h + v_i^A u_A^h = \delta_i^h$$

In view of the equations (3.2) and (3.3), we have

$$(l_i^h v_h^a) u_b^i = \delta_b^a , (l_i^h v_h^a) u_B^i = 0$$
$$(m_i^h v_h^A) u_b^i = 0 , (m_i^h v_h^A) u_B^i = \delta_B^A$$

Thus we have

$$l_i^h v_h^a = v_i^a , l_i^h v_h^A = 0$$
$$m_i^h v_h^a = 0 , m_i^h v_h^A = v_i^A \tag{3.4}$$

Since $fm = 0$, we have $f_i^h m_h^j = 0$. Contracting with v_j^A and using (3.4), we get

$$f_i^h v_h^A = 0$$

Again since $l_j^h u_a^j = u_a^h$, therefore

$$l_j^h u_a^j v_i^a = u_a^h v_i^a$$

or $$l_j^h (\delta_i^j - u_A^j v_i^A) = u_a^h v_i^a$$

Thus we have

$$l_i^h = u_a^h v_i^a$$

Similarly we can show that

$$m_i^h = u_A^h v_i^A$$

Let us now define

$$g_{ji} = v_j^a v_i^a + v_j^A v_i^A$$

Then g_{ji} is globally defined positive definite Riemannian metric relative to which (u_b^h, u_B^h) form an orthogonal frame and

$$v_j^a = g_{ji} u_a^i, v_j^A = g_{ji} u_A^i$$

Let us further put

$$l_{ji} = v_j^a v_i^a, m_{ji} = v_j^A v_i^A$$

Thus

$$l_{ji} + m_{ji} = g_{ji}$$

The following equations can be proved easily

$$l_j^t l_i^s g_{ts} = l_{ji}$$
$$l_j^t m_i^s g_{ts} = 0$$
$$m_j^t m_i^s g_{ts} = m_{ji}$$

If we put

$$G_{ji} = \frac{1}{2}(g_{ji} + m_{ji} + f_t^s f_s^t g_{ij})$$

then G_{ji} is globally defined Riemannian metric and satisfies

$$v_j^A = G_{ji} u_A^i \quad \text{and} \quad m_{ji} = m_j^t G_{ti}$$

Now

$$G(u_a, u_A) = \frac{1}{2}\{g(u_a, u_A) + m(u_a, u_A) + f_t^s f_s^t u_a^i u_A^j\} \tag{3.5}$$

Since L^* and M^* are orthogonal with respect to Riemannian metric g, hence in view of above equation (3.5), it follows that L^* and M^* are also orthogonal with respect to G. Hence we have the theorem.

Theorem 3.1 *Let M be a $2n$ dimensional differentiable manifold equipped with general even order structure of rank $2n$. Then there exist complementary distributions L^* and M^* and a positive definite Riemannian metric G with respect to which L^* and M^* are orthogonal.*

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