Theoretical Mathematics & Applications, vol.1, no.1, 2011, 37-48
ISSN: 1792-9687 (print), 1792-9709 (online)
International Scientific Press, 2011

New Relations between Thickness and Outer Thickness of a Graph and its Arboricity

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Abstract

The thickness of a graph is the minimum number of planar into which the graph can be decomposed. Determining the thickness of a graph is known too NP-complete problem. The outer thickness of a graph is minimum number of outer planar into which the graph can be decomposed. Outer thickness is one of the classical and standard measures of non-outer planarity of graphs. We conjecture that determining the outer thickness of a graph is also NP-complete. Arboricity of a graph is the minimum number of edge-disjoint forests whose union is G. In this paper, we show the new relations between thickness and outer thickness of a graph and its arboricity.

Mathematics Subject Classification : 97K30, 05C10
Keywords: Thickness, Outer thickness, Arboricty

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Article Info: *Revised* : August 11, 2011. *Published online* : October 31, 2011

1 Introduction

In VLSI circuit design, a clip id represented as a hyper graph corresponding to the nets connecting the cells, a chip-designer has to place the marco cells on a printed circuit bord (which usually consist of superimposed layers) , according to several designing rules. One of these requirements is to avoid crossing, since crossing lead to undesirable signals, it is therefore desirable to find ways to handle wire crossing of the graph representing the chip. In practice, crossing-wires must be laid out in different layers. There are two approaches for distributing the nets to the layers.

According to the first method, one of the wires must change its layer with help of so-called contact cuts when ever a crossing between two wires occurs. Unfortunately, the presence of too many contact cuts leads to an increase in area and consequently to a higher probability of faulty chips. Therefore, a requirement of this manufacturing method is to reduce crossing as much as possible. If a large number of crossings are unavoidable, a second approach is appropriate. The representing graph is decomposed into planar sub graph each completely embedded. In one layer which is not used by the outer planar sub graph.

Since no contact cuts are used, the manufacturing cost measure of this method is the number of layers. An application of this approach was given by Aggarwal, Klawe and Shor [1].

Indeed both approaches have a graph theoretic counterpart. In the first one we look for the minimum number of edge-crossing needed in a graph-embedding the so-called crossing number cr (G) of a graph. In the second approach, the minimum number of planar sub graph (graphs), whose union is (are) the original graph, is requested. This number is called the thickness $\theta(G)$ of the graph G. Similarly we define outer thickness. Determining the thickness of a graph is known too NP-complete problem [23, 21, 16, 4, 6, 8, 13, 17-19].

2 Characterization of Planar and Outer Planer Graphs

Definition 2.1. *A* graph *H* is said to be homemorphic from *G* if either $H \cong G$ or *H* is isomorphic to a subdivision of G. *A* graph G_1 is homemorphic with G_2 if there exists a graph a graph G_3 such that G_1 and G_2 are both homemorphic from G_3.

Corollary 2.1. If a graph *G* has sub graph that is homemorphic from K_5 or $K_{3,3}$, then *G* is non planar.
Proof. Trivial. ◊

Theorem 2.1 [14]. *A* graph is planar if and only if it does not contain a sub graph which is homemorphic from K_5 or $K_{2,3}$.

Theorem 2.2 [7]. *A* graph is outer planar if and only if it does not contain a sub graph which is homemorphic from K_4 or $K_{2,3}$.

Corollary 2.2 [20]. Let *G* be a connected planar graph with n vertices, m edge, and f faces, the we have for the plane embedding of G that $n-m+f=2$.

Corollary 2.3 [20]. If $l(f_i)$ does not the length of face F_i in a plane graph *G*, then $2E=\sum l(f_i)$.

Definition 2.2. If a graph $G'=(V,E')$ is an outer planer sub graph of *G* such that every graph G'' obtained from G' by adding on edge from $E \setminus E'$ is non- outer planer, then G' is called a maximal outer planer sub graph of *G*.

Definition 2.3. If there is no planar sub graph $G''=(V,G'')$ of *G* $|E''|>|E'|$, then G' is a maximum planar sub graph.

A maximal planar sub graph is maximal in the sense that adding edge is not possible and the maximum planar sub graph is maximal with respect to the cardinality of its edge set.

Definition 2.4. If a graph $G' = (V, E')$ is a outer planer sub graph of G such that every graph G'' obtained from G' by adding on edge from $E \setminus E'$ is non- outer planer, then G' is called a maximal outer planer sub graph of G.

Definition 2.5. Let $G' = (V, E')$ be a maximal outer planer sub graph of G. If there is no outer planer sub graph $G'' = (V, E'')$ of G with $|E''| > |E'|$, then G' is a maximum outer planer sub graph.

Theorem 2.3. If G is a simple planar graph with at least three vertices, then $E(G) \leq 3n - 6$. If also G is triangle-free, then $E(G) \leq 2n - 4$.

Proof. It suffices to consider connected graphs; otherwise we could add edges. Euler's formula will relate n and E if we can dispose of f.

Corollary 2.3. provides an inequality between E and F. every face $\{f_i\}$ be the list of face lengths, this yields $2E \sum f_i \geq 3f$.

But $n - m + f = 2 \rightarrow f = 2 - n + E$

$\rightarrow \quad 2E \geq 6 - 3n - 3E \rightarrow E \leq 3n - 6$.

When G is triangle- free, the faces length at least 4. In this case

$$2E \sum f_i \geq 4f \rightarrow 2E \geq 8 - 4n + 4E \rightarrow E \leq 2n - 4. \qquad \Diamond$$

Theorem 2.4 [14]. Let $G' = (V, E')$ be a maximum outer planar sub graph of a graph $G = (V, E)$, then $E \leq 2n - 3$. If G is triangle-free, then $|E| < {}^{3n}\!/_{2} - 2$.

Theorem 2.5 [10]. The maximum outer planar sub graph of Q_n contains $3 \times 2^{n-1} - 2$ edge.

3 Thickness

Definition 3.1 the thickness of a graph, denoted by $\theta(G)$, is the minimum of planar sub graph in to which the graph can be decomposed. Evidently, $\theta(G)=1$, if and only if G is planar.

Theorem 3.1. If $G=(V,E)$ is a graph with $(|V|=n>2$ and $|E|=m$. Then :

i) $\theta(G)\geq\left\lceil\frac{m}{3n-6}\right\rceil$

ii) $\theta(G)\geq\left\lceil\frac{m}{2n-4}\right\rceil$, if G has no triangle.

Proof. By Theorem 2.3, the denominator is the maximum size of each planar sub graph. The pigeonhole principle then yields the inequality. ◊

Theorem 3.2 [3,9,12]. If $G=(V,E)$ is a graph with $(|V|=n>10$) and $|E|=m$ maximal degree d, then

i) $\theta(k_n)\leq\left\lfloor\frac{n+7}{6}\right\rfloor$

ii) $\theta(G)\leq\left\lfloor\sqrt{\frac{m}{3}}+\frac{7}{6}\right\rfloor$

iii) $\theta(G)\leq\left\lceil\frac{d}{2}\right\rceil$.

Theorem 3.3 [5]. The thickness of the complete bipartite graph $k_{m,n}$ is $\theta(k_{m,n})\leq\left\lceil\frac{m.n}{2(m+n-2)}\right\rceil$ except if m and n are both odd, $m\leq n$ and there is an integer k satisfying $n=\left\lceil\frac{2k(m-2)}{m-2k}\right\rceil$.

Theorem 3.4 [5]. The thickness of the complete bipartite graph $k_{n,n}$ is $\theta(k_{n,n}) = \left\lfloor \frac{n+5}{4} \right\rfloor$.

Theorem 3.5 [3,15]. Let G be a graph with m edges, then it holds that $\theta(G) \le \left\lceil \sqrt{\frac{2m}{3} + \frac{3}{2}} \right\rceil$.

Wessal [22] gave lower and upper bounds for the thickness of a graph as a function of the minimum and maximum degree. The upper bound was independently given also by Halton [12].

Theorem 3.6 [9]. Let G be a graph with minimum degree δ and maximum degree Δ then $\left\lceil \frac{\delta+1}{4} \right\rceil \le \theta(G) \le \left\lceil \frac{\Delta}{2} \right\rceil$.

Theorem 3.7 [15]. The thickness of the hypercube Q_n is $\theta(Q_n) = \left\lceil \frac{n+1}{4} \right\rceil$.

3.1 Algorithm for the thickness problem

Now we describe the basic approach to get approximation for thickness. For a detailed description of the extracting method see algorithm (Thick).

Algorithm (THICK)

1. $P \leftarrow 0$ and $t \leftarrow 1$
2. While $E \neq \emptyset$ do
3. Find a planar sub graph of $G' = (V, E_t)$ of $G = (V, E)$;
4. $E \leftarrow E \setminus E_t$;
5. $P \leftarrow P \cup \{E_t\}$;
6. $t \leftarrow t + 1$
7. return p;

4 Outer Thickness

Definition 4.1. The outer thickness of a graph, denoted by $\theta_0(G)$, is the minimum number of outer planar sub graph in to which the graph can be decomposed.

Theorem 4.1. Let $G=(V,E)$ be a graph with $|V|=n$ and $|E|=m$. Then $\theta_0(G) \geq \left\lceil \frac{m}{2n-3} \right\rceil$. If also G has no triangle then $\theta_0(G) \geq \left\lceil \frac{m}{3n/2-2} \right\rceil$.

Proof. By Theorem 2.4, The denominator is maximum size of each outer planar sub graph. The pigeonhole principle yields the inequality. ◊

Theorem 4.2 [10]. For complete graphs, $\theta_0(k_n) = \left\lceil \frac{n+1}{4} \right\rceil$, except that $\theta_0(k_7) = 3$.

Theorem 4.3 [11]. For complete bipartite graphs, with $m \leq n$
$\theta_0(k_{m,n}) = \left\lceil \frac{mn}{2m+n-2} \right\rceil$.

Theorem 4.4 [20]. $(G) \leq \sqrt{\frac{m}{8}} + 0(1)$ for an arbitrary graph G with m edges.

Theorem 4.5 [20]. For a graph with minimum degree δ and maximum degree Δ, it holds that

$$\left\lceil \frac{\delta}{4} \right\rceil \leq \theta_0(G) \leq \left\lceil \frac{\Delta}{2} \right\rceil.$$

Similarly, it is easy to modify (OThick) to approximate outer thickness by changing the "outer planar" in step 3 instead of planar. See algorithm (OThick).

4.1 Algorithm (OThick)

1. $P \leftarrow 0$ and $t \leftarrow 1$
2. While $E \neq \emptyset$ do
3. Find a outer planar sub graph of $G' = (V, E_t)$ of $G = (V, E)$;
4. $E \leftarrow E \setminus E_t$;
5. $P \leftarrow P \cup \{E_t\}$;
6. $t \leftarrow t + 1$
7. return p;

5 Arboricity

Definition 5.1. The arboricity of *G*, denoted by $\gamma(G)$, is the minimum number of edge-disjoint forests whose union is *G*.
Nash – williams gave the exact solution for arboricity.

Theorem 5.1 [9]. For any graph *G*, $\gamma(G) = \max\left\lceil \frac{e_H}{n_H - 1} \right\rceil$, where *H* ranges over all non-trival induced sub graph of *G*.

Theorem 5.2 [2]. If *G* is d-regular, then

$$\gamma(G) = \left\lceil \frac{d+1}{2} \right\rceil = \left\lceil \frac{e}{n-1} \right\rceil \text{ and } \left\lceil \frac{\delta+1}{2} \right\rceil \leq \gamma(G) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil.$$

5.1 Relations among thickness and arboricity

The arboricity and thickness of a graph *G* closely related in size; namely $\theta(G) \leq \gamma(G) \leq 3\theta(G)$. The first inequality follows from the Definitions, and the second follows from Theorems 5.1 and 5.2, with together imply that any planar graph has arboricity at most 3.

5.2 New relation among outer thickness and arboricity

It is well-know that outer thickness and arboricity within a constant factor of each other $\gamma(G) \le 2\theta_0(G)$.

Now we show that

$$\gamma(G) \le \frac{3}{2}\theta_0(G).$$

Claim 1. If G be a graph then

$$\gamma(G) \le \frac{3}{2}\theta_0(G).$$

Proof. According to Theorem 2.4

$$m \le 3n/2 - 2 = \frac{3n-4}{2} \Rightarrow$$

$$m \le \frac{3n-4}{2} < \frac{3n-3}{2} = \frac{3}{2}(n-1) \xrightarrow{theorem(5.1)} \gamma(G) \le \frac{3}{2}\theta_0(G).$$

Also it has been shown that a planar graph can be divided into two outer planar graphs, therefore $\theta_0(G) \le 2\theta(G)$. ◊

5.3 New relation among thickness and arboricity

We show that

$$\gamma(G) \le 4\theta(G).$$

Claim 2. If G is a graph then

$$\gamma(G) \le 4\theta(G).$$

Proof. According to theorem (2.4) $m \le 2n-4$.

$$m \le 2n-4 < 2n-2 = 2(n-1) \to m < 2(n-1) \xrightarrow{theorem(5.1)} \gamma(G) \le 2\theta(G)$$

Also since $\theta(G) \le \theta_0(G)$ and $\theta_0(G) \le 2\theta(G)$, then

$$\gamma(G) \le 2\theta_0(G) \le 2\times 2\theta(G) = 4\theta(G).$$ ◊

5 Conclusion

In this paper, we presented some results concerning the thickness and outer thickness of a graph. In particular, bound on the arboricity of a graph is given. It seems that bound are not unique.

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