

The Existence of the Moments of the Cauchy Distribution under a Simple Transformation of Dividing with a Constant

Johnson Ohakwe¹ and Bright Osu²

Abstract

In this paper, we establish the existence of the moments of a Cauchy distribution with parameters, a and b denoted by Cauchy (a, b) , via a simple transformation of dividing with a suitable constant. As a result of this transformation every Cauchy (a, b) would be distributed on the interval $[-1, 1]$. The results for the first four crude moments were given in view of their importance in obtaining the very important statistical measures namely variance, skewness and kurtosis.

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¹ Department of Mathematics and Statistics, Faculty of Sciences,
Anambra State University, P.M.B. 02, Uli, Anambra State, Nigeria,
e-mail: ohakwe.johnson@yahoo.com

² Department of Mathematics, Faculty of Biological and Physical Sciences,
Abia State University, P.M.B. 2000, Uturu, Abia State, Nigeria,
e-mail: megaobrait@yahoo.com

1 Introduction

The probability density function of a Cauchy distribution (cd) with the location parameter, a and scale parameter b is given by

$$f(x|a,b) = \frac{b}{\pi(b^2 + (x-a)^2)}, \quad -\infty < x < \infty, \quad b > 0. \quad (1)$$

Equation (1) is referred to as Cauchy (a,b) . The standard form of (1) can be obtained by replacing a with 0 and b with 1 and is given by

$$f(x|0,1) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty \quad (2)$$

The Cauchy distribution represents an extreme case and serves as counter examples for some well accepted results and concepts in statistics. For example the central limit theorem does not hold for the limiting distribution of the mean of a random sample from a Cauchy distribution [3]. Because of this special nature, some authors consider the Cauchy distribution as a pathological case. However, it can be postulated as a model for describing data that arise as n realizations of the ratio of two normal random variables. Other applications given in literature: [4] found that Cauchy distribution describes the distribution of velocity differences induced by different vortex elements. An application of the Cauchy distribution to study the polar and non-polar liquids in porous glasses is given in [5]. In [1] pointed out that the Cauchy distribution describes the distribution of hypocenters on focal spheres of earthquakes. It is shown in the paper by [7] that the source of fluctuations in contact window dimensions is variation in contact resistivity, and the contact resistivity is distributed as a Cauchy random variable.

In spite of the wide applications of the Cauchy distribution in various fields, the non-existence of its moments is of huge concern inasmuch as it has undefined expectation value and higher moments diverge. For the expectation value, the integral

$$E(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{1+x^2} dx \quad (3)$$

is not completely convergent, that is

$$\lim_{a \rightarrow \infty, b \rightarrow \infty} \frac{1}{\pi} \int_{-a}^b \frac{x}{1+x^2} dx \tag{4}$$

does not exist. However the principal values

$$\lim_{a \rightarrow \infty} \frac{1}{\pi} \int_{-a}^a \frac{x}{1+x^2} dx \tag{5}$$

does exist and is equal to zero [6]. In any case the convention is to regard the expectation value of the Cauchy distribution as undefined.

Attempts had previously been made to solve the problem of non-definition of the Cauchy distribution through truncation. For a symmetric truncation $-y < x < y$, the renormalized probability density function given by

$$f(x) = \frac{1}{2 \arctan y} \cdot \frac{1}{1+x^2} \tag{6}$$

which has expectation value, $E(x) = 0$, and variance, $Var(x) = \frac{x}{\arctan x} - 1$,

third central moment $\mu_3 = 0$, and fourth central moment $\mu_4 = \frac{x(x^2 - 3)}{3 \arctan x} + 1$.

The fraction of the original Cauchy distribution within the symmetric interval is

$$f(x) = \frac{2}{\pi} \arctan x.$$

Now the crucial question is “why would someone truncate?” simply because of the need to obtain a distribution whose moments are mathematically tractable. Therefore the purpose of this paper is to employ a simple transformation on a finite distribution of a Cauchy random variable in order to obtain a distribution whose moments are mathematically tractable. For this purpose we would consider the Cauchy random variable X , whose distribution is given in equation (1) and whose realizations are from the non-compact real space (\mathbb{R}) . That is, a real space (\mathbb{R}) without the two points, $-\infty$ and ∞ [2]. That is $X \in \mathbb{R} : -r < x < q$, with $r, q \in \mathbb{R}$ or

$$X = x_{-r}, \dots, x_{-1}, x_0, x_1, \dots, x_q \quad (7)$$

If we let

$$|x_m| = \text{Max} |x_{-r}, \dots, x_{-1}, x_0, x_1, \dots, x_q| \quad (8)$$

and dividing (8) by $|x_m|$, we obtain the distribution of Y , whose values will lie in the interval $-1 < y < 1$, where

$$Y = \frac{X}{|x_m|} = \frac{(x_{-r}, \dots, x_{-1}, x_0, x_1, \dots, x_q)}{|x_m|} \quad (9)$$

Under the transformation in (9), the probability density function (pdf) of Y is given by

$$f(y) = \frac{b\phi}{\pi} \cdot \frac{1}{b^2 + (y-a)^2}, \quad -1 < y < 1 \quad (10)$$

where ϕ is the pdf stabilization term which can be obtained by evaluating the integral in (10), equating to unity and solving for ϕ . If

$$\frac{b\phi}{\pi} \int_{-1}^1 \left(\frac{1}{b^2 + (x-a)^2} \right) dx = \frac{2b\phi}{\pi} \int_0^1 \left(\frac{1}{b^2 + (x-a)^2} \right) dx = 1 \Rightarrow \frac{2b\phi}{\pi} \left[\arctan\left(\frac{1-a}{b}\right) \right]_0^1 = 1$$

which implies that

$$\phi = \frac{\pi}{2b \arctan\left(\frac{1-a}{b}\right)}$$

Therefore the pdf of Y is given by

$$f(y) = \frac{1}{2 \arctan\left(\frac{1-a}{b}\right)} \cdot \frac{1}{b^2 + (y-a)^2}, \quad -1 < y < 1 \quad (11)$$

Having obtained the pdf of Y , the next task is to obtain the first four moments of Y , $E(Y^k)$, $k=1,2,3,4$. Interest is limited to these moments because the basic statistical measures namely; mean, variance, skewness and kurtosis are obtained from them. For the purpose of establishing the moments, we would adopt the notation

$$K = \frac{1}{2 \arctan\left(\frac{1-a}{b}\right)} \quad (12)$$

2 Some Moments of the Cauchy Distribution Under Transformation

2.1 First Moment E(Y)

By definition

$$\begin{aligned} E(Y) &= \int y f(y) dy = K \int_{-1}^1 \frac{y}{b^2 + (y-a)^2} dy \\ &= K \left[\frac{1}{2} \ln(b^2 + y^2 - 2ya + a^2) + \frac{a}{b} \arctan\left(\frac{y-a}{b}\right) \right]_{-1}^1 \end{aligned}$$

thus,

$$E(Y) = \frac{1}{2 \arctan\left(\frac{1-a}{b}\right)} \left[\frac{1}{2} \{ \ln(1+a^2+b^2-2a) - \ln(1+a^2+b^2+2a) \} + \frac{a}{b} \{ \arctan\left(\frac{1-a}{b}\right) - \arctan\left(\frac{-1-a}{b}\right) \} \right]$$

2.2 Second Moment E(Y²)

$$\begin{aligned} E(Y^2) &= K \int y^2 f(y) dy = K \int_{-1}^1 \frac{y^2}{b^2 + (y-a)^2} dy = 2K \int_0^1 \frac{y^2}{b^2 + (y-a)^2} dy \\ &= 2K \left[y + a \ln(a^2 + b^2 + y^2 - 2ya) + \frac{a^2 - b^2}{b} \arctan\left(\frac{y-a}{b}\right) \right]_0^1 \end{aligned}$$

This implies that

$$E(Y^2) = \frac{1}{\arctan\left(\frac{1-a}{b}\right)} \left[1 + a \{ \ln(1+a^2+b^2-2a) - \ln(a^2+b^2) \} + \frac{a^2 - b^2}{b} \{ \arctan\left(\frac{1-a}{b}\right) - \arctan\left(\frac{-a}{b}\right) \} \right]$$

2.3 Third Moment $E(Y^3)$

By definition,

$$\begin{aligned} E(Y^3) &= K \int y^3 f(y) dy = K \int_{-1}^1 \frac{y^3}{b^2 + (y-a)^2} dy = 2K \int_0^1 \frac{y^3}{b^2 + (y-a)^2} dy \\ &= K \left[\frac{1}{2} y^2 + 2ya + \frac{3a^2 - b^2}{2} \ln(a^2 + b^2 + y^2 - 2ya) + \frac{a^3 - 3ab^2}{b} \arctan\left(\frac{1-a}{b}\right) \right]_{-1}^1 \end{aligned}$$

therefore

$$E(Y^3) = \frac{1}{2 \arctan\left(\frac{1-a}{b}\right)} \left[4a + \frac{3a^2 - b^2}{2} \{ \ln(1 + a^2 + b^2 - 2a) - \ln(1 + a^2 + b^2 + 2a) \} + \frac{a^3 - 3ab^2}{b} \{ \arctan\left(\frac{1-a}{b}\right) - \arctan\left(\frac{-1-a}{b}\right) \} \right]$$

2.4 Fourth Moment $E(Y^4)$

$$\begin{aligned} E(Y^4) &= K \int y^4 f(y) dy = K \int_{-1}^1 \frac{y^4}{b^2 + (y-a)^2} dy = 2K \int_0^1 \frac{y^4}{b^2 + (y-a)^2} dy \\ &= 2K \left[\frac{1}{3} y^3 + a y^2 - b^2 y + 3a^2 y + 2a(a^2 - b^2) \ln(a^2 + b^2 + y^2 - 2ya) + b(b^2 - 6) \arctan\left(\frac{y-a}{b}\right) \right]_0^1 \end{aligned}$$

therefore

$$E(Y^4) = \frac{1}{\arctan\left(\frac{1-a}{b}\right)} \left[\frac{1}{3} + a - b^2 + 3a^2 + 2a(a^2 - b^2) \ln(1 + a^2 + b^2 - 2a) + b(b^2 - 6) \arctan\left(\frac{1-a}{b}\right) - 2a(a^2 - b^2) \ln(a^2 + b^2) - b(b^2 - 6) \arctan\left(\frac{-a}{b}\right) \right]$$

Having obtained the moments, $E(Y^k)$ using the transformed distribution, one can obtain those of X using

$$(x_m)^k \cdot E(Y^k)$$

Lemma 2.1 *The following expression for the mean, variance, skewness for $\gamma_1(a,b)$ and kurtosis for $\gamma_2(a,b)$ of a random variable Y defined in (9) are given as follows:*

$$E(Y) = K_1(Y) = \frac{1}{g}(\varphi(a,b) + \psi(a,b))$$

$$\text{Var}(Y) = K_2(Y) = \frac{2}{g}(\varphi^{(1)}(a,b) + \psi^{(1)}(a,b))$$

$$\gamma_1(a,b) = \frac{K_3(Y)}{[K_2(Y)]^{\frac{3}{2}}} = \frac{\varphi^{(2)}(a,b) + \psi^{(2)}(a,b)}{[\varphi^{(1)}(a,b) + \psi^{(1)}(a,b)]^{\frac{3}{2}}}$$

$$\gamma_2(a,b) = \frac{K_4(Y)}{[K_2(Y)]^2} = \frac{\varphi^{(3)}(a,b) + \psi^{(3)}(a,b)}{[\varphi^{(1)}(a,b) + \psi^{(1)}(a,b)]^2}$$

Proof. The first four cummulants $K_i(Y)$, $i = 1, 2, 3, 4$ are given respectively by

$$K_1(Y) = \frac{1}{g}(\varphi(a,b) + \psi(a,b)),$$

where

$$g = 2 \tan^{-1}\left(\frac{1-a}{b}\right)$$

$$\varphi(a,b) = \ln\left(\frac{(1+a^2+b^2-2a)^{\frac{1}{2}}}{1+a^2+b^2+2a}\right) \quad \text{and} \quad \psi(a,b) = \frac{a}{b} \left\{ \tan^{-1}\left(\frac{1-a}{b}\right) + \tan\left(\frac{1-a}{b}\right) \right\}.$$

$$K_2(Y) = \frac{2}{g}(\varphi^{(1)}(a,b) + \psi^{(1)}(a,b)),$$

where

$$\varphi^{(1)}(a,b) = 1 + \ln\left(\frac{1+a^2+b^2-2a}{a^2+b^2}\right)$$

and

$$\psi^{(1)}(a,b) = \frac{a^2 - b^2}{b} \left\{ \tan^{-1}\left(\frac{1-a}{b}\right) + \tan\left(\frac{1-a}{b}\right) \right\}$$

$$K_3(Y) = \frac{1}{g} \left(\varphi^{(2)}(a,b) + \psi^{(2)}(a,b) \right),$$

where

$$\varphi^{(2)}(a,b) = 4a + \ln \left(\frac{(1+a^2+b^2-2a)^{\frac{3a^2-b^2}{2}}}{1+a^2+b^2-2a} \right)$$

and

$$\psi^{(2)}(a,b) = \frac{a^3 - 3ab^2}{b} \left\{ \tan^{-1} \left(\frac{1-a}{b} \right) + \tan \left(\frac{1-a}{b} \right) \right\} .$$

Finally, $K_4(Y) = \left(\varphi^{(3)}(a,b) + \psi^{(3)}(a,b) \right)^{\frac{2}{g}}$, where

$$\varphi^{(3)}(a,b) = \frac{1}{3} + a - b^2 + 3a^2 + \ln \frac{(1+a^2+b^2-2a)^{\frac{2a(a^2-b^2)}{2}}}{a^2+b^2}$$

and

$$\psi^{(3)}(a,b) = b(b^2 - 6) \left\{ \tan^{-1} \left(\frac{1-a}{b} \right) + \tan \left(\frac{a}{b} \right) \right\} .$$

From the above four expressions for the moments of Y , we have that

$$\lim_{a \rightarrow 0, b \rightarrow 0} \gamma_1(a,b) = 0 \quad \text{and} \quad \lim_{a \rightarrow 0, b \rightarrow \infty} \gamma_1(a,b) = 0$$

Hence the limiting values of the kurtosis as a tends to zero and b tends to infinity are given by

$$\lim_{a \rightarrow 0, b \rightarrow 0} \gamma_2(a,b) = \frac{1}{3} \quad \text{and} \quad \lim_{a \rightarrow 0, b \rightarrow \infty} \gamma_2(a,b) = -1. \quad \square$$

3 Conclusion

Considering the non-existence of the moments of the Cauchy distribution with parameters a and b denoted by X such that $X \sim \text{Cauchy}(a,b)$, $-\infty < -r < x < q < \infty$, the fundamental findings of this study is that, we can obtain the moments of a Cauchy distribution through a simple transformation of dividing

with a suitable constant which converts X to be distributed on the interval $[-1, 1]$ denoted by Y such that $Y \sim \text{Cauchy}(a, b)$, $-1 < y < 1$. The beauty of this study is that having obtained the moments of $Y \sim \text{Cauchy}(a, b)$, $-1 < y < 1$, we can easily generate those of those of $X \sim \text{Cauchy}(a, b)$, $-\infty < -r < x < q < \infty$.

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