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# Decomposition of a vector measure

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**Abstract**

In this paper we show the vector form of the decomposition of a measure. We consider a bounded vector measure $m$ on $K(G;E)$ and we prove that it decomposes into two measures, one of which is absolutely continuous with respect to the Haar measure and the other foreign to the Haar measure.

**Keywords:** Vector measure; Haar measure; singular measures; absolute continuity

## 1 Introduction

In this article we prove and extend in vector measure case a theorem due to Radon-Nikodym in the single case of positive measure. First we prove a theorem which allow us to get a vector measure at each we have a complex measure and finally we get and prove the decomposition of any vector measure.

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## 2 Preliminaries Notes

**Definition 2.1.** *Let $G$ be a locally compact group and $K(G;E)$ the space of continuous functions with compact support on $G$ in $E$.*
*We call a vector measure on $G$ with respect to the Banach spaces $E$ and $F$; a linear map:*

$$\begin{aligned} m: \quad K(G;E) &\to F \\ f &\mapsto m(f) \end{aligned}$$

*such as $\forall K$ compact of $G \quad \exists a_K > 0, \|m(f)\|_F \leq a_K \|f\|_\infty$, where $\|.\|_F$ designates the norm on the Banach space $F$ and $\|f\|_\infty = sup\{\|f(t)\|_E, t \in K\}$, the norm on $K(G;E)$.*

**Definition 2.2.** *Let $E$ be a set and $\mathfrak{B}$ a sigma algebra of subsets of $E$. $m$ a bounded complex or vector measure and $\mu$ a positive measure on $\mathfrak{B}$.*
*We say that $m$ is absolutely continue with respect to $\mu$ if*

$$\forall A \in \mathfrak{B} \quad such\,as\,\mu(A) = 0 \quad then \quad m(A) = 0$$

*we note: $m \ll \mu$*

**Definition 2.3.** *Two measures $\nu$ and $\mu$ are foreign (or singular)if there exists a partition $(E_1, E_2)$ of $E$ such as*

$$|\mu|(E_1) = 0 \quad and \quad |\nu|(E_2) = 0$$

*we note : $\mu \perp \nu$.*

Let we see the following theorem called Lebesgue-Radon-Nikodym.

**Theorem 2.1.** *Let $\mu$ be a positive measure; any real or complex measure $\nu$ can be uniquely written in the form : $\nu = \nu_a + \nu_s$ where $\nu_a$ is positive and absolutely continuous with respect to $\mu$ and $\nu_s$ singular; positive and foreign to $\mu$.*

The following theorem demonstrated allowed us to make the transition from a complex measure to the vector measure.

**Theorem 2.2.** *E and F two Banach spaces, G a locally compact group Let $\nu \in K(G,E)$ be a complex measure $\sigma$-finite, $w \in F$ a vector The mapping*

$$\begin{array}{rll} m: & K(G,E) & \to F \\ & f & \mapsto w\nu(f) \text{ is a vector measure.} \end{array}$$

*Proof.* Show before that $w\nu(f) \in F$.
$\nu(f) \in \mathbb{C}$ is a scalar and $w \in F$ a vector so $w\nu(f) \in F$.
Then $m$ is linear because $\nu$ is linear.
$\nu$ being a complex Radon measure we have $\forall K \, compact \, of \, G \quad \exists a_k$ such as $|\nu(f)| \leqslant a_k \|f\|_\infty$ we have:

$$\begin{aligned} \|m(f)\|_F &= \|w\nu(f)\|_F \\ &\leqslant \|w\|_F |\nu(f)| \\ &\leqslant \|w\|_F \times a_k \|f\|_\infty \\ &\leqslant M_k \|f\|_\infty \quad with \quad M_K = \|w\| a_k, \end{aligned}$$

hence $m$ is continuous and therefore $m$ is a vector measure.

□

# 3 Main Result

**Theorem 3.1.** *Any vector measure $m$ on $K(G,E)$ decomposes uniquely into*

$$m = m_a + m_s \quad with \quad m_a \ll \mu \quad and \quad m_s \perp \mu$$

*Proof.* *The uniqueness.*
Suppose there exists $m'_a$ et $m'_s$ such as $m = m'_a + m'_s = m_a + m_s$ with $m_a \ll \mu$ and $m_s \perp \mu$ on one hand and $m'_a \ll \mu \, and \, m'_s \perp \mu$ on the second .
which equals $m'_a - m_a = m_s - m'_s$
Like $m_s \perp \mu$ then there exists $G_1 \subset G$ such as $\mu(G_1) = 0$ and $m_s(\bar{G}_1) = 0$
Like $m'_s \perp \mu$ then there exists $G_2 \subset G$ such as $\mu(G_2) = 0$ and $m'_s(\bar{G}_2) = 0$
$(m_s - m'_s)(\bar{G}_1 \cap \bar{G}_2) = 0$

$\mu(G_1 \cup G_2) = \mu(G_1) + \mu(G_2) = 0$
Since $m_a \ll \mu$ and $m'_a \ll \mu$ then $m_a(G_1 \cup G_2) = 0$ and $m'_a(G_1 \cup G_2) = 0$

$$\begin{aligned}(m'_a - m_a)(G_1 \cup G_2) &= m'_a(G_1 \cup G_2) - m_a(G_1 \cup G_2) \\ &= 0 \quad \text{hence} \\ m'_a - m_a &= 0 \Rightarrow m'_a = m_a.\end{aligned}$$

We got $m'_a - m_a \ll \mu$ so $m_s - m'_s = 0$ on $G \Rightarrow m_s = m'_s$, hence the uniqueness.
*Existence.*
Let $\nu$ be a complex measure on $K(G; E)$, according to Theorem 2.1;
there exists $\nu_a \ll \mu \quad and\, \nu_s \perp \mu$ such as $\nu = \nu_a + \nu_s$.

$$\forall f \in K(G,E) \quad \forall w \neq 0 \in F, w\nu(f) = w\nu_a(f) + w\nu_s(f)$$

$$\begin{aligned}\nu_a \ll \mu \Leftrightarrow \mu(A) = 0 \quad &\Rightarrow \quad \nu(A) = 0 \quad \forall\, A \subset G. \\ &\Rightarrow \quad w\nu(A) = 0 \quad \forall\, w \neq 0 \\ &so \quad w\nu_a \ll \mu\end{aligned}$$

$$\begin{aligned}\nu_s \perp \mu \Leftrightarrow \mu(A) = 0 \qquad &and \quad \nu_s(\bar{A}) = 0 \quad \forall\, A \subset G \\ &and \quad w\nu_s(\bar{A}) = 0 \quad \forall\, w \neq 0 \\ &so \qquad w\nu_s \perp \mu\end{aligned}$$

According to Theorem 2.2; $w\nu_a(f)$ and $w\nu_s(f)$ are the vectors measures belonging to $F$ Banach space, from which we put:

$$\begin{aligned}m_a(f) &= w\nu_a(f) \\ m_s(f) &= w\nu_s(f)\end{aligned}$$

thus giving $m(f) = m_a(f) + m_s(f)$ and consequently $m = m_a + m_s$.

□

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