Upper Bounds for Ruin Probability in a Controlled Risk Process under Rates of Interest with Homogenous Markov Chains

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Abstract
This paper explores recursive and integral equations for ruin probability of a controlled risk process under rates of interest with homogenous Markov chains. We assume that claim and rates of interest are homogenous Markov chains, take a countable number of non-negative values. Generalized Lundberg inequalities for ruin probability of this process are derived via a recursive technique. Recursive equations for finite time ruin probability and an integral equation for ultimate ruin probability are presented, from which corresponding probability inequalities and upper bounds are obtained. An illustrative numerical example is discussed.

Mathematics Subject Classification: 62P05, 60G40, 12E05

Keywords: ruin probability, homogenous Markov chain, controlled risk process

1 Introduction

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Article Info: Received: April 12, 2017. Revised: May 11, 2017
Published online: September 15, 2017
The ruin problem in stochastic environments has been studied by many researchers [9], [10]. In classical risk model, the claim number process was assumed to be a Poisson process and the individual claim amounts were described as independent and identically distributed random variables. In recent years, the classical risk process has been extended to more practical and real situations. For most of the investigations treated in risk theory, it is very significant to deal with the risks that rise from monetary inflation in the insurance and finance market, and also to consider the operation uncertainties in administration of financial capital. Teugels and Sundt [16], [17] studied ruin probability under the compound Poisson risk model with the effects of constant rate. Yang [19] given both exponential and non – exponential upper bounds for ruin probabilities in a risk model with constant interest force and independent premiums and claims. Xu and Wang [18] given upper bounds for ruin probabilities in a risk model with interest force and independent premiums and claims with Markov chain interest rate. Cai [1], [2] considered the ruin probabilities in two risk models, with independent premiums and claims and used a first – order autoregressive process to model the rates of in interest. Cai and Dickson [3] built Lundberg inequalities for ruin probabilities in two discrete- time risk process with a Markov chain interest model and independent premiums and claims. P. D. Quang [11] established Lundberg inequalities using the recursive technique for ruin probabilities in two risk model with homogenous Markov chain premiums when claims and interest rates sequences are independent. P. D. Quang [12] used martingale approach to build upper bounds for ruin probabilities in a risk model with interest force and independent interest rates and premiums when claims is a Markov chain. P. D. Quang [13] used martingale approach to build upper bounds for ruin probabilities in a risk model with interest force and independent interest rates and Markov chain claims and Markov chain premiums. P. D. Quang [14] used martingale approach to build upper bounds for ruin probabilities in a risk model with interest
force and independent claims, Markov chain premiums and Markov chain
interests. P. D. Quang [15] also used recursive approach to build upper bounds for
ruin probabilities in a risk model with interest force and Markov chain premiums,
Markov chain claims, while the independent interest rates.
In addition, many papers studied an insurance model where the risk process can be
controlled by proportional reinsurance. The performance criterion is to choose
reinsurance control strategies to bound the ruin probability of a discrete-time
process with a Markov chain interest. Controlling a risk process is a very active
area of research, particularly in the last decade; see [4, 5, 6, 7], for instance.
Nevertheless obtaining explicit optimal solutions is a difficult task in a general
setting. Maikol A. Diasparra and Rosaria Romera [8] obtained generalized
Lundberg inequalities for the ruin probabilities in a controlled discrete-time risk
process with a Markov chain interest.
In this article, we extend the model considered by Maikol A. Diasparra and
Rosaria Romera [8] to introduce homogenous Markov chain claims and
homogenous Markov chain rates of interest.

2 Preliminary Notes
Let \( Y_n \) be the \( n \)–th claim payment. The random variable \( Z_n \) stands for the length
of the \( n \)–th period, that is, the time between the occurrence of the claims \( Y_{n-1} \)
and \( Y_n \). Let \( \{I_n\}_{n \geq 0} \) be the interest rate process. We assume that \( Y_n, Z_n, I_n \) are
defined on the probability space \((\Omega, A, P)\). We consider a discrete – time
insurance risk process in with the surplus process \( \{U_n\}_{n \geq 1} \) with initial surplus \( u \)
can be written as
\[
U_n = U_{n-1}(1 + I_n) + C(b_{n-1})Z_n - h(b_{n-1}, Y_n), \text{ for } n \geq 1.
\] (2.1)
We make several assumptions.
Assumption 2.1. \( U_o = u \geq 0 \).
Assumption 2.2. \( \{Y_n\}_{n \geq 0} \) is an homogeneous Markov chain, such that for any \( n \) the values of \( Y_n \) are taken from a set of non-negative numbers \( G_y = \{y_1, y_2, \ldots, y_n, \ldots\} \) with \( Y_0 = y_i \) and

\[
p_{ij} = P\left[ \omega \in \Omega : Y_{n+1} (\omega) = y_j \middle| Y_n (\omega) = y_i, n \in N, y_i \in G_y, y_j \in G_y \right],
\]

where \( 0 \leq p_{ij} \leq 1 \), \( \sum_{j=1}^{\infty} p_{ij} = 1 \).

Assumption 2.3. \( \{I_n\}_{n \geq 0} \) is an homogeneous Markov chain, such that for any \( n \) the values of \( I_n \) are taken from a set of non-negative numbers \( G_i = \{i_1, i_2, \ldots, i_m, \ldots\} \) with \( I_0 = i_r \) and

\[
q_{rs} = P\left[ \omega \in \Omega : I_{m+1} (\omega) = i_s \middle| I_m (\omega) = i_r, m \in N, i_r \in G_i, i_s \in G_i \right],
\]

where \( 0 \leq q_{rs} \leq 1 \), \( \sum_{s=1}^{\infty} q_{rs} = 1 \).

Assumption 2.4. \( \{Z_n\}_{n \geq 0} \) is a sequence of independent and identically distributed non-negative continuous random variables with the same distributive function

\[
F(z) = P(\omega \in \Omega; Z_0 (\omega) \leq z).
\]

Assumption 2.4. We denote by \( C(b) \) the premium left for the insurer if the retention level \( b \) is chosen, where \( 0 < C(b) \leq c, b \in B \).

The process can be controlled by reinsurance, that is, by choosing the retention level (or proportionality factor or risk exposure) \( b \in B \) of a reinsurance contract for one period, where \( B := [b_{min}, 1], \ b_{min} \in (0, 1) \) will be introduced below. The premium rate \( c \) is fixed.

Assumption 2.5. We denote the function \( h(b,y) \) with values in \([0, y]\) specifies the fraction of the claim \( y \) paid by the insurer, and it also depends on the retention level \( b \) at the beginning of the period. Hence \( y - h(b,y) \) is the part paid by the reinsurer. The retention level \( b = 1 \) stands for control action no reinsurance.
In this article, we consider the case of proportional reinsurance, which means that
\[ h(b, y) = b y, \quad \text{with} \quad b \in B. \] (2.2)

Usually, the constant \( b_{\min} \) in Assumption 2.4 is chosen by
\[ b_{\min} := \min \{ b \in (0, 1] \mid C(b) > 0 \}. \]

Assumption 2.6. We suppose that \( \{ Y_n \}_{n \geq 0}, \{ Z_n \}_{n \geq 0} \) and \( \{ I_n \}_{n \geq 0} \) are independent.

Assumption 2.7. We consider Markovian control policies \( \pi = \{ a_n \}_{n \geq 1} \), which at each time \( n \) depend only on the current state, that is, \( a_n(U_n) := b_n \) for \( n \geq 0 \).

Abusing notation, we will indentify functions \( a : X \to B \), where \( X = \mathbb{R} \cup \ell, B \) is the decision space.

Consider an arbitrary initial state \( U_0 = u \geq 0 \) and a control policy \( \pi = \{ a_n \}_{n \geq 1} \).

Then, by iteration of (2.1) and assuming (2.2), it follows that for \( n \geq 1 \), \( U_n \) satisfies
\[ U_n = u \prod_{l=1}^{n} (1 + I_l) + \sum_{l=1}^{n} \left( C(b_{n-1}) Z_l - b_{l-1} Y_l \prod_{m=l+1}^{n} (1 + I_m) \right) \] (2.3)

The ruin probability when using the policy \( \pi \), given the initial surplus \( u \), and the initial claim \( Y_o = y_i \), the initial interest rate \( I_o = i_r \) with Assumption 2.1 to 2.7 is defined as
\[ \psi^\pi(u, y, i_r) = P^\pi \left( \bigcup_{k=1}^{\infty} (U_k < 0) \mid U_0 = u, Y_o = y_i, I_o = i_r \right) \] (2.4)

which we can also express as
\[ \psi^\pi(u, y, i_r) = P^\pi \left( U_k < 0 \text{ for some } k \geq 1 \mid U_0 = u, Y_o = y_i, I_o = i_r \right) \] (2.5)

Similarly, the ruin probabilities in the finite horizon case with Assumption 2.1 to 2.7, are given by
\[ \psi_n^\pi(u, y, i_r) = P^\pi \left( \bigcup_{k=1}^{n} (U_k < 0) \mid U_0 = u, Y_o = y_i, I_o = i_r \right) \] (2.6)

Firstly, we have
Upper Bounds for Ruin Probability in a Controlled Risk Process under

\[ \psi_1^\pi(u, y, i, r) \leq \psi_2^\pi(u, y, i, r) \leq \ldots \leq \psi_n^\pi(u, y, i, r) \leq \ldots \]  
(2.7)

and with any \( n \in N \),

\[ \psi_n^\pi(u, y, i, r) \leq 1. \]  
(2.8)

Thus, from (2.7) and (2.8), we obtain

\[ \lim_{n \to \infty} \psi_n^\pi(u, y, i, r) = \psi^\pi(u, y, i, r). \]

We denote by \( \Pi \) the policy space. A control policy \( \pi^* \) is said to be optimal if for any initial \((Y_0, I_0) = (y_i, i_r)\), we have

\[ \psi^\pi(u, y, i, r) \leq \psi^*(u, y, i, r) \text{ for all } \pi \in \Pi. \]

3 Main Results

3.1. Integral Equation for Ruin Probability

We now construct recursive equation for finite time ruin probabilities and an integral equation

**Theorem 3.1.** Given model (2.1) and Assumptions 2.1 to 2.7, for \( n = 1, 2, \ldots \), we have

\[
\psi_{n+1}^\pi(u, y, i, r) = \sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \rho_0 q_{1s} \left\{ \begin{array}{c}
\int_0^{b_y - u(l+i_s)} \frac{b_y - u(l+i_s)}{C(b_o)} \, dF(z) + \int_{b_y - u(l+i_s)}^{\infty} \psi_n^\pi(u(l+i_s) - b_o y_j + C(b_o)z, y, i_s) \, dF(z) \end{array} \right\}, \quad (3.1)
\]

and

\[
\psi_n^\pi(u, y, i, r) = \sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \rho_0 q_{1s} \left\{ \begin{array}{c}
\int_0^{b_y - u(l+i_s)} \frac{b_y - u(l+i_s)}{C(b_o)} \, dF(z) + \int_{b_y - u(l+i_s)}^{\infty} \psi_n^\pi(u(l+i_s) - b_o y_j + C(b_o)z, y, i_s) \, dF(z) \end{array} \right\}. \quad (3.2)
\]

Where throughout this paper:

i) If \( v \leq 0 \text{ then } F(v) = 0, \)
ii) If \( v \leq 0 \) then \( \int_{v}^{+\infty} dF(z) = \int_{0}^{+\infty} dF(z) \),

iii) If \( v \leq 0 \) then \( \int_{0}^{v} \psi(z, y_i, i_r) dF(z) = 0 \).

**Proof.**

We consider \( Y_1(\omega) = y_j, I_1(\omega) = i_s, (\omega \in \Omega) \) and

\[
B = \{ \omega \in \Omega : U_0(\omega) = u, Y_0(\omega) = y_j, I_0(\omega) = i_r \},
\]

\[
A_{js} = \{ \omega \in \Omega : Y_1(\omega) = y_j, I_1(\omega) = i_s \},
\]

\[
A_1 = \{ \omega \in \Omega : Z_1(\omega) < \frac{U_0(\omega)(1 + I_1(\omega)) - b_0 Y_1(\omega)}{C(b_o)} \},
\]

\[
\overline{A}_1 = \{ \omega \in \Omega : Z_1(\omega) \geq \frac{U_0(\omega)(1 + I_1(\omega)) - b_0 Y_1(\omega)}{C(b_o)} \}.
\]

Let \( V_k = u(Y_k, Z_k) = b_o Y_k - C(b_o) Z_k \). From (2.1), we have

\[
U_1 = U_0(1 + I_1) - V_1 = u(1 + I_1) - b_o Y_1 + C(b_o) Z_1
\]

Therefore

\[
P^\pi(\omega \in \Omega : U_1(\omega) < 0 \mid A_1 \cap A_{js} \cap B) = 1
\]

\[
\Rightarrow P^\pi(\omega \in \Omega : \bigcup_{k=1}^{n+1} U_k(\omega) < 0 \mid A_1 \cap A_{js} \cap B) = 1. \tag{3.3}
\]

In addition,

\[
P^\pi(\omega \in \Omega; U_1(\omega) < 0 \mid \overline{A}_1 \cap A_{js} \cap B) = 0. \tag{3.4}
\]

Let \( \{ \tilde{Y}_n \}_{n \geq 0}, \{ \tilde{Z}_n \}_{n \geq 0}, \{ \tilde{I}_n \}_{n \geq 0} \) be independent copies of \( \{ Y_n \}_{n \geq 0}, \{ Z_n \}_{n \geq 0}, \{ I_n \}_{n \geq 0} \),

\[
\{ Z_n \}_{n \geq 0}, \{ I_n \}_{n \geq 0} \text{ with } \tilde{Y}_0(\omega) = Y_1(\omega) = y_j, Z_0(\omega) = Z_1(\omega), \tilde{I}_0(\omega) = I_1(\omega) = i_s
\]

and \( \tilde{V}_k = b_o \tilde{Y}_k - C(b_o) \tilde{Z}_k \),
\[ \tilde{U}_n = \tilde{U}_o \prod_{l=1}^{n} (1 + \tilde{I}_l) + \sum_{l=1}^{n} \left( C(b_{l-1}) \tilde{Z}_l - b_{l-1} \tilde{Y}_l \prod_{m=l+1}^{n} (1 + \tilde{I}_m) \right). \]

Thus (2.3) and (3.4) imply

\[ \mathbb{P}^{\pi} \left( \omega \in \Omega : \bigcup_{k=1}^{n+1} U_k(\omega) < 0 \left| A_1 \cap A_{j_s} \cap B \right. \right) = \mathbb{P}^{\pi} \left( \omega \in \Omega : \bigcup_{k=2}^{n+1} U_k(\omega) < 0 \left| A_1 \cap A_{j_s} \cap B \right. \right) \]

\[ = \mathbb{P}^{\pi} \left( \omega \in \Omega : \bigcup_{k=2}^{n} \left[ U_k(\omega)(1+I_k(\omega)) - b_{m-1}Y(k) + C(b_{m})Z(k) \prod_{m=m+1}^{k}(1+I_m(\omega)) \right] \right. \]

\[ + \sum_{m=2}^{k} (C(b_{m-1})Z_m(\omega) - b_{m-1}Y_m(\omega) \prod_{p=m+1}^{k}(1+I_p(\omega)) < 0 \left| A_1 \cap A_{j_s} \cap B \right. \right) \]

\[ = \mathbb{P}^{\pi} \left( \omega \in \Omega : \bigcup_{k=1}^{n} \tilde{U}_o(\omega) \prod_{m=2}^{k}(1+\tilde{I}_m(\omega)) + \sum_{m=2}^{k} (C(b_{m-1})\tilde{Z}_m(\omega) - b_{m-1}\tilde{Y}_m(\omega) \prod_{p=m+1}^{k}(1+\tilde{I}_p(\omega)) < 0 \left| A_1 \cap A_{j_s} \cap B \right. \right), (3.5) \]

Now, from (2.1) implies

\[ \psi_{n+1}(u, y_i, i_r) = \mathbb{P}^{\pi} \left( \omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k(\omega) < 0 \left| B \right. \right) \]

\[ = \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} \sum_{i_r} \mathbb{P}^{\pi} \left( \omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k(\omega) < 0 \left| A_{ij} \cap B \right. \right. \right) \]

\[ = \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} \sum_{i_r} \mathbb{P}^{\pi} \left( \omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k(\omega) < 0 \left| A_1 \cap A_{ij} \cap B \right. \right. \right) \mathbb{P} \left( A_1 \left| A_{ij} \cap B \right. \right) + \]

\[ + \left. \mathbb{P}^{\pi} \left( \omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k(\omega) < 0 \left| A_1 \cap A_{ij} \cap B \right. \right. \right) \mathbb{P} \left( A_1 \left| A_{ij} \cap B \right. \right) \right) \]

(3.6)

From (3.3), we have

\[ \mathbb{P}^{\pi} \left( \omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k(\omega) < 0 \left| A_1 \cap A_{ij} \cap B \right. \right) \mathbb{P} \left( A_1 \left| A_{ij} \cap B \right. \right) \]

\[ = \mathbb{P}^{\pi} \left( \omega \in \Omega : \frac{b_oy_j - u(1 + i_s)}{C(b_o)} \right) \]
\[
\frac{b_0y_j-u(1+i_s)}{C(b_o)} = \int_0^{+\infty} \text{d}F(z), \tag{3.7}
\]

and from (3.5), we have

\[
P^\pi \left( \omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k(\omega) + 1 - \hat{I}_m(\omega)) + \sum_{m=2}^{k} \left( \sum_{p=m+1}^{k} (1+\hat{I}_m(\omega)) \right) \right) \leq [A_1 \mid A_{ij} \cap B].
\]

Combining (3.7) and (3.8), therefore (3.6) may be written

\[
\psi_{n+1}(u, y, i, r) = \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij}q_{rs} \left[ \int_0^{+\infty} \text{d}F(z) + \int_0^{+\infty} \psi_{n}(u(1+i), -b_0y_j + C(b_o)z, y_j, i_s) \text{d}F(z) \right] \tag{3.9}
\]

When \( n = 0 \), we have

\[
\psi_{1}(u, y, i, r) = \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij}q_{rs} \left[ \frac{b_0y_j-u(1+i_s)}{C(b_o)} \right]. \tag{3.10}
\]

From the dominated convergence theorem, the integral equation for \( \psi_{n+1}(u, y, i, r) \) in Theorem 3.1 then follows immediately by letting \( n \to \infty \) in (3.9).

### 3.2. Inequalities for Ruin Probability

We now establish inequalities for the ruin probability corresponding to (2.4) and (2.6), respectively. We first prove the following Lemma.
Lemma 3.1. Given model (2.1) and Assumptions 2.1 to 2.7, and
\[ E^x \left[ (b_o Y_1 - C(b_o) Z_i) | Y_o = y_i \right] < 0, \]
and
\[ P^x \left[ b_o Y_1 - C(b_o) Z_i > 0 | Y_o = y_i \right] > 0, \]  \hspace{1cm} (3.11)

For any \( y_i \in G_y \), then there exists a unique positive constant \( R_i \) satisfying
\[ E^x \left[ e^{-R_i Y_i - b_o Y_i} | Y_o = y_i \right] = 1. \]  \hspace{1cm} (3.12)

Proof.
Let the function
\[ f_i(t) = E^x \left[ e^{-(C(b_o) Z_i - b_o Y_i)} Y_1 = y_i \right] - 1, \quad t \in (0; + \infty), \]
We have
\[ f_i'(t) = E^x \left[ \frac{\partial}{\partial t} \left( e^{-t(C(b_o) Z_i - b_o Y_i)} \right) Y_1 = y_i \right], \]
\[ f_i''(t) = E^x \left[ (b_o Y_1 - C(b_o) Z_i)^2 e^{-t(C(b_o) Z_i - b_o Y_i)} Y_1 = y_i \right] \geq 0. \]

Which implies that
\[ f_i(t) \] is a convex function with \( f_i(0) = 0 \),  \hspace{1cm} (3.13)
and
\[ f_i'(0) = E^x \left[ b_o Y_1 - C(b_o) Z_i Y_1 = y_i \right] < 0. \]  \hspace{1cm} (3.14)

As \( P^x \left[ b_o Y_1 - C(b_o) Z_i > 0 | Y_o = y_i \right] > 0 \), we can find some constant \( \delta > 0 \) such that
\[ P^x \left[ b_o Y_1 - C(b_o) Z_i > \delta > 0 | Y_o = y_i \right] > 0. \]

We therefore have
\[ f_i(t) = E^x \left[ e^{-t(C(b_o) Z_i - b_o Y_i)} Y_1 = y_i \right] - 1 \]
\[ \geq E^x \left[ \left( e^{-t(C(b_o) Z_i - b_o Y_i)} \right) Y_1 = y_i \right] 1_{[b_o Y_1 - C(b_o) Z_i > \delta]} \frac{1}{e^{\delta}} - 1 \geq e^{\delta} - 1, \]
implies that
\[ \lim_{t \to +\infty} f_i(t) = +\infty, \]  \hspace{1cm} (3.15)
From (3.13), (3.14) and (3.15) there exists a unique positive constant $R_i$ satisfying (3.12).

Now consider

$$R_o = \inf \{ R_i > 0 : E^\pi \left( e^{-R_i \left[ C(b_i)Z_i - b_i Y_i \right]} \bigg| Y_o = y_i, \gamma_i \in G_\gamma \right) \leq 1, y_i \in G_y \}.$$

**Remark 3.1.** $E^\pi \left[ e^{-R_i \left[ C(b_i)Z_i - b_i Y_i \right]} \bigg| Y_o = y_i \right] \leq 1.$

Using Lemma 3.1 and Theorem 3.1, we have a probability inequality for $\psi^\pi(u, y_i, i_r)$ by an inductive approach as follows.

**Theorem 3.2.** Given model (2.1) and Assumptions 2.1 to 2.7, under the conditions of Lemma 3.1 and $R_o > 0$, we have that

$$\psi^\pi(u, y_i, i_r) \leq \beta E^\pi \left[ e^{-R_o u(1 + i_r)} \bigg| I_o = i_r \right] \tag{3.16}$$

For any $u > 0$, $y_i \in G_Y$ and $i_r \in G_I$, where

$$\beta = \inf_{t > 0} \frac{e^{R_o C(b_o) t} \int_0^t e^{-R_o C(b_o) z} dF(z)}{F(t)}, 0 \leq \beta \leq 1.$$

**Proof.**

a) if $\inf_{t > 0} \frac{e^{R_o C(b_o) t} \int_0^t e^{-R_o C(b_o) z} dF(z)}{F(t)} < +\infty$.

Firstly, we have

$$\beta = \inf_{t > 0} \frac{e^{R_o C(b_o) t} \int_0^t e^{-R_o C(b_o) z} dF(z)}{F(t)} = \inf_{t > 0} \frac{e^{R_o C(b_o) (t-z)} dF(z)}{F(t)} \geq \inf_{t > 0} \frac{dF(z)}{F(t)} = 1.$$
Upper Bounds for Ruin Probability in a Controlled Risk Process under...
\[\sum_{j} \sum_{s} p_{ij} q_{rs} E \left( \frac{b_o y_j - u(1 + i_s)}{C(b_o)} \right) = 0\]

\[\leq \beta \sum_{j} \sum_{s} p_{ij} q_{rs} e^{R_o \left[ b_o y_j - u(1 + i_s) \right]} E \left( e^{-R_o C(b_o)Z_i} \right) \tag{3.19}\]

Combining (3.18) and (3.19), we imply

\[
\psi_{i}^\pi (u, y_i, i_r) \leq \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij} q_{rs} e^{R_o \left[ b_o y_j - u(1 + i_s) \right]} E \left( e^{-R_o C(b_o)Z_i} \right)
\]

\[
= \beta E^\pi \left[ e^{R_o \left[ b_o Y_{i} - u(1+I_{r}) \right]} \right] Y_o = y_i, I_o = i_r, E^\pi \left[ e^{-R_o C(b_o)Z_i} \right]
\]

\[
= \beta E^\pi \left[ e^{-R_o \left[ C(b_o)Z_i - b_o Y_{i} \right]} \right] Y_o = y_i, E^\pi \left[ e^{-R_o \left[ u(1+I_{r}) \right]} \right] I_o = i_r \leq \beta E^\pi \left[ e^{-R_o \left[ u(1+I_{r}) \right]} \right] I_o = i_r, \tag{3.20}
\]

Under an inductive hypothesis, we assume

\[
\psi_{n}^\pi (u, y_i, i_r) \leq \beta E^\pi \left[ e^{-R_o \left[ u(1+I_{r}) \right]} \right] I_o = i_r. \tag{3.21}
\]

So inequality (3.30) implies (3.21) holds with \(n = 1\). We have

\[
\psi_{n}^\pi (u(1 + i_s) - b_o y_j + C(b_o)z, y_j, i_s) \leq \beta E^\pi \left[ e^{-R_o \left[ (u(1+i_s) - b_o y_j + C(b_o)z) \right]} \right] I_o = i_r
\]

For \( y_i \in G_y \) and \( i_r \in G_I \), \( u(1 + i_s) - b_o y_j + C(b_o)z \geq 0, I_i \geq 0 \) then

\[
\psi_{n}^\pi (u(1 + i_s) - b_o y_j + C(b_o)z, y_j, i_s) \leq \beta E^\pi \left[ e^{-R_o \left[ (u(1+i_s) - b_o y_j + C(b_o)z) \right]} \right] I_o = i_r
\]

\[
\leq \beta e^{-R_o \left[ (u(1+i_s) - b_o y_j + C(b_o)z) \right]}. \tag{3.22}
\]

So from Lemma 3.1, (3.9), (3.17) and (3.22), we obtain

\[
\psi_{n+1}^\pi (u, y_i, i_r) = \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij} q_{rs} \left\{ \int_{0}^{b_o y_j - u(1+i_s)} \frac{b_o y_j - u(1+i_s)}{C(b_o)} dF(z) + \int_{b_o y_j - u(1+i_s)}^{+\infty} \psi_{n}^\pi (u(1+i_s) - b_o y_j + C(b_o)z, y_j, i_s) dF(z) \right\}
\]
\[\sum_{j} \sum_{s} p_{g_{rs}} \left\{ \int_{0}^{+\infty} dF(z) + \int_{b_{0}y_{j} - u(l + i_{s}) \over C(b_{o})}^{+\infty} \psi_{n}^{s}(u(l + i_{s}) - b_{o}y_{j} + C(b_{o})z, y_{j}, i_{s})dF(z) \right\} \]

\[+ \sum_{j} \sum_{s} p_{g_{rs}} \left\{ \int_{0}^{+\infty} dF(z) + \int_{b_{0}y_{j} - u(l + i_{s}) \over C(b_{o})}^{+\infty} \psi_{n}^{s}(u(l + i_{s}) - b_{o}y_{j} + C(b_{o})z, y_{j}, i_{s})dF(z) \right\}. \tag{3.23} \]

Because \((j, s) \in K_{1}: -b_{o}y_{j} - u(l + i_{s}) \over C(b_{o}) \leq 0\) then

\[F\left( -b_{o}y_{j} - u(l + i_{s}) \over C(b_{o}) \right) = 0,\]

\[\int_{b_{0}y_{j} - u(l + i_{s}) \over C(b_{o})}^{+\infty} \psi_{n}^{s}(u(l + i_{s}) - b_{o}y_{j} + C(b_{o})z, y_{j}, i_{s})dF(z) = \int_{0}^{+\infty} \psi_{n}^{s}(u(l + i_{s}) - b_{o}y_{j} + C(b_{o})z, y_{j}, i_{s})dF(z). \]

Combining with (3.22), we have

\[\sum_{j} \sum_{s} p_{g_{rs}} \left\{ \int_{0}^{+\infty} dF(z) + \int_{b_{0}y_{j} - u(l + i_{s}) \over C(b_{o})}^{+\infty} \psi_{n}^{s}(u(l + i_{s}) - b_{o}y_{j} + C(b_{o})z, y_{j}, i_{s})dF(z) \right\} \]

\[= \sum_{j} \sum_{s} p_{g_{rs}} \left\{ \int_{0}^{+\infty} \psi_{n}^{s}(u(l + i_{s}) - b_{o}y_{j} + C(b_{o})z, y_{j}, i_{s})dF(z) \right\}. \tag{3.24} \]

Using (3.17) and (3.24), we have
\[
\sum_{j} \sum_{s} p_{j} q_{rs} \left\{ F\left( \frac{b_{o}y_{j} - u(1+i_{s})}{C(b_{o})} \right) + \int_{0}^{+\infty} \psi_{n}^{\pi}(u(1+i_{s}) - b_{o}y_{j} + C(b_{o})z, y_{j}, i_{s}) dF(z) \right\} \\
\leq \beta \sum_{j} \sum_{s} p_{j} q_{rs} \int_{0}^{+\infty} e^{-R_{o}[u(1+i_{s}) - b_{o}y_{j} + C(b_{o})z]} dF(z) + \int_{b_{o}y_{j} - u(1+i_{s}) \over C(b_{o})}^{+\infty} e^{-R_{o}[u(1+i_{s}) - b_{o}y_{j} + C(b_{o})z]} dF(z) \\
= \beta \sum_{j} \sum_{s} p_{j} q_{rs} \int_{0}^{+\infty} e^{-R_{o}[u(1+i_{s}) - b_{o}y_{j} + C(b_{o})z]} dF(z). \tag{3.25}
\]

From (3.24) and (3.35), we obtain
\[
\psi_{n+1}(u, y_{i}, i_{s}) \leq \beta \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{j} q_{rs} \int_{0}^{+\infty} e^{-R_{o}[u(1+i_{s}) - b_{o}y_{j} + C(b_{o})z]} dF(z) \\
= \beta E_{[u_{i}]} \left[ e^{-R_{o}[b_{o}y_{i} - u(1+I_{i})]} \right] \left[ Y_{0} = y_{i}, I_{0} = i_{r} \right] E_{[u_{I_{i}}]} \left[ e^{-R_{o}C(b_{o})Z_{i}} \right] \\
= \beta E_{[u_{i}]} \left[ e^{-R_{o}[C(b_{o})Z_{i} - b_{o}y_{i}]} \right] \left[ Y_{0} = y_{i}, I_{0} = i_{r} \right] \leq \beta E_{[u_{I_{i}}]} \left[ e^{-R_{o}u(1+I_{i})} \right] \left[ I_{0} = i_{r} \right].
\]

Consequently
\[
\psi_{n+1}(u, y_{i}, i_{s}) \leq \beta E_{[u_{i}]} \left[ e^{-R_{o}u(1+I_{i})} \right] \left[ I_{0} = i_{r} \right],
\]

Such that inequality (3.21) holds for any \( n = 1, 2, 3, \ldots \) and inequality (3.16) follows by letting \( n \to \infty \) in inequality (3.21).

\[
e^{R_{o}C(b_{o})t} \int_{0}^{t} e^{-R_{o}C(b_{o})z} dF(z)
\]

b) If \( \inf_{t>0} \frac{e^{R_{o}C(b_{o})t}}{F(t)} = +\infty \Leftrightarrow \beta = 0. \)

With any \( \varepsilon > 0: \)
\[
\frac{e^{R_{o}C(b_{o})t}}{F(t)} \geq \varepsilon \quad \text{and} \quad F(v) \leq \frac{1}{\varepsilon} \int_{0}^{v} e^{-R_{o}C(b_{o})z} dF(z).
\]
We also prove similar such that a), we obtain
\[ \psi^n(u, y, i_r) \leq \frac{1}{\varepsilon} E^{\pi} \left[ e^{-R_o u(1+I)} | I_o = i_r \right]. \]  
(3.26)

Let \( n \to +\infty \) in inequality (4.16), we imply
\[ \psi^n(u, y, i_r) \leq \frac{1}{\varepsilon} E^{\pi} \left[ e^{-R_o u(1+I)} | I_o = i_r \right]. \]  
(3.27)

Let \( \varepsilon = n (n \in \mathbb{N}^* ) \) then (3.27) becomes
\[ \psi^n(u, y, i_r) \leq \frac{1}{n} E^{\pi} \left[ e^{-R_o u(1+I)} | I_o = i_r \right]. \]  
(3.28)

letting \( n \to \infty \) in inequality (3.28), we have
\[ \psi^n(u, y, i_r) \leq 0 = \beta E^{\pi} \left[ e^{-R_o u(1+I)} | I_o = i_r \right]. \]

Thus, inequality (3.16) holds when \( \beta = 0 \).

**Remark 3.2.** Let \( A(u, i_r) = \beta E^{\pi} \left[ e^{-R_o u(1+I)} | I_o = i_r \right]. \) From \( I_o \geq 0, \beta \leq 1 \), we have
\[ A(u, i_r) = \beta E^{\pi} \left[ e^{-R_o u(1+I)} | I_o = i_r \right] \leq \beta e^{-R_o u} \leq e^{-R_o u}. \]

So an upper bound for the ruin probability from inequality (3.16) is better than \( e^{-R_o u} \).

**4 Numerical Example**

In this section we give a numerical example to illustrate the bounds of \( \psi^n(u, y, i_r) \) derived in Section 3.

Let \( \{Z_n\}_{n \geq 0} \) be a sequence of independent and identically distributed non-negative continuous random variables with the same distributive function \( F(z) = 1 - e^{-0.25z} (z \geq 0) \).
Let \( \{Y_n\}_{n \geq 0} \) be a homogeneous Markov chain such that for any \( n \), \( Y_n \) take values in \( \{1, 3\} \) with \( Y_1 \) having a distribution:

<table>
<thead>
<tr>
<th>( Y_1 )</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>0.4</td>
<td>0.6</td>
</tr>
</tbody>
</table>

and matrix \( P = \begin{bmatrix} p_{ij} \end{bmatrix}_{2 \times 2} \) is given by
\[
P = \begin{bmatrix} 0.3 & 0.7 \\ 0.2 & 0.8 \end{bmatrix}
\]

Let \( \{Y_n\}_{n \geq 0} \) be a homogeneous Markov chain such that for any \( n \), \( I_n \) take values in \( \{0, 1; 0, 15\} \) with \( I_1 \) having a distribution:

<table>
<thead>
<tr>
<th>( I_1 )</th>
<th>0.1</th>
<th>0.15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>0.35</td>
<td>0.65</td>
</tr>
</tbody>
</table>

and matrix \( Q = \begin{bmatrix} q_{ij} \end{bmatrix}_{2 \times 2} \) is given by
\[
Q = \begin{bmatrix} 0.25 & 0.75 \\ 0.6 & 0.4 \end{bmatrix}
\]

Then, we have
\[
E(Y_1|Y_o = 1) = 1.03 + 3.07 = 2.4; E(Y_1|Y_o = 3) = 1.02 + 3.08 = 2.6; E(X_1) = \frac{1}{0.25} = 4.
\]

We chose \( \pi = \{\alpha_n\}_{n \geq 0} \) với \( a_n = 1 \) nên \( b_o = 1, C(b_o) = 1 \), therefore
\[
E(Y_1|Y_o = y_i) < (E(Z_i), y_i \in G_y).
\]

In the other hand,
\[
P(Y_1 - X_1 > 0|Y_o = 1) > 0, P(Y_1 - X_1 > 0|Y_o = 3) > 0.
\]

Combining (4.1), (4.2) imply that Lemma 2.1 holds.

Next, we solve equation (3.12).

Firstly, we have
\[
E[e^{R(Y_1 - Z_i)}|Y_o = y_i] = E[e^{R_Y|Y_o = y_i}]E[e^{-RZ_i}|i = 1, 2).
\]
where
\[
E\left[e^{-R_i} \right] = 0.25 \int e^{-(R_i + 0.25)x} \, dx = \frac{0.25}{R_i + 0.25} (i = 1, 2).
\]

and
\[
E\left[e^{R_i Y_i} \mid Y_o = 1\right] = e^{R_i} \cdot P\left[Y_1 = 1 \mid Y_o = 1\right] + e^{3R_i} \cdot P\left[Y_1 = 3 \mid Y_o = 1\right]
\]
\[
= 0.3e^{R_i} + 0.7e^{3R_i}
\]
\[
E\left[e^{R_i Y_i} \mid Y_o = 3\right] = e^{R_i} \cdot P\left[Y_1 = 1 \mid Y_o = 3\right] + e^{3R_i} \cdot P\left[Y_1 = 3 \mid Y_o = 3\right]
\]
\[
= 0.2e^{R_i} + 0.8e^{3R_i}
\]

Respective equation (3.12) for \( R_1, R_2 \) by
\[
0.3e^{R_1} + 0.7e^{3R_1} = 4R_1 + 1 \tag{4.3}
\]
\[
0.2e^{R_2} + 0.8e^{3R_2} = 4R_2 + 1 \tag{4.4}
\]

Using Maple, we find respective root of (3.12) for \( R_1, R_2 \), by
\[
R_1 \approx 0.33878; R_2 \approx 0.28124
\]

Hence, \( R_o = \min \{R_1, R_2\} = 0.28124 \).

We can apply the result of Theorem 3.2 for \( \psi^u (u, y, i_r) \)
\[
\psi^u (u, y, i_r) \leq E^\pi \left[ e^{-R_{,u(1+I)}} \mid I_o = i_r \right] = g(u, i_r), i_r \in G_i. \tag{4.5}
\]

where
\[
g(u; 0.1) = E\left[e^{-R_{,u(1+I)}} \mid I_o = 0.1\right]
\]
\[
= e^{-1R_{,u}} \cdot P[I_1 = 0.1 \mid I_o = 0.1] + e^{-1.15R_{,u}} \cdot P[I_1 = 0.15 \mid I_o = 0.1]
\]
\[
= 0.2e^{0.5R_{,u}} + 0.0e^{-7R_{,u}}
\]
\[
g(u; 0.15) = E\left[e^{-R_{,u(1+I)}} \mid I_o = 0.15\right]
\]
\[
= e^{-1R_{,u}} \cdot P[I_1 = 0.1 \mid I_o = 0.15] + e^{-1.15R_{,u}} \cdot P[I_1 = 0.15 \mid I_o = 0.15]
\]
\[ = 0, \phi^1 e^{Ku} + \phi^4 e^{Ku} \]

Table 4.1 shows values upper bounds \( g(u,i_r) \) of \( \psi^\pi(u,y_i,i_r) \) for a range of value of \( u \)

<table>
<thead>
<tr>
<th>u</th>
<th>( g(u;0,1) )</th>
<th>( g(u;0,15) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.726228</td>
<td>0.729814</td>
</tr>
<tr>
<td>2</td>
<td>0.527426</td>
<td>0.532654</td>
</tr>
<tr>
<td>3</td>
<td>0.38306</td>
<td>0.388775</td>
</tr>
<tr>
<td>4</td>
<td>0.27822</td>
<td>0.283774</td>
</tr>
<tr>
<td>5</td>
<td>0.202082</td>
<td>0.207141</td>
</tr>
<tr>
<td>6</td>
<td>0.146785</td>
<td>0.15121</td>
</tr>
<tr>
<td>7</td>
<td>0.106624</td>
<td>0.110387</td>
</tr>
<tr>
<td>8</td>
<td>0.077454</td>
<td>0.080588</td>
</tr>
<tr>
<td>9</td>
<td>0.056266</td>
<td>0.058836</td>
</tr>
<tr>
<td>10</td>
<td>0.040876</td>
<td>0.042958</td>
</tr>
<tr>
<td>15</td>
<td>0.008276</td>
<td>0.008919</td>
</tr>
<tr>
<td>20</td>
<td>0.001677</td>
<td>0.001854</td>
</tr>
</tbody>
</table>

5 Conclusion

Theorem 3.2 provide recursive equations for \( \psi^\pi_n(u,y_i,i_r) \) and an integral equation for \( \psi^\pi(u,y_i,i_r) \), by using a recursive technique. Using Lemma 3.1 and Theorem 3.2, we obtain a probability inequality for \( \psi^\pi(u,y_i,i_r) \) by an inductive approach. An illustrative numerical example is discussed.
References


