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Doubly Stochastic Reduced Form Credit Risk Model and Default Probability Uncertainty – a Technical Toolkit

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Abstract

Reduced form credit risk models provide a versatile platform to model credit risks and to quantify the interplay between the stochastic dimension of default probabilities and credit spread levels. This article gives a brief introduction to the required technical foundations and discusses the approach to examine uncertainties regarding default probabilities and credit spreads which has been established by [1]. The intention of this article is to help academic researcher as well as practitioners to understand related research projects, to do new research on this question or to improve credit risk models used in financial institutions.

Mathematics Subject Classification: G12; G13
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1 Introduction

Reduced form credit risk models provide a versatile platform to model credit risk and credit security prices. These models cannot only be used to model risks from a financial market practitioner's perspective but they can also be used to analyze a wide range of questions in the field of financial market research. This article gives a brief introduction to the technical foundations required to understand and to work with reduced form credit risk models. Based on this, it is in detail elaborated how the interplay between a possibly stochastic variation of default probabilities and the level of credit spreads can be quantified by estimating these models under different probability measures. This approach has been applied by [1] for the first time and they show for three sovereigns' credit spreads, that the risk premium which refers to this "second" risk dimension is a highly relevant driver of these spreads. A similar study has been presented by [2] and they also come to the conclusion that the stochastic variation of the default probability is a very important driver of the credit spreads included in their sample.

These studies provide very interesting insights into credit spreads' development and the results suggest to pursue further financial market research based on the applied approach as well as to extend existing credit risk models to account for the relevance of the "second dimension" risk premium. Accordingly, a understanding of this approach would be very helpful for researchers and practitioners from wide range of professional backgrounds. The mentioned articles do however not elaborate on the details of the applied approach but focus on the application. For many readers and potential applicators who do not have a strong academic focus on quantitative credit risk models, this discussion of the basic approach itself is not detailed enough to get the necessary understanding which is required to interpret related results or to pursue new research within the applied modelling framework. This article is intended to fill this gap and to provide researchers and practitioners a understanding of the doubly stochastic credit risk models and especially of second dimension risk premium analysis in this framework.

The second section gives a general introduction to doubly stochastic reduced form credit risk models. Cox-Ingersoll-Ross (CIR) diffusions (c.f. [3]) are introduced as possible modelling choice for the second stochastic dimensions. The

third section shows how pricing formulas can be derived based on that model. In the fourth section, it is elucidated how second dimension risk premium analysis can be conducted based on doubly stochastic reduced form models. The last section discusses a possible estimation strategy for the presented model framework.
This presentation is based on the characteristics of the CIR type diffusion. This diffusion type has been chosen for the presentation of examples because it is very easy to handle and formulas for the first two moments are well known in closed form. However, the results can directly be transferred to differently specified models. The model setting is applied to credit default swaps (CDS) as an example for credit securities which has also be chosen for the model estimation in [1] and [2].

2 Doubly stochastic reduced form framework

2.1 Basic ideas

Reduced form credit risk models go back to [4], [5] and [6]. The basic idea of the reduced form approach is to model a default as a jump of a stochastic (Poisson) process. This implies that default time is viewed as the stopping time of that process. A helpful feature of this class of "reduced form" models is the direct link between the underlying Poisson parameter and the default probability. The following introduction builds on [7] and [8].

To establish the **basic setting of a reduced form model** a measure space $(\Omega_1, \mathcal{F}_1, P_1)$ with the corresponding filtration $\mathcal{F}_{1,s}$, a measurable space $(M_1, \mathcal{M}_1)$ and an index set $S \neq \emptyset$ be defined. In addition, a Poisson process

$$\mathcal{P}oi = (Poi_s, s \in S) \tag{1}$$

is defined as a family of measurable mappings between probability and measure space:

$$Poi_s : (\Omega_1, \mathcal{F}_1, P_1) \rightarrow (M_1, \mathcal{M}_1) \tag{2}$$

$$\omega_1 \mapsto Poi_s(\omega_1) \tag{3}$$

with $\omega_1 \in \Omega_1$. Poi_s counts the number of *events* up to time s. In the present case, $Poi_s = 1$ means that a credit event has already occurred at time s, while $Poi_s = 0$ denotes that it has not. The increments $Poi_{s_1} - Poi_{s_0}$ are for $s_0, s_1 \in S$ and $s_1 - s_0 \geq 0$ independently Poisson distributed, the Poisson parameter depends on the length of the respective period $[s_0, s_1]$ only and Markov property is satisfied accordingly. At the first point in time, the process value be almost surely zero and the process be supported by the probability space introduced above. The intensity parameter of this Poisson process is denoted by λ_s with $s \in S$. The probability distribution $Pr^{Poi}(Poi_{s_0+t} = 0|Poi_{s_0} = 0)$ of the process value in $[s_0, s_0+t] \subset S$ conditioned on $Poi_{s_0} = 0$ is accordingly given by the poisson probability distribution $POI(j|ev)$ for $j = 0$ with ev denoting the expected value. This implies in closed form:

$$Pr^{Poi}(Poi_{s_0+t} = 0|Poi_{s_0} = 0) = POI(j = 0|ev = \lambda_{s_0,s_0+t}) = e^{-\lambda_{s_0,s_0+t}}. \quad (4)$$

This implies in turn (as the default time denoted as $\tau \in S$ is in this context also stopping time for Poi_s[2]) that

$$Pr^{Poi}(Poi_{s_0+t} > 0|Poi_{s_0} = 0) = 1 - e^{-\lambda_{s_0,s_0+t}}. \quad (5)$$

If λ_s is constant for all $s \in [0, t]$, one can rewrite $\lambda_{s_0,s_0+t} = \lambda_{\widehat{t}} \times t$ for all $\widehat{t} \in [s_0, s_0 + t]$. For non constant λ_s, one rewrites

$$\lambda_{s_0,s_0+t} = \int_{s_0}^{s_0+t} \lambda_s ds. \quad (6)$$

The filtration $\mathcal{F}_{1,s}$ is generated by realizations of the underlying process $\mathcal{P}oi$ prior to time s:

$$\mathcal{F}_{1,s} = \sigma\{Poi_t : 0 \leq t \leq s\}. \quad (7)$$

So far, the intensity has been assumed to be deterministic. This does not seem to be plausible for real world applications. Therefore a **second stochastic dimension** is added and diffusions are introduced as stochastic drivers of the default intensities. Diffusions are stochastic differential equations characterized by a specific functional form, which will be introduced in detail

[2]It is assumed that the model holds only up to the first credit event.

later. A Poisson process with stochastic intensity is called "Cox" process and the framework then becomes "doubly stochastic" (c.f. [9]).

To introduce this "second stochastic dimension" in the model set up, a probability space $(\Omega_2, \mathcal{F}_2, P_2)$ with corresponding filtration $\mathcal{F}_{2,s}$ and a measurable space $(M_2, \mathcal{M}_2)$ with $M_2 \subseteq \mathbb{R}^n$ for $n \in \mathbb{N}^+$ denoting a multivariate state vector be defined. The index set $S \neq \emptyset$ is still the same as in the subsection before. Finally, a Brownian motion $B_s \in \mathbb{R}^n$ and the following "diffusion" process $\mathcal{Y} = (Y_s, s \in S)$ is defined as a family of measurable mappings between probability and measure space:

$$Y_s : (\Omega_2, \mathcal{F}_2, P_2) \rightarrow (M_2, \mathcal{M}_2) \tag{8}$$

$$\omega_2 \mapsto Y_s(\omega_2). \tag{9}$$

Y_s be moreover distinguished by the family of transition probability laws $Pr^{\mathcal{Y}}(Y_{s_0+t}|Y_{s_0+t-1}, .., Y_{s_0})$ and satisfies the Markov law, i.e.

$$Pr^{\mathcal{Y}}\left(Y_{s_0+t} = m_{2,s_0+t}|Y_{s_0+t-1} = m_{2,s_0+t-1}, Y_{s_0+t-2} = m_{2,s_0+t-2}, \cdots, Y_{s_0} = m_{2,s_0}\right) \tag{10}$$

$$= Pr^{\mathcal{Y}}\left(Y_{s_0+t} = m_{2,s_0+t}|Y_{s_0+t-1} = m_{2,s_0+t-1}\right) \tag{11}$$

with $s_0, s_0+1, \cdots, s_0+t \in S$, $t \geq 2$ and $m_{2_{s_0}}, m_{2,s_0+1}, \cdots, m_{2,s_0+t} \in M_2$ with $\mathcal{F}_{2,s_0} \subseteq \mathcal{F}_{2,s_0+1} \subseteq \cdots \subseteq \mathcal{F}_{2,s_0+t}$. Intuitively, one can say that the filtration $\mathcal{F}_{2,s_0}$ – containing the information provided by all realization of Y_s up to time $s_0 \in S$ – does not provide more information on the future development of Y_s than the single realization of Y_{s_0}.

The change in the process is moreover determined by a stochastic differential equation of the following form:

$$dY_s = \mu_{Y_s} ds + \sigma_{Y_s} dB_s \tag{12}$$

with $\mu : M_2 \rightarrow \mathbb{R}^n$ and $\sigma : M_2 \rightarrow \mathbb{R}^{n\times n}$. The change in the "diffusion" process Y_s is therefore explained by a deterministic part consisting of a so called drift parameter μ_{Y_s}, which is weighted by the respective time horizon, and a stochastic part. The stochastic component is driven by the change in the previously introduced Brownian motion B_s. The diffusion process Y_s is the solution to the stochastic differential equation of the diffusion type.
In the doubly stochastic framework, the intensity λ_s is assumed to depend on

the “state vector” Y_s in linear form:

$$\lambda_s = \tilde{\rho}_0 + \tilde{\rho}_1 Y_s, \tag{13}$$

with $\tilde{\rho}_0 \in \mathbb{R}^1$ and $\tilde{\rho}_1 \in \mathbb{R}^n$. In the most simple and therefore most frequently applied case, the state vector is one dimensional, respectively $Y_s = \lambda_s$. λ_s itself is then the only state variable driven by the underlying diffusion. This implies $Y_s \in \mathbb{R}$ and one dimensionality of both the drift and the diffusion coefficients in the underlying stochastic differential equation.

2.2 Modelling the intensity process

The set of possible specifications of a diffusion – i.e. the functional forms the coefficients μ_{Y_s} and σ_{Y_s} are assumed to be defined by – is rather large. In this article we introduce one special specification which is very frequently applied in Quantitative Finance: the square root model by Cox-Ingersoll-Ross (CIR) which has a rather simple form and is particularly popular for short term interest rate modelling:

$$d\lambda_s = (\mu_0 - \mu_1 \lambda_s) + \sigma_1 \sqrt{\lambda_s} dB_s \tag{14}$$

with B_s denoting a Brownian motion and μ_0, μ_1 and σ_1 being constant coefficients. This complies with $\mu_{Y_s} = \mu_0 - \mu_1 \lambda_s$ and $\sigma_{Y_s} = \sigma_1 \sqrt{\lambda_s}$.

The CIR process is only defined for positive process values. Moreover, the process is non-negative for *i)* $\mu_0 > 0$ and *ii)* $\mu_1 > 0$. Then, the stochastic differential equation also has a “unique strong solution”[3] for every starting point Y_0 ([12]) and the conditional distribution of Y_t approaches the gamma distribution for large t ([3]). A CIR process satisfying the “Feller”-condition *iii)* $2\mu_0 > \sigma_1^2$ is also strictly positive ([13]). For *iv)* $0 < \mu_0 < \sigma_1^2$, the zero bound can be reached, but it is directly reflecting ([12]). Because the diffusion coefficient σ_1 tends to zero when the process values approaches to zero. The change in the process then becomes deterministic with the mean reverting drift part being the only relevant determinant. The zero bound is, moreover, “absorbing” ([12]) for $\mu_0 = 0$. For $\mu_0 < 0$, the process is “pushed” out of

[3]This means: $\mathbb{E}\left[\int_{s_0}^{s_0+t} |Y_s^2| ds < \infty\right]$ for all $s \in [s_0, s_0 + t]$ with $s \in S$ (c.f. [9] or [11]).

the defined domain $((\mathbb{R}^+)^N)$. This makes CIR diffusions with negative drift coefficients a rather abstract concept and will not be discussed in this section.

For CIR processes, satisfying conditions $i), ii)$ as well as condition $iv)$ or condition $iii)$, the probability distribution of the process value conditioned on a previous value is known in closed form ([3]). For univariate cases, the conditional first two moments are known in closed form. This can be very helpful for analyzing estimated models with respect to the second dimension risk premium. The respective formulas for the conditional expectations and the conditional variance can be found in [3] or in [11]:

$$\mathbb{E}\left(Y_{s+t}|Y_s\right) = \frac{\mu_0}{\mu_1} + \left(Y_s - \frac{\mu_0}{\mu_1}\right) e^{-\mu_1 t} \tag{15}$$

$$Var\left(Y_{s+t}|Y_s\right) = Y_s \frac{\sigma_0^2\left(e^{-\mu_1 t} - e^{-2\mu_1 t}\right)}{\mu_1} + \frac{\mu_0 \sigma_0^2\left(1 - e^{-2\mu_1 t}\right)}{2\mu_1^2} \tag{16}$$

$$Cov\left(Y_{s+t_1}, Y_{s+t_2}|Y_s\right) = Y_{s_0} \frac{\sigma_0^2}{2\mu_1} e^{-\mu_1(t_1+t_2)}\left(e^{2\mu_1 t_2} - 1\right) \tag{17}$$

for $t_2 \geq t_1$. The conditional expectations are linear in y_s and the coefficient multiplied with Y_s is $\exp^{-\mu_1 t}$. This reflects a stronger persistence of the process for weak mean reversion. Moreover, the level of the conditional variance is proportional to σ_0^2 and the persistence of the conditional variance increases with μ_1/σ_0^2.

3 Pricing formulas in the doubly stochastic reduced form framework

For the derivation of pricing formulas, the filtration $\mathcal{F}_{2,s}$ needs to be specified in more detail, similar to $\mathcal{F}_{1,s}$. It is the σ-algebra generated by the realization of the diffusion process $\mathcal{Y}$ prior to s:

$$\mathcal{F}_{2,s} = \sigma\{Y_t : 0 \leq t \leq s\} \tag{18}$$

So far, two different probability spaces have been introduced: one referring to stochastic movement in the underlying intensity λ_s and one directly referring to the random jumps of the Poisson process. Both probability spaces are now combined to a single one. This is necessary for the calculation of expected values, which depend both on possible jumps given certain jump intensities, and on the future (stochastic) developments of the underlying intensity. A new sample space $\Omega = \Omega_1 \times \Omega_2$, a new sigma algebra $\mathcal{F} = \sigma\{\mathcal{F}_1 \vee \mathcal{F}_2\}$[4] and the respective filtration $\mathcal{F}_s$ are introduced. Moreover, a probability measure P is introduced which satisfies all general requirements regarding probability measures with respect to $\mathcal{F}$ and $\mathcal{F}_s$, i.e.: $P(\Omega) = 1$, $P(F) < \infty$ for all $F \in \mathcal{F}$ as well as countable additivity for disjoint collections (c.f. [14]).

Based on this framework, pricing formulas for future payoffs, which depend on the respective credit risks, are now derived. This can be used to deduce pricing formulas for credit securities. One important input for net present values[5], which will be used for deriving pricing formulas, is still missing: the discount rate r_s and the respective discount factor for any $t \in \mathcal{R}^+$:

$$\nu_{s_0,s_0+t} = e^{-\int_{s_0}^{s_0+t} r_s ds}. \tag{19}$$

The expected return is however usually not observable and is therefore usually substituted by the risk free-rate. The **concept of risk neutrality** is applied.

This presumption implies that the expected payoffs can be discounted by the risk free rate in order to obtain market prices. This may seem odd at first glance as real world investors are usually assumed to be risk averse and the real world market prices should ceteris paribus be inferior to the ones obtained from a model based on the risk free rate. It will be shown that the assumption of risk neutral investors is only a hypothetical auxiliary construct, not leading to model prices which generally are below real market prices. Instead, the pricing formulas are further adapted.

The mechanics behind this are shown based on the value of a zero bond ZB_{s_0,s_0+t} in time s_0 with an underlying default process driven by λ_s, a payment C_s summing to the face value c at maturity $s_0 + t$, if no default has occurred. It is assumed that the payoffs sum up to zero in the case of default. In other words, there is no recovery. The rate expected by risk averse market investors

[4]In this context, "$\vee$" denotes the union of σ-fields.

[5]Net present value refers to the current value of future payoffs

be r_s for all $s \in [s_0, s_0+t]$. Therefore, the following equation holds :

$$\begin{aligned} ZB_{s_0,s_0+t} &= \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t} r_s ds} C_{s_0+t} | \mathcal{F}_{s_0}\right] \\ &= \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t} r_s ds} e^{-\int_{s_0}^{s_0+t} \lambda_s ds} c | \mathcal{F}_{s_0}\right] \\ &= e^{-\int_{s_0}^{s_0+t} r_s ds} c\mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t} \lambda_s ds} | \mathcal{F}_{s_0}\right] \\ &= e^{-\int_{s_0}^{s_0+t} r_s ds} c\mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t} \lambda_s ds} | \mathcal{F}_{2,s_0}\right] \end{aligned} \tag{20}$$

The final transformation basically says that the current price of the zero bond ZB_{s_0,s_0+t} equals the discounted expected payoff. The expectation still included does not directly refer to the question whether a default occurs, but it refers to the future development of λ_s. The expectation is therefore only conditioned on the part of the filtration which refers to the development of λ_s, namely $\mathcal{F}_{2,s}$. The return is factored out because it is assumed to be deterministic. A detailed proof was presented by [5].

This equation includes several unknown variables: both λ_s and r_s are - in opposition to r_s^f - not directly observable for any $s \in S$. Just substituting r_s by r_s^f is not an appropriate approach to reduce the numbers of unknown variables to one, because the equation then should not hold anymore since

$$\mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t} \lambda_s + r_s ds} | \mathcal{F}_{2,s_0}\right] c < \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t} \lambda_s + r_s^f ds} | \mathcal{F}_{2,s_0}\right] c. \tag{21}$$

A standard trick in the context of risk neutral pricing is to adapt λ_s in a way that the expected payoffs discounted by the risk free discount rate are in accordance with the observed market prices of the respective zero bonds ([9]). For the presentation of this step in the present model framework, the intensity is assumed to be deterministic.

The risk premium, which is originally defined as the difference between expected return and risk free return, is roughly speaking assigned to the default intensity which is then denoted as "risk neutral"-default intensity $\lambda_s^{\mathbb{Q}}$, whereas the actual default intensity is denoted as $\lambda_s^{\mathbb{P}}$. $\lambda_s^{\mathbb{Q}}$ is the intensity process which would be implied as true intensity process in market prices of zero bonds, if these were observed in a risk neutral world. $\lambda_s^{\mathbb{Q}}$ should ceteris paribus be higher than $\lambda_s^{\mathbb{P}}$ to counterbalance the lower discount rate and one has

$$\mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t} \lambda_s^{\mathbb{P}} + r_s ds} | \mathcal{F}_{2,s_0}\right] c = \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t} \lambda_s^{\mathbb{Q}} + r_s^f ds} | \mathcal{F}_{2,s_0}\right] c \tag{22}$$

with $\lambda_s^{\mathbb{Q}} \geq \lambda_s^{\mathbb{P}}$ and $r_s \geq r_s^f$ for all $s \in S$. The pricing formula for the zero bond is then given by

$$ZB_{s_0,s_0+t} = \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t} \lambda_s^{\mathbb{Q}} + r_s^f ds} | \mathcal{F}_{2,s_0}\right] c \tag{23}$$

So far, the difference in measures applies in a framework with a deterministic intensity. The original framework originally is, however, doubly stochastic and that implies a second source of risk: this "second dimension" risk refers to the uncertainty regarding current and future default intensity levels. Risk averse investors may expect a risk premium for this kind of uncertainty in addition to a premium for the risk of a default given certain intensity levels. From the perspective of a bond buyer, it is not guaranteed – in this context – that this source of risk leads to an increase in the expected return. The respective uncertainty is also relevant for (short) sellers of credit securities or investors in credit securities as a sudden drop in default probabilities should ceteris paribus lead to an increase the prices of bonds and to a decrease of insurance prices. The "second dimension" risk premium could – in other words – become negative. This may rather be the case for units with particular low anticipated default probabilities: Investors may – for example – rather insure people against the unlikely default of such a unit instead of insuring themselves or instead of betting on the occurrence of a credit event. The risk premium for the parties that profit from higher intensities might then dominate the risk premium from the other side. The main part of the debate in this chapter is, however, restricted to increases in returns due to the second dimension of risk respectively a positive second dimension risk premium because the empirical results in [1] and [2] suggest this to be the more relevant case.

It seems reasonable to consider both kinds of risk and the respective premia separately as they are indeed related, but not in 1:1 relation. It might, for example, be the case that the expected intensity levels and the respective default risk premium are particularly low, while the variance of the intensity and the respective "second dimension" risk premium are very high. On the other hand, it might be the case, that the expected intensity levels and the respective risk premium are very high, while the uncertainty regarding the intensity level respectively the second dimension risk premium is very low.

The presented approach therefore has to be further adapted to equate the expected payoff of the zero bond, which is discounted based on the risk free rate, and the observed market prices. Consequently, two new measures with respect to $\lambda_s^{\mathbb{Q}}$ respectively two different versions of P_2 are introduced which both refer to the variation in the risk neutral intensity $\lambda_s^{\mathbb{Q}}$ but not – at least not directly – to the actual intensity $\lambda_s^{\mathbb{P}}$. The measure $\widehat{\mathbb{P}}$ refers to the actual movement of the risk neutral intensity $\lambda_s^{\mathbb{Q}}$. The measure $\widehat{\mathbb{Q}}$, on the other hand, refers to the distribution of $\lambda_s^{\mathbb{Q}}$, which the expectations in pricing equation 23 are built on, so the pricing formula still holds in the context of stochastic intensities. It refers, in other words, to the expectations with respect to $\lambda_s^{\mathbb{Q}}$, that would be implied by market spreads in a world that is second dimension risk neutral.

Under the new (second dimension) risk neutral measure $\widehat{\mathbb{Q}}$, the expectations with respect to future $\lambda_s^{\mathbb{Q}}$ are from now on denoted as $\mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right]$. This term differs only from $\mathbb{E}_{s_0}^{\widehat{\mathbb{P}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right]$, if market participants' expected returns change due to the uncertainty regarding $\lambda_s^{\mathbb{Q}}$. If a risk premium is only demanded by investors for taking the default risk *per se* – i.e. the risk existing no matter whether the default probability is deterministic or not – there should only be a difference between $\lambda_s^{\mathbb{Q}}$ and $\lambda_s^{\mathbb{P}}$, but not between the two expectations with respect to the future development of $\lambda_s^{\mathbb{Q}}$.

With a discount factor based on the risk free rate r_s^f based on that framework the pricing formula of this zero bond becomes:

$$
\begin{aligned}
ZB_{s_0,s_0+t} &= \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s+r_s ds}|\mathcal{F}_{2,s_0}\right]c \\
&= \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}+r_s^f ds}|\mathcal{F}_{2,s_0}\right]c \\
&= \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right]ZB_{s_0,s_0+t}^f c. \qquad (24)
\end{aligned}
$$

The next section discusses how an estimated model can be analyzed with respect to the second dimension risk premium. In this section, the type of payoffs to be priced is extended first:

So far, the valuation of credit payments was based on the assumption of zero payments in the case of default, i.e. there was no recovery. This will be different now and the **pricing of recovery payments** is introduced. In this context, one has to think about the valuation of a payment that is executed in the case of default right after the default occurred. This be exemplified based on a payment obligation with payoff Z_τ. This obligation pays the amount z if the underlying unit defaults before maturity $s_0 + t$ and nothing otherwise.

The payment is moreover supposed to be executed right after default time τ. The value of that default payment DP_{s_0,s_0+t} at time s_0 is

$$DP_{s_0,s_0+t} = \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+\tau} r_s ds} Z_\tau | \mathcal{F}_{s_0}\right]. \tag{25}$$

The payoff of this obligation may be positive at each point in time until maturity because a default may occur in each point in time. The expectation therefore refers at each particular point in time until maturity to the question whether a default occurs just at that time and not to the question whether a default occurs anytime until maturity. This implies an expectation regarding the level of the intensity at each point conditioned on the fact that no default hast occured yet.

[5] shows that the discounted expectation of the payment can be rewritten as

$$\begin{aligned}\mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+\tau} r_s ds} Z_\tau | \mathcal{F}_{s_0}\right] &= \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[\int_{s_0}^{s_0+t} \lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s} \lambda_u^{\mathbb{Q}} + r_u^f du} z ds | \mathcal{F}_{2,s_0}\right] \\ &= z\mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[\int_{s_0}^{s_0+t} \lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s} \lambda_u^{\mathbb{Q}} + r_u^f du} ds | \mathcal{F}_{2,s_0}\right] \end{aligned} \tag{26}$$

The expectations denoted by $\mathbb{E}_s^{\widehat{\mathbb{Q}}}$ now again only refer to the future development of $\lambda_s^{\mathbb{Q}}$. Again the expectation based on the true distribution law of $\lambda_s^{\mathbb{Q}}$ would only equate this pricing formula if market participants' return expectations did not change because of the uncertainty with respect to $\lambda_s^{\mathbb{Q}}$. Based on this formula, one can easily derive an equation, which links the previously introduced risk neutral pricing formula and the value DP_{s_0,s_0+t} of a contract with maturity $s_0 + t$ paying off Z_s in all $s \in [s_0, s_0 + t]$ with $Z_s = z$ if $s = \tau$ and $Z_s = 0$ otherwise:

$$DP_{s_0} = \mathbb{E}_{s_0}\left[\left(\int_{s_0}^{s_0+t} Z_s e^{-\int_{s_0}^{s} r_u du}\right) | \mathcal{F}_{s_0}\right] \tag{27}$$

$$= \int_{s_0}^{s_0+t} \mathbb{E}_{s_0}\left[Z_s | \mathcal{F}_{s_0}\right] e^{-\int_{s_0}^{s} r_u du} \tag{28}$$

$$= z\int_{s_0}^{s_0+t} \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[\left(\lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s} \lambda_u^{\mathbb{Q}} + r_u du}\right) | \mathcal{F}_{2,s_0}\right] \tag{29}$$

$$= z\int_{s_0}^{s_0+t} ZB_{s_0,s}^f \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[\left(\lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s} \lambda_u^{\mathbb{Q}} du}\right) | \mathcal{F}_{2,s_0}\right], \tag{30}$$

with discount factor $ZB^f_{s_0,s}$ denoting a risk-free zero bond issued in s_0 and with maturity s. Now, the **pricing of credit default swaps** (CDS) is discussed as an example. Before this specific functional link between default intensity λ_s and CDS spreads is presented, the functionality of this class of credit securities is introduced.

CDS are insurance contracts between two parties with respect to the default of a third party. This basically means that the insurer or CDS seller pays a certain amount to the insurance or CDS buyer if the third party defaults. The insured party in return pays a semi- or quarter-annual payment – which is usually called "spread" payment (denoted by $SP_{s_0}(M)$ for a CDS issued in s_0 and maturity M in years) – until the contract ends. This is either the case when maturity $s_0 + M$ is reached or after a possible default of the respective third party. The spread is constant for one single CDS contract. Historical data of CDS spreads usually refer to newly issued contracts. Accordingly, s_0 usually complies with the index for CDS spread time series.

The amount to be paid by the insurance seller in the case of default depends on the proportion of debt which is not repaid by the third party in the context of default. This share is called the "loss rate" LR. In the present framework, LR is defined with respect to the face value of an ordinary bond. If a third party is, for example, only able to pay back 50% of the issued bonds' face value, the seller of a CDS referring to this defaulting unit as third party has to pay 50% of the respective CDS contract's face value. This would usually lead to a payment of 50 cents per contract as the face value of an ordinary CDS contract is one. LR is in the following assumed to be constant for the respective third party[6]. LR is identical for all CDS contracts with respect to the same third party. It is finally important to notice that the insured person does not necessarily hold a security issued by the respective third party.

For the pricing of newly issued CDS contracts, the single spread payment claims can be considered as $2 \times M$ zero bonds, with maturity $\frac{n}{2}$ and face values $SP_{s_0}(M)$, with $n \in \{1, \cdots, 2 \cdot t\}$ for a CDS maturity of $s_0 + M$, $[s_0, s_0 + M] \subset S$, $M \in \mathbb{N}^+ \cup \{0.5\}$ and semi-annually spread payment. n denotes the number of the respective spread payment. This set up implies $s_n - s_{n-1} = 0.5$ for all

[6]This is of course a simplifying assumption and assuming the loss rate to be stochastic and uncertain would be more realistic. An additional risk premium for uncertainty with respect to the loss rate would then be possible. This might be a field for future research.

$n \geq 1$. The value SV_{s_0,s_n} of one single payment obligation to be paid in s_n is in s_0 based on the pricing formulas for defaultable zero bonds:

$$SV_{s_0,s_n} = \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[e^{-\int_{s_0}^{s_n}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right]ZB_{s_0,s_n}^f SP_{s_0}(s_0+t). \tag{31}$$

The value $SV_{s_0}^{total}(M)$ of the whole set of spread payments $SP_{s_0}(s_0+t)$ referring to a CDS contract issued in s_0 with maturity s_0+t is then in s_0:

$$SV_{s_0}^{total}(M) = SP_{s_0}(M)\sum_{n=1}^{2t}\left(\mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[e^{-\int_{s_0}^{s_n}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right]ZB_{s_0,s_n}^f\right). \tag{32}$$

For valuation of the spread payment counterpart, i.e. the insurance obligation, one can go back to the recovery payments presented in the previous section. The insurance obligation again refers to a possible payment at each point in time until maturity. This payment sums up to zero, if the respective third party has not defaulted yet and it is positive right at the point in time the default occurs. The payoff is now denoted by INS_s. The amount paid in this case of default is LR. The value of the insurance claim from the perspective of the CDS buyer is denoted by $VINS_{s_0}(M)$ and can be obtained based on the following formula:

$$VINS_{s_0}(M) = \mathbb{E}_{s_0}\left[\int_{s_0}^{s_0+t} e^{-\int_{s_0}^{s}r_s ds}INS_s|\mathcal{F}_{s_0}\right] \tag{33}$$

$$= LR\left[\int_{s_0}^{s_0+t} ZB_{s_0,s}^f\mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[\lambda_s^{\mathbb{Q}}e^{-\int_{s_0}^{s}\lambda_u^{\mathbb{Q}}du}|\mathcal{F}_{2,s_0}\right]ds\right]. \tag{34}$$

The "market" spread $SP_{s_0}(M)$ is then the one that equates the values of both payment sides, namely the value of total spread payments $SV_{s_0}^{total}(M)$, and the value of the insurance claim $VINS_{s_0}(M)$. The following equation is supposed to hold accordingly (c.f. [8]):

$$SP_{s_0}(M)\sum_{n=1}^{2M}\left(\mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[e^{-\int_{s_0}^{s_0+0.5n}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right]ZB_{s_0,s_0+0.5n}^f\right)$$

$$= LR\left[\int_{s_0}^{s_0+M} ZB_{s_0,s}^f\mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[\lambda_s^{\mathbb{Q}}e^{-\int_{s_0}^{s}\lambda_u^{\mathbb{Q}}du}|\mathcal{F}_{2,s_0}\right]ds\right]. \tag{35}$$

So far, two versions of P_2 have been introduced: $\widehat{\mathbb{Q}}$ and $\widehat{\mathbb{P}}$. Now, the notation of the CIR diffusions, which determine the distribution law of $\lambda_s^{\mathbb{Q}}$, is extended to distinguish between the diffusions under both measures (c.f. [1]).

This is done referring to the CDS pricing formula. Then, it is shown in the context of the CDS pricing formula 35, how the coefficients of the respective stochastic differential equation can be interpreted with respect to the second dimension risk premium.

4 The second dimension risk premium and diffusions under both measures

In the previous section, the difference between $\widehat{\mathbb{Q}}$ and $\widehat{\mathbb{P}}$ has already been discussed. The difference between both measures refers to the distribution law of $\lambda_s^{\mathbb{Q}}$. The distribution law of the diffusion process $\lambda_s^{\mathbb{Q}}$ is generally determined by an underlying stochastic differential equation like the CIR diffusion. Considering these two ingredients of the model set up, it seems to be reasonable to adjust the notation of the respective diffusion accordingly. The diffusion determining the distribution law under $\widehat{\mathbb{Q}}$ is denoted in the following way:

$$d\lambda_s^{\mathbb{Q}} = \left(\mu_0^{\widehat{\mathbb{Q}}} - \mu_1^{\widehat{\mathbb{Q}}}\lambda_s^{\mathbb{Q}}\right) ds + \sigma_1\sqrt{\lambda_s^{\mathbb{Q}}} dB_s^{\widehat{\mathbb{Q}}}. \tag{36}$$

The true distribution law of $\lambda_s^{\mathbb{Q}}$ is given by:

$$d\lambda_s^{\mathbb{Q}} = \left(\mu_0^{\widehat{\mathbb{P}}} - \mu_1^{\widehat{\mathbb{P}}}\lambda_s^{\mathbb{Q}}\right) ds + \sigma_1\sqrt{\lambda_s^{\mathbb{Q}}} dB_s^{\widehat{\mathbb{P}}}. \tag{37}$$

Drift coefficients and Brownian motion differ in both equations, while the diffusion coeffient is identical. The reason for that lies in equation 15: only the drift coefficient and the respective value of the process itself go into the formula for the conditional expectation. And the expectations regarding the intensities are what matters in the “second dimension” risk premium context. This is shown based on the CDS pricing formula 35 and the idea of a positive second dimension risk premium introduced before:

The “first dimension” risk premium, i.e. the premium with respect to the default risk per se (i.e. given a specific deterministic series of intensities), is already taken into account by substituting $\lambda_s^{\mathbb{P}}$ by $\lambda_s^{\mathbb{Q}}$. Because of the uncertainty with respect to $\lambda_s^{\mathbb{Q}}$, the discount factor $ZB_{s,s+t}^f$ may, however, still be larger (or smaller) than the discount factor based on the expected return, even after this substitution. In other words, the discount factor $ZB_{s,s+t}^f$ might only be the appropriate one without any further adjustments, if there is no "second dimension" risk premium in this model. In the following, this is shown referring to the case of positive second dimension risk premia. To adjust for the effect of the lower discount factor respectively the higher discount rate, positive payoffs have to get lower weights and negative payoffs have to get higher weights[7]. This is the case, if the expectations regarding future intensities, which are conditioned on the current intensity levels, tend to be higher. Then, the negative payoff in the default case is more likely and the actual payment of all single spreads is more unlikely. The reasoning for a negative second dimension risk premium works accordingly.

This can be shown based on the expectations with respect to functions depending on the intensity process, which are included in formula 35 as well. The expectation with respect to the first function ($e^{-\int_{s_0}^{s_0+0.5n} \lambda_s^{\mathbb{Q}} ds}$) refers to the probability that a default has not occurred yet at the point in time chosen as higher boundary of the included integral. This figure is lower if expected future intensities are higher – both intuitively and based on mathematical reasoning[8]. Accordingly, single positive payoffs are weighted by lower weights if the expected future intensities are higher – which is in accordance with the presented economic reasoning.

The relation between future intensities and the level of the second function ($\lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s} \lambda_u^{\mathbb{Q}} du}$) is not directly clear. The intensities' expected values enter this function in two ways: the function decreases in the intensity, which goes into the exponential function negatively, and it increases with the intensity, by which the exponential function is multiplied. Considering the economic meaning of this function, this is reasonable: As discussed before, the function value refers to the probability that the default has not yet occurred at the point in time chosen as upper border in the included integral, but occurs just right then. There is, moreover, an integral built over that function. This integral over the function refers to the probability that the default occurs at any point in time between the time chosen as lower boundary of the outer integral

[7] In a risk neutral world, the observed spreads and loss rates would only be reasonable from a no aribtrage pricing point of view, if the expected values of $\lambda_s^{\mathbb{Q}}$ respectively the expected default probabilities were higher (than they actually are). The actualexpectations regarding future intensities would be as pessimistic as they are when based under the diffusion referring to $\widehat{\mathbb{Q}}$.

[8] The intensity goes into the exponential function negatively.

and the time chosen as higher boundary of the outer integral. The insurance payment is, in other words, weighted higher if the expectations of the future default intensity tend to be higher. This is again in accordance with the presented economic reasoning. The risk neutral expectations regarding the future values of the intensities therefore have to be higher (compared to expectations based on the true distribution law), the stronger the expected return (after taking into account the "first dimension" risk premium) exceeds the risk free return[9].

The established positive relation between the second dimension risk premium and the expected values of the intensities can also be explained in a less complicated fashion based on the temporary assumption that there is no first dimension risk premium (i.e. $\lambda_s^{\mathbb{P}} = \lambda_s^{\mathbb{Q}}$) and the zero bond pricing formula 24, which refers to the price of a zero bond without recovery. If the second dimension risk premium is zero as well, the following version of the pricing equation 24 holds:

$$
\begin{aligned}
ZB_{s_0,s_0+t} &= \mathbb{E}_{s_0}^{\widehat{\mathbb{P}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right] ce^{-\int_{s_0}^{s_0+t} r_s ds} \\
&= \mathbb{E}_{s_0}^{\widehat{\mathbb{P}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right] ce^{-\int_{s_0}^{s_0+t} r_s^f ds} \qquad (38)
\end{aligned}
$$

If there exists a positive second dimension risk premium, the risk-free rate is not equal to the expected return ($r_s > r_s^f$) and the equation 38 does not hold anymore. As described before, one can adjust for the difference between the discount factors resulting from r_s^f respectively r_s by introducing the risk-free measure $\widehat{\mathbb{Q}}$:

$$
\begin{aligned}
ZB_{s_0,s_0+t} &= \mathbb{E}_{s_0}^{\widehat{\mathbb{P}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right] ce^{-\int_{s_0}^{s_0+t} r_s ds} \qquad (39) \\
&= \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right] ce^{-\int_{s_0}^{s_0+t} r_s^f ds}. \qquad (40)
\end{aligned}
$$

If the expected return is higher (lower) than the risk-free return because of a positive (negative) second dimension risk premium, the intensity values which are expected under the measure $\widehat{\mathbb{Q}}$ should exceed (be inferior to)[10] the values expected under $\widehat{\mathbb{P}}$.

Accordingly, the difference between the conditional expectations of the intensity under both measures directly measures the "second dimension" risk

[9] The opposite is the case if the second dimension risk premium is negative.
[10] The intensity goes into the exponential function negatively.

premium. Formula 15 shows how the drift coefficients impact the conditional expectations. If the ratio $\frac{\mu_0}{\mu_1}$ (i.e. the mean reversion) is the same under both measures, a comparison of the drift parameter μ_1 is sufficient to evaluate the difference in the conditional expectations. A larger value for μ_1 implies a larger conditional expectation (closer to the mean reversion level), if the value of the intensity, which the expectation is conditioned on, is below the mean reversion level. The opposite holds if the value of the intensity is above the mean reversion level. If $\frac{\mu_0}{\mu_1}$ is higher and μ_1 is smaller under one measure, still not a general statement can be made. In most cases, the conditional expectation would be larger under the previously described measure. It might, however, still be the case that – if the intensity value is below the mean reversion level under both measures – the conditional expectation is higher under the described measure.

The difference between both measures with respect to the "second dimension" risk premium is therefore optimally evaluated with reference to the actual time series of $\lambda_s^{\mathbb{Q}}$. Based on the CIR coefficients under both measures and this time series, conditional expectations can be calculated for all horizons t. The difference between the resulting conditional expected values can then be evaluated:

$$\mathbb{E}_{s_0}^{\widehat{\mathbb{P}}}\left[\lambda_{s_0+t}^{\mathbb{Q}}|\mathcal{F}_{2,s_0}\right] - \mathbb{E}_{s_0}^{\widehat{\mathbb{P}}}\left[\lambda_{s_0+t}^{\mathbb{Q}}|\mathcal{F}_{2,s_0}\right] \tag{41}$$

Another reasonable approach to evaluate the relevance of the "second dimension" risk premium is the following: the model implied CDS spreads can be calculated based on the respective time series of $\lambda_s^{\mathbb{Q}}$. The expectations can be calculated based on both $\widehat{\mathbb{Q}}$ leading to "true" model spreads $\widehat{SP}_{s_0}$ and $\widehat{\mathbb{P}}$ leading to "wrong" model spreads $\widehat{SP}_{s_0}^{\widehat{\mathbb{P}}}$. The latter is calculated based on this formula:

$$\widehat{SP}_{s_0}^{\widehat{\mathbb{P}}}(M) = \frac{\widehat{LR}\left[\int_{s_0}^{s_0+M} ZB_{s_0,s}^f \mathbb{E}_{s_0,\widehat{\mu}_0^{\widehat{\mathbb{P}}},\widehat{\mu}_1^{\widehat{\mathbb{P}}},\widehat{\sigma}_1}^{\widehat{\mathbb{P}}}\left[\widehat{\lambda}_s^{\mathbb{Q}} e^{-\int_{s_0}^{s}\widehat{\lambda}_u^{\mathbb{Q}}du}|\mathcal{F}_{2,s_0}\right] ds\right]}{\sum_{n=1}^{2M}\left(\mathbb{E}_{s_0,\widehat{\mu}_0^{\widehat{\mathbb{P}}},\widehat{\mu}_1^{\widehat{\mathbb{P}}},\widehat{\sigma}_1}^{\widehat{\mathbb{P}}}\left[e^{-\int_{s_0}^{s_0+0.5n}\widehat{\lambda}_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right] ZB_{s_0,s_0+0.5n}^f\right)} \tag{42}$$

with $\widehat{LR}, \widehat{\mu}_0^{\widehat{\mathbb{P}}}, \widehat{\mu}_1^{\widehat{\mathbb{P}}}, \widehat{\sigma}_1$ denoting estimated coefficients, $\mathbb{E}_{s_0,\widehat{\mu}_0^{\widehat{\mathbb{P}}},\widehat{\mu}_1^{\widehat{\mathbb{P}}},\widehat{\sigma}_1}^{\widehat{\mathbb{P}}}$ denoting the resulting expectation and $\widehat{\lambda}_s$ denoting the estimated intensity process. The true model spreads are accordingly calculated as

$$\widehat{SP}^{\widehat{\mathbb{Q}}}_{s_0}(M) = \frac{\widehat{LR}\left[\int_{s_0}^{s_0+M} ZB^f_{s_0,s}\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_0,\widehat{\mu}_0^{\mathbb{Q}},\widehat{\mu}_1^{\mathbb{Q}},\widehat{\sigma}_1}\left[\widehat{\lambda}^{\mathbb{Q}}_s e^{-\int_{s_0}^{s}\widehat{\lambda}^{\mathbb{Q}}_u du}|\mathcal{F}_{2,s_0}\right] ds\right]}{\sum_{n=1}^{2M}\left(\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_0,\widehat{\mu}_0^{\mathbb{Q}},\widehat{\mu}_1^{\mathbb{Q}},\widehat{\sigma}_1}\left[e^{-\int_{s_0}^{s_0+0.5n}\widehat{\lambda}^{\mathbb{Q}}_s ds}|\mathcal{F}_{2,s_0}\right] ZB^f_{s_0,s_0+0.5n}\right)}. \tag{43}$$

A great difference between the true and wrong model spreads implies that the "second dimension" risk premium is an important driver of credit spreads.

Finally, the difference in the undelying diffusions under both measures can be – as by [1] – evaluated based on the **Girsanov theorem**. This standard theorem is frequently used in the Quantitative Finance stock price- or short term rate context and is introduced in the next paragraphs:

Consider a measure space $\left(\widehat{\Omega}, \widehat{\mathcal{P}}, \mathcal{F}\right)$. $\widehat{B}_s$ be Brownian motion under probability measure $\widehat{\mathcal{P}}$, Θ_t be an adapted process to the resulting filtration $\mathcal{F}_s$, the index set S be the same as before and a process Z_t be defined as

$$Z_s = e^{\left[-\int_0^s \Theta_t d\widehat{B}_t - \frac{1}{2}\int_0^s \Theta_t^2 dt\right]} \tag{44}$$

for $t \in S$ and $s \geq t$. $\widehat{\mathcal{P}}$ be, moreover, related to the second probability measure $\tilde{\mathcal{P}}$ with Z_s being Radon-Nykodin derivative linking these two measures:

$$\frac{d\tilde{\mathcal{P}}}{d\widehat{\mathcal{P}}} = Z_s \tag{45}$$

According to the Girsanov theorem, under mild technical conditions, $\tilde{B}$ defined as $\tilde{B}_s = \widehat{B}_s + \int_0^s \Theta_t dt$ is a Brownian motion under the measure $\tilde{\mathcal{P}}$. In equity modelling, the variable Θ_s is frequently considered to be the market price of risk. Applying this approach to the presented framework shall elucidate its reasonability. Θ_s be in this context denoted by η_s and the Radon-Nykodin derivative relating $\widehat{\mathbb{Q}}$ and $\widehat{\mathbb{P}}$ be defined by

$$\widehat{Z}_s = e^{\left[-\int_0^s \eta_t dB_t^{\widehat{\mathbb{Q}}} - \frac{1}{2}\int_0^s \eta_t^2 dt\right]} \tag{46}$$

for $t \in S$ and $s \geq t$ so that

$$\frac{d\widehat{\mathbb{P}}}{d\widehat{\mathbb{Q}}} = Z_s. \tag{47}$$

This implies that

$$d\lambda^{\mathbb{Q}}_s = \left(\mu_0^{\widehat{\mathbb{P}}} - \mu_1^{\widehat{\mathbb{P}}}\lambda^{\mathbb{Q}}_s\right) ds + \sigma_1\sqrt{\lambda^{\mathbb{Q}}_s}\left(dB^{\widehat{\mathbb{Q}}}_s + \eta_s ds\right). \tag{48}$$

$\sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}\eta_s$ accordingly gives the difference in change in $\lambda_s^{\mathbb{Q}}$ between $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{Q}}$. The greater η_s, the greater is the increase of $\lambda_s^{\mathbb{Q}}$ under $\widehat{\mathbb{Q}}$ compared to the increase under $\widehat{\mathbb{P}}$. η_s is therefore another reasonable measure for the size of the "second dimension" risk premium. A negative value for η_s would refer to situations in which the insurance buyer expects a price reduction for the possibility of changes in the default intensity as the insurance may be worthless in the case of a sudden decrease in default intensities.

η_s is in the following assumed to depend on $\lambda_s^{\mathbb{Q}}$ in a specific functional form. This step is line with the literature on quantitative equity modelling (c.f. [15], [16], [17]). The specific form is chosen based on the plausible assumption that the difference in change should increase linearly in the level of the underlying intensity (c.f. [18] and [19]). η_s already goes into the change of $\lambda_s^{\mathbb{Q}}$ as a factor multiplied by $\sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}$. To obtain a linear form, it is accordingly assumed that η_s depends on $\lambda_s^{\mathbb{Q}}$ in the following way:

$$\eta_s = \frac{\rho_0}{\sqrt{\lambda_s^{\mathbb{Q}}}} + \rho_1\sqrt{\lambda_s^{\mathbb{Q}}}. \tag{49}$$

This results in the actual difference in change of $\lambda_s^{\mathbb{Q}}$ being given by

$$\sigma_1\left(\rho_0 + \rho_1\lambda_s^{\mathbb{Q}}\right) \tag{50}$$

which is a linear function in $\lambda_s^{\mathbb{Q}}$ as it is supposed to be. This implies the following link between ρ_0, ρ_1 and the CIR coefficients under both measures:

$$\rho_0 = \frac{\mu_0^{\widehat{\mathbb{Q}}} - \mu_0^{\widehat{\mathbb{P}}}}{\sigma_1} \tag{51}$$

$$\rho_1 = \frac{\mu_1^{\widehat{\mathbb{P}}} - \mu_1^{\widehat{\mathbb{Q}}}}{\sigma_1}. \tag{52}$$

Accordingly, the coefficients ρ_0 and ρ_0 can be derived from the CIR coefficients and a time-series of the process η_s can be calculated and analyzed.

5 Estimation procedure

The estimation of this model-framework implies the estimation of the following coefficients: $\{\widehat{\mu}_0^{\widehat{\mathbb{P}}}, \widehat{\mu}_1^{\widehat{\mathbb{P}}}\widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1, \widehat{LR}\}$. After the parameters under $\widehat{\mathbb{Q}}$ have been estimated, the coefficients under $\widehat{\mathbb{P}}$ can be estimated based on frequently discussed time-series methods for diffusion processes. Therefore, this second step is not discussed in detail in this paper.

5.1 Estimation of the diffusion parameters under $\widehat{\mathbb{Q}}$

To estimate the distribution law of $\lambda_s^{\mathbb{Q}}$ under the risk neutral measure $\widehat{\mathbb{Q}}$ is a challenging task since only a set of spread time series $SP_{s_0}(M)$ and approximations for the risk neutral discount factors $ZB^f_{s_0,s_0+s}$[11] are directly observable. A loss rate LR is frequently assumed ex-ante as well. However, [1] demonstrate that LR is suggest based on the term structure of CDS identifiable and show that the typically assumed loss rate level of 70 percent is sometimes far from the loss rate equating the pricing formula in their model[12].

The suggested iterative procedure is – as mentioned before – restricted to models driven by affine diffusion processes (c.f. [21]) since the theory on affine processes is exploited to substitute the expectations included in formula 55. [6] show that expectations with respect to transforms of such affine processes can be depicted in exponential linear form depending on the value of the state process at the point in time when the expectation is built in. The coefficients of this function can be obtained as solutions to given ODEs that depend on the parameters of the underlying diffusions.

Adapting the results in [22] to the expectations included in the CDS pricing formula, one yields

$$\mathbb{E}\left[e^{\int_{s_0}^{s_1}\lambda_s^{\mathbb{Q}}ds}|\lambda_{s_0}^{\mathbb{Q}}\right] = e^{\alpha_{s_1-s_0}+\beta_{s_1-s_0}\lambda_{s_0}^{\mathbb{Q}}} \tag{53}$$

$$\mathbb{E}\left[\lambda_s e^{\int_{s_0}^{s}\lambda_u^{\mathbb{Q}}ds}|\lambda_{s_0}^{\mathbb{Q}}\right] = e^{\alpha_{s_1-s_0}+\beta_{s_1-s_0}\lambda_{s_0}^{\mathbb{Q}}}(A_{s_1-s_0}+B_{s_1-s_0})\lambda_{s_0}^{\mathbb{Q}} \tag{54}$$

with $\alpha_{s_1-s_0}, \beta_{s_1-s_0}, A_{s_1-s_0}$ and $B_{s_1-s_0}$ being solutions to ODEs. The coefficients depend on the parameter of the diffusion equation driving $\lambda_s^{\mathbb{Q}}$ under the respective measure.

[11] For example based on the yield-curve for AAA-bonds which is published on a daily basis by the ECB

[12] Considering for example sovereign data shows that loss rates can strongly vary. Historical data as published by [20] reflect a wide range of loss rates, ranging from 1.9% in the case of Belize in 2006, to 82 % in the Russian case.

Knowledge regarding the diffusion coefficients would therefore allow to substitute the expectations in the CDS pricing formula by the exponential linear functions depending on the intensity's current realization $\lambda_{s_0}^{\mathbb{Q}}$ only. The coefficients of this exponential linear form are, however, still unknown as the diffusion coefficients are not known either. The set of coefficients $\{\widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1, \widehat{LR}\}$ is therefore assumed ex-ante and the resulting ODEs are solved to get a series of coefficients for the exponential linear form. The expectations in the pricing formula are then substituted by the respective exponential linear functions depending on the realization of $\lambda_{s_0}^{\mathbb{Q}}$ and an estimation $\widehat{\lambda}_{s_{0_i}}^{\mathbb{Q}}$ can then be obtained for each observation $s_{0_i} \in [s_{0_1}, s_{0_2}.., s_{0_N}]$ with N denoting the respective sample size: define

$$\begin{aligned} & f(\lambda_{s_0}^{\mathbb{Q}} | \widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1^{\widehat{\mathbb{Q}}}, \widehat{LR}) \\ & = SP_{s_0}(M) \sum_{n=1}^{2M} \left(\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_0, \widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1^{\widehat{\mathbb{Q}}}} \left[e^{-\int_{s_0}^{s_0+0.5n} \lambda_s^{\mathbb{Q}} ds} | \lambda_{s_0}^{\mathbb{Q}} \right] ZB^f_{s_0, s_0+0.5n} \right) \\ & - \widehat{LR} \left[\int_{s_0}^{s_0+M} ZB^f_{s_0,s} \mathbb{E}^{\widehat{\mathbb{Q}}}_{s_0, \widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1^{\widehat{\mathbb{Q}}}} \left[\lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s} \lambda_u^{\mathbb{Q}} du} | \lambda_{s_0}^{\mathbb{Q}} \right] ds \right]. \end{aligned} \tag{55}$$

$\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_0, \widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1^{\widehat{\mathbb{Q}}}}$ denotes expectations built in s_0 under $\widehat{\mathbb{Q}}$ depending on the set of coefficients $\{\widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1^{\widehat{\mathbb{Q}}}\}$. For each time step $s_{0_i} \in [s_{0_1}, s_{0_2}.., s_{0_N}]$, one searches for $\widehat{\lambda}_{s_{0_i}}^{\mathbb{Q}}$ which satisfies $f(\widehat{\lambda}_{s_{0_i}}^{\mathbb{Q}} | \widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1, \widehat{LR}) = 0$. The extracted time series $\widehat{\lambda}_{s_{0_i}}^{\mathbb{Q}}$ is then however depending on the ex-ante determined coefficient set and it is therefore probably biased. This bias is, however, still going to be corrected: spreads from contracts with other maturities (i.e. in the present case 1,3,7 and 10 years) are taken and the sum of squared distances between these observed spreads $SP_{s_{0_i}}(M)$ and the model spreads $\widehat{SP}_{s_{0_i}}(M)$ based on the time series of intensities estimated in our first step is minimized by choosing a new set of coefficients. Model spreads can in this context be calculated based on this formula:

$$\widehat{SP}_{s_{0_i}}(M) = \frac{\widehat{LR} \left[\int_{s_{0_i}}^{s_{0_i}+M} ZB^f_{s_{0_i},s} \mathbb{E}^{\widehat{\mathbb{Q}}}_{s_{0_i}, \widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1^{\widehat{\mathbb{Q}}}} \left[\widehat{\lambda}_s^{\mathbb{Q}} e^{-\int_{s_{0_i}}^{s} \widehat{\lambda}_u^{\mathbb{Q}} du} | \lambda_{s_{0_i}}^{\mathbb{Q}} \right] ds \right]}{\sum_{n=1}^{2M} \left(\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_{0_i}, \widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1^{\widehat{\mathbb{Q}}}} \left[e^{-\int_{s_{0_i}}^{s_{0_i}+0.5n} \widehat{\lambda}_s^{\mathbb{Q}} ds} | \lambda_{s_{0_i}}^{\mathbb{Q}} \right] ZB^f_{s_{0_i}, s_{0_i}+0.5n} \right)} \tag{56}$$

and the minimization problem is accordingly given by

$$\underbrace{min}_{\{\widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1, \widehat{LR}\}} \sum_{M \in \{1,3,7,10\}} \sum_{s_{0_i} \in \{s_{0_1},..,s_{0_N}\}} \left[\widehat{SP}_{s_{0_i}}(M) - SP_{s_{0_i}}(M) \right]^2. \tag{57}$$

This new set of coefficients $\{\widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1, \widehat{LR}\}$ is, however, again biased as it depends in turn on the time series of intensities estimated based on the coefficient values, which were chosen ex-ante. The estimation has therefore not been completed yet. The new set of coefficients is subsequently used for estimating a times series $\widehat{\lambda}_{s_{0_i}}^{\mathbb{Q}}$ which is again based on the time series of $SP_{s_{0_i}}(5)$. The estimated time series $\widehat{\lambda}_{s_{0_i}}^{\mathbb{Q}}$ is in turn used for the estimation of a new coefficient set by comparing model spreads $\widehat{SP}_{s_{0_i}}(M)$ with the actual spreads $SP_{s_{0_i}}(M)$ for $M \in [1, 3, 7, 10]$. Both steps are afterwards repeated until the estimates of the coefficients and the intensities converge. All variables are identified (c.f. [1]). The final estimates of the coefficients and the time-series of intensities are characterized by approximately equating the pricing formula in each observation date s_{0_i} for each maturity M. The ODEs resulting in the coefficients of the exponential linear form for the conditional expectations have thereby of course to be solved over and over again. On the one hand, this can be done numerically but there are on the other hand fortunately also analytical solutions available that were presented by [23].

5.2 Estimation of the diffusion parameter under $\widehat{\mathbb{P}}$

After having estimated $\{\widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1, \widehat{LR}\}$ as well as a times series of intensities $\lambda_{s_{0_i}}^{\widehat{\mathbb{Q}}}$, the set of CIR drift coefficients under the historical measure $\widehat{\mathbb{Q}}$ can be estimated. The diffusion coefficient σ under $\widehat{\mathbb{P}}$ is the same as under $\widehat{\mathbb{Q}}$ and therefore only $\{\mu_0^{\widehat{\mathbb{P}}}, \mu_1^{\widehat{\mathbb{P}}}\}$ are left for estimation under $\widehat{\mathbb{Q}}$. There are many publications which deal with the estimation of stochastic differential equations based on time-series data. For the CIR case, the estimation is particularly simple: The transition probability distribution of the CIR process is known to be a non-central χ^2-distribution. [12] present closed form representations for probability distributions of diffusion process realization $\lambda_{s_0+t}^{\mathbb{Q}}$ based on the underlying CIR coefficient and conditioned on a specific previous realization $\lambda_{s_0}^{\mathbb{Q}}$. The set of possible approaches to estimate the drift parameters under the historical measure based on the times series of extracted risk neutral intensities is accordingly wide, including maximum-likelihood estimators (MLE), quasi-maximum-likelihood estimators (QML) or methods-of-moments estimators (MoM).

6 Conclusion

This article introduces doubly stochastic reduced form credit risk models. Based on this, it shown in detail, how the relevance of the second dimension risk premium can be assessed based on these models. It is elucidated that the diffusions driving the stochastic variation of the default intensity can be interpreted directly, in combination with a time-series of default intensities, or based on the Girsanov theorem. Moreover, an estimation strategy for reduced form credit risk models is described. Based on this presentation, the reader is intended to better understand ongoing research, to conduct related research on its own or to complement credit risk models used by practitioners.

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