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Time-Varying Cross-Hedge Effectiveness: A Local Cointegration Approach

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Abstract

The dynamic nature of many asset price processes and the lack of perfect hedging assets can lead to unstable hedge ratios over time, necessitating the re-estimation and rebalancing of cross-hedges. Cross-hedging occurs when a portfolio or asset is hedged with a statistically related yet not identical underlying derivative. Ordinary Least Squares regression is an oft applied method for estimating constant minimum-variance hedge ratios to curb price volatility or manage a market-neutral porfolio. However, constant estimates are often unsuitable under cross-hedging where the dependence structure between the two assets change over time. Rather than traditional correlation-based hedging, this paper focuses on cointegration-based cross-hedging with respect to the equilibrium between asset prices. We apply and test the out-of-sample efficacy of models that enable the cointegrating vector, or hedge ratio between two nonstationary price series, to vary over time. Models are estimated across daily data for selected equity, bond and commodity pairs. Rolling-window regression, exponentially-weighted moving average and Dynamic Linear Models (Gaussian Linear State-Space Models) are investigated. Results show that time-varying parameter models have superior out-of-sample hedging performance compared to constant parameter methods.

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This finding is confirmed through extensive Monte Carlo simulation. In practice, this reduction in basis risk comes with incurred transaction costs from routine hedge reblancing.

Keywords: Market Neutral; Cointegration; Time-Varying Model; Dynamic Linear Model; Kalman Filter; Hedging

1 Introduction

Ordinary Least Squares (OLS) regression is an oft applied statistical method for estimating constant minimum-variance hedge ratios to curb price volatility or manage a market-neutral portfolio. However, the dynamic nature of many asset price processes and the lack of perfect hedging assets can lead to unstable hedge ratios over time, necessitating the re-estimation and rebalancing of hedges. In practice, linear hedging between two assets can be estimated based on the correlation across returns. Instead of correlated returns, this paper extends another common practice of hedging with cointegrated prices series. We apply and test the out-of-sample hedge effectiveness of statistical models that enable the cointegrating vector, or hedge ratio, to vary over time across various financial instruments. We investigate models with parameters that are allowed to adapt while absorbing new information (or observations) in an on-line fashion, namely rolling window regression (RWR), exponentially weighted moving average (EWMA) and Dynamic Linear Models (DLM - also known as Linear Gaussian State-Space Models). The empirical analysis is conducted on daily data from the following instruments, with hypothetical use-cases following:

Asset Class	**Instrument**	**Hedge**
Equities	Apple Stock (AAPL)	Powershares QQQ Trust (QQQ)
Bonds	High-Yielding Bond	Low-Yielding Bond
Commodities	U.S. Gulf Coast Jet Fuel	West Texas Intermediate Crude Oil Futures

Equities: A trader wishes to neutralize systematic risk to exploit the mean-reverting spread between the stock and ETF

Bonds: A large bank hedges bond price volatility to capture the interest rate differential

Commodities: A commercial airline imperfectly hedges jet fuel price volatility by participating in highly liquid WTI crude oil futures markets The importance of effective hedging bears significant relevance in portfolio and risk management. When constructing portfolios to mitigate certain risks or isolate particular exposures, maintaining market-neutrality is a challenging yet important objective. Hedging linear relationships can be viewed as a regression problem, that is, we aim to minimize the variance of such a portfolio to maximize the benefits of correlation between the pair. The practical value of hedging solutions has led to deep literature on this subject. Traditionally, constant hedge ratios are estimated via ordinary least squares (OLS) regression on returns, with the slope coefficient equaling the hedge ratio (e.g. Ederington (1979); Anderson and Danthine (1980)). However, this procedure is only appropriate if the assumption of constant variance in the distribution of asset returns holds true- an overwhelmingly large body of literature shows that it does not. There is well established evidence of heteroskedasticity often encountered in asset returns (Park & Bera (1987)). This non-constant variance leads to non-constant covariation in multivariate settings such as that of hedge estimation, therefore an interest in extending the traditional OLS hedging model to those which can account for time-varying variance and covariation exists.

Triantafyllopoulos & Montana (2009) apply state-space modeling and Kalman filtering in a real-time statistical arbitrage framework to capture the cointegrated nature between two exchange traded funds. Park & Jei (2006) and Bera et al (1997) studied the hedging effectiveness of corn and soybean futures contracts on spot prices with bivariate GARCH models and found that the variance of hedge ratios is inversely related to hedging effectiveness. Kroner & Sultan (1993) also estimate time-varying hedge ratios for foreign exchange futures using a bivariate error correction model with a GARCH error structure. The authors found that time-varying hedge ratios outperform the conventional models both in-sample and out of sample. By applying various constant and time-varying hedge ratios to Indian stock and commodity futures markets, Kumar et al. (2008) finds further evidence of time-varying hedge ratios reducing variance compared to constant hedge ratio models. Myers (1991) analyzes hedging in futures markets and concludes that both simple and relatively complex models that take advantage of all relevant conditioning information available to traders, e.g. time-varying parameter models lead to better hedging compared to traditional OLS hedging.

While hedge ratios are traditionally constructed on asset returns due to their stationary nature, we assess hedge ratios on prices under the local cointegration framework. Local cointegration, also defined as time-varying or functional cointegration, has been explored in the literature, though not as thoroughly as returns-based hedging. Park & Hahn (1999) model U.S. automobile demand using cointegration with time-varying coefficients, such that the coefficient evolves smoothly over time and is estimated nonparametrically. More recently, Bierens & Martins (2010) apply time-varying vector error correction models to the purchasing power parity hypothesis of international prices and nominal exchange rates, and find evidence of time-varying cointegration. The authors estimate the time-varying coefficients using Chebyshev time polynomials. Xiao (2009) proposes a functional cointegration model, which allows the cointegrating vector to vary stochastically through both kernel and local polynomial estimation. Wagner (2010) applies cointegration in a state-space setting. The objective of this paper is to provide further evidence of the utility of time-varying models to manage financial risk by applying three practical models that allow the cointegrating vector to vary over time under the local cointegration framework. The first section briefly discusses spurious regression and cointegration. The second section of this paper provides detail on the three time-varying models of interest, namely rolling-window regression, exponentially weighted moving average, and dynamic linear models. Section 3 summarizes the empirical analysis and performance evaluation. In this section, daily, every other day, and weekly (5 day) hedge rebalancing performance are evaluated with results documented for comparison between dynamic models and against the traditional OLS. Results show that these methods can greatly reduce the variance of the of the hedging model residuals, also known as basis risk. This performance improvement, however, is only possible in a practice by incurring increased transaction costs due to portfolio rebalancing. Section 4 contains extensive results from simulation studies to confirm the robustness of model estimates and hedge effectiveness of locally cointegrating prices. Lastly, we conclude with a summary of the research and findings.

2 Overview of Cointegration

Cointegration, popularized by the work of Engle and Granger (1987), is a model-free phenomenon which occurs when two (or more) stochastic processes are non-stationary, but some linear combination of said processes is stationary. Let $\{y_1, y_2, ...y_p\}$ denote a set of p vectors, each with an equal number of observations $t_1, t_{2,}...t_T$. Then the set p is said to be cointegrated if each vector $\{y_1, y_2, ...y_p\}$ taken individually is $I(1)$, e.g. integrated of order 1; a non-stationary process that becomes stationary when differenced once, while some linear combination of the series $\gamma' p$ is $I(0)$, or stationary for some non-zero vector γ. Specifically, a set of series, all integrated of order n, (in our case integrated of order 1), are said to be cointegrated if and only if some linear combination of the series, with non-zero weights, is integrated of order less than n (Murray, 1994). For example, take the bivariate case where processes y_t and x_t both follow non-stationary random walks.

$$y_t \sim I(1), \quad x_t \sim I(1). \tag{2.1}$$

If these series are cointegrated, there exists

$$z_t = y_t - \gamma x_t \quad z_t \sim I(0), \tag{2.2}$$

such that z_t follows a stationary $I(0)$ process. The γ parameter, known as the cointegrating vector, can be estimated by $\hat{\gamma}$ via least-squares spurious regression through the origin (note the lack of an intercept term) of one random walk onto another:

$$y_t = \hat{\gamma} x_t + z_t. \tag{2.3}$$

$$\hat{\gamma} = \frac{\sum_{i=1}^{t} x_t y_t}{\sum_{i=1}^{t} x_t^2} \tag{2.4}$$

In application, y_t, x_t, and z_t can all be tested for unit roots (or lack thereof in the case of z_t if y_t and x_t are actually cointegrated) using statistical tests for stationarity. The Engle-Granger Representation Theorem states that $x_t$and $y_t$cointegrate if and only if there exists an error correction model (ECM) for either x_t, y_t, or both. For example, let $z_t = y_t - \hat{\gamma} x_t$ be a stationary relation between $I(1)$ variables as shown above. Then there exists a stationary ARMA model for z_t. Assume for simplicity an AR(2),

$$z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + \epsilon_t, \qquad \phi(1) = 1 - \phi_1 - \phi_2 > 0. \tag{2.5}$$

This is equivalent to

$$(y_t - \gamma x_t) = \phi_1(y_{t-1} - \gamma x_{t-1}) + \phi_2(y_{t-2} - \gamma x_{t-2}) + \epsilon_t \tag{2.6}$$

$$y_t = \gamma x_t + \phi_1 y_{t-1} - \phi_1 \gamma x_{t-1} + \phi_2 y_{t-2} - \phi_2 \gamma x_{t-2} + \epsilon_t, \tag{2.7}$$

or

$$\Delta y_t = \gamma \Delta x_t + \phi_2 \gamma \Delta x_{t-1} - \phi_2 \Delta y_{t-1} - (1 - \phi_1 - \phi_2)\{y_{t-1} - \gamma x_{t-1}\} + \epsilon_t. \tag{2.8}$$

Unlike hedging based on correlation, cointegration-based hedging provides a robust alternative. Correlated hedging requires assets to move in tandem while cointegration implies that two price series cannot wander off in opposite directions for very long without eventually reverting to a mean distance. It does not necessarily require that on a daily basis the two prices have to move in synchrony - what it implies is that there exists some long run equilibrium relationship between the two series.

2.1 Local Cointegration

The objective of this paper is to provide robust and practical evidence that minimum-variance hedging based on prices can be improved by increasing the flexibility of our models. Specifically, by allowing for time-varying covariance structure between nonstationary price series, we capture information that lets one routinely update their knowledge of the underlying process. In this paper, we define Local Cointegration as cointegration that holds under a time-varying cointegrating vector, or simply,

$$z_{t+q} = y_{t+q} - \gamma_t x_{t+q} \quad z_{t+q}, \sim I(0) \quad y_{t+q} \sim I(1) \quad x_{t+q} \sim I(1), \tag{2.9}$$

where q denotes q-periods ahead. This can be be any number in theory, but we study and assess the domain of $q = \{0, 1, 2, 5\}$. When $q = 0$, the cointegrating vector updates contemporaneously while $q > 0$ holds a constant cointegrating vector γ_t for q periods before updating. Under such circumstances, if the residual series z_t is stationary, we define the process as locally cointegrating. Even if tests for constant cointegration fail (or marginally pass), the multivariate process between two nonstationary series may be cointegrated under short durations which the time-varying nature of γ_t captures - leading to a stationary residual series. Though structurally and intuitively simple, this time-varying coefficient approach to price-based hedging has valuable practical implications for risk management and trading as evidenced by our study.

3 Time-Varying Parameter Models

Under theoretical assumptions of covariance-stationarity or fixed long run equilibria, constant model parameters can be estimated to determine whether or not a cointegrating relationship exists to implement a hedge. Particularly with financial time series, these relationships and therefore static model parameter estimates are not constant. This introduces complexity to the estimation problem. To estimate hedge performance of time-varying model parameters, Rolling Window Regression (RWR), Exponentially Weighted Moving Average (EWMA) models and Dynamic Linear Models (DLM) are investigated. In this section we provide an overview of the models and how the cointegration-based hedges are constructed.

3.1 Rolling Window Regression

Often referred to as the “poor man’s” time-varying parameter model, a rolling linear regression is simply the moving-average counterpart to linear regression. For a window with $n < T$, the rolling window linear regression (RWR) model may be expressed as (Zivot & Wang, 2003)

$$y_t(n) = x_t(n)\gamma_t(n) + z_t(n), \qquad t = n, ..., T, \tag{3.1}$$

Where $y_t(n)$ is an $n \times 1$ vector of observations (asset prices) on the response, $x_t(n)$ is an $n \times k$ matrix of explanatory variables (in our case, the $n \times 1$ vector of hedging asset's price observation), $\gamma_t(n)$ is a $k \times 1$ cointegrating vector (or hedge ratio) and $z_t(n)$ is the $n \times 1$ vector of stationary error/residual terms, e.g. the hedge basis. The n observations are the n most recent observations from time $t-n+1$ to time t, akin to an n-period moving average, but here we have an n period moving regression. The parameters can be estimated (Zivot & Wang, 2003) such that

$$\hat{\gamma}_t(n) = [x_t(n)'x_t(n)]^{-1}x_t(n)'y_t(n), \tag{3.2}$$

$$\hat{\sigma}_t^2(n) = \frac{1}{n-k}\hat{z}_t(n)'\hat{z}_t(n) = \frac{1}{n-k}[y_t(n) - x_t(n)\hat{\gamma}_t(n)]'[y_t(n) - x_t(n)\hat{\gamma}_t(n)], \tag{3.3}$$

$$VAR_{\rightarrow\infty}(\hat{\gamma}_t(n)) = \hat{\sigma}_t^2(n)[x_t(n)'x_t(n)]^{-1}. \tag{3.4}$$

3.2 Exponentially Weighted Moving Average

In terms of the traditional OLS model, the hedge ratio can be estimated as

$$\hat{\gamma} = \frac{COV(x_t, y_t)}{VAR(x_t)}. \tag{3.5}$$

We apply this approach to determining time-varying hedge ratios in an exponentially weighted setting via EWMA. The unconditional covariance matrix of our two series represented can be estimated as

$$\hat{\Sigma} = \frac{1}{T-1}\sum_{t=1}^{T}(y_t - \bar{y})(x_t - \bar{x}), \tag{3.6}$$

where $\hat{\Sigma}$ denotes the covariance matrix of x_t and y_t. Time-variation in the covariance matrix is introduced by weighting more recent observations heavily relative to past observations through exponential smoothing,

$$\hat{\Sigma}_t = (1-\lambda)(y_t - \bar{y})(x_t - \bar{x}) + \lambda\hat{\Sigma}_{t-1}, \tag{3.7}$$

$$\hat{\Sigma}_t = \begin{bmatrix} \hat{\sigma}_y^2 & \hat{\sigma}_{xy} \\ \hat{\sigma}_{xy} & \hat{\sigma}_x^2 \end{bmatrix}$$

where $0 < \lambda < 1$is the weight parameter, $\hat{\sigma}_y^2$, $\hat{\sigma}_x^2$ and $\hat{\sigma}_{xy}$ are the variance of y, x, and their covariance estimates at time t, respectively. The larger λ, the more weight is given to previous observations and less to the most recent observation. Financial risk institution $RiskMetrics^{TM}$ implements EWMA with $\lambda = 0.94$, as we shall in our hedge performance testing. To initialize the EWMA the estimate of the entire sample covariance matrix $\hat{\Sigma}_1 = \hat{\Sigma}$ is used. For a given λ and an initial estimate $\hat{\Sigma}_1$, $\hat{\Sigma}_t$ can be computed recursively. Under the assumption that the joint distribution of the observed asset prices x_t and y_t, is multivariate normal with mean zero and covariance matrix Σ_t, where the mean μ_t is a function of parameter Θ, then λ and Θ can be estimated jointly via Maximum Likelihood because the log-likelihood function of the data is

$$ln\, L(\Theta, \lambda) \propto -\frac{1}{2}\sum_{t=1}^{T}|\Sigma_t| - \frac{1}{2}\sum_{t=1}^{T}(y_t - \bar{y})(x_t - \bar{x})'\Sigma_t^{-1}(y_t - \bar{y})(x_t - \bar{x}), \tag{3.8}$$

which can be evaluated recursively by substituting $\hat{\Sigma}_t$ for Σ_t (Tsay, 2010).

3.3 Dynamic Linear Models & The Kalman Filter

Dynamic Linear Models (Kalman ,1960 and Anderson & Moore, 1979) follow a Bayesian estimation philosophy for estimating time-varying parameters. This method, theorized quite some time ago, has gained recent popularity dude to advances in technology and computing power. Recent treatments of Dynamic Linear Models and Kalman Filtering (Kalman, 1960) were developed in 2001 (see Durbin & Koopman, 2001). The idea is that an observation y_t at time t depends on an underlying *state vector (hedge ratio)* γ_t and the independent variable X_t. We treat γ_t as a random state rather than a constant coefficient as done in simple linear regression, and this state can vary over time. Under the Kalman Filter, this is a Gaussian Process where the joint distribution of all parameters $(...\gamma_{t-2}, \gamma_{t-1}, \gamma_t, \gamma_{t+1,}\gamma_{t+2}..., \gamma_{t-2}, \gamma_{t-1}, y, y_{t+1}, y_{t+2}, ...)$ is multivariate normal. The Gaussian assumption can be relaxed under extensions of the Kalman Filter such as the Particle Filter. We refer the reader to Bishop (2006) and Kitagawa & Gersch (1996) for further treatment of filtering methods. Modified from Tsay's (2010) treatment of state-space models and the Kalman Filter, the linear Gaussian state-space DLM can be written as a hierarchical model given by

$$y_t = \alpha_t + \gamma_t x_t + z_t, \quad z_t \sim N(0, P_\epsilon), \tag{3.9}$$

$$\alpha_t = R\alpha + w_t \quad w_t \sim N(0, Q_w), \tag{3.10}$$

$$\gamma_t = T\gamma_{t-1} + u_t, \quad u_t \sim N(0, Q_u), \tag{3.11}$$

such that z_t, w_t and u_t are two independent Gaussian white noise series, and are independent of both $E(y_t)$, $E(\alpha_t)$ and $E(\gamma_t)$ at time $t > 0$, respectively. The estimation of the parameters that specify a Dynamic Linear Model is quite involved. Taken from Shumway & Stoffer (2000), here we use Maximum Likelihood Estimation (MLE), where θ denotes the parameter set. The likelihood is computed using innovations $\epsilon_1, \epsilon_2, ..., \epsilon_n$ defined by

$$z_t = y_t - E(y_t|y_{t-1}) = y_t - \alpha_t - \gamma_t X_t^{t-1}, \tag{3.12}$$

with covariance matrix $\Sigma_t = var(z_t)$. Ignoring the constant, we can write the log likelihood function to be maximized, $L_Y(\theta)$ as

$$-lnL_Y(\theta) = \frac{1}{2}\sum_{t=1}^{n} log|\Sigma_t(\theta)| + \frac{1}{2}\sum_{t=1}^{n} z_t(\theta)'\Sigma_t(\theta)^{-1}z(\theta). \tag{3.13}$$

Solving this function is not a trivial task, hence various recursive and algorithmic approaches have been presented (Gupta and Mehra,1974). For deeper detail, we refer the reader to Shumway & Stoffer (2000).

DLM parameters can be estimated with the Kalman Filter, a forwards-backwards recursive algorithm. Essentially, the Kalman Filter is the continuous-state-space analogue of the Hidden Markov Model, which deals with a discrete state-space. For extensive treatment of the Kalman Filter algorithm there is deep literature on the subject, one referral is Kitagawa & Gersch (1996). The following is a high level treatment of the Kalman Filter algorithm, taken from West & Harrison (1997).

The "Generation Step"

Without loss of generality, suppose $\alpha_t = 0$. At time t, calculate a "prior" mean and variance for the quantities at time t. The expectation of γ_t at time t is b_t so the expectation of γ_t at time $t-1$ is $b_{t|t-1} = T_t b_{t-1}$. The value of the state vector is not observable, but at any time there exists a mean vector and covariance matrix for it. The variance of γ_t at time t is S_t, so the variance of γ_t at time $t-1$ is $S_{t|t-1} = T_t S_{t-1} T_t' + Q_t$. At time $t-1$ the expectation of Y_t is

$$F_t = x_t b_{t|t-1}, \tag{3.14}$$

the variance of y_t is

$$D_t = x_t S_{t|t-1} x_t' + P_t, \tag{3.15}$$

and the covariance of γ_t and Y_t is

$$C_t = S_{t|t-1} x_t'. \tag{3.16}$$

So, at time $t-1$,

$$E\begin{pmatrix}\gamma_t \\ y_t\end{pmatrix} = \begin{pmatrix} b_{t|t-1} \\ F_t \end{pmatrix}, \tag{3.17}$$

and

$$Var\begin{pmatrix}\gamma_t \\ y_t\end{pmatrix} = \begin{pmatrix} S_{t|t-1} & C_t \\ C_t' & D_t \end{pmatrix}. \tag{3.18}$$

The "Observation Step"

At time t, y_t is observed. Beliefs about γ_t are updated. Under the Gaussian assumption, then this is done by applying Baye's rule. The updated mean for γ_t is

$$b_t = b_{t|t-1} + C_t D_t^{-1}(y_t - F_t), \tag{3.19}$$

and the updated variance matrix for γ_t is

$$S_t = S_{t|t-1} - C_t D_t^{-1} C_t'. \tag{3.20}$$

Note that the variance matrices P_t and Q_t are known/given. Although they have t subscripts, P, Q, x, and T would often remain constant.

Updating

When some new data is observed, first a generation step then an observation step is carried out to update the state vector.

Forecasting

A generation step on its own gives a one-step-ahead forecast. Forecasts can be generated further into the future by a sequence of generation steps without observation steps. For example, suppose the data at time t is observed. One-step-ahead forecasts can be found.

$$E\begin{pmatrix}\gamma_{t+1}\\ y_{t+1}\end{pmatrix} = \begin{pmatrix}Tb_t\\ xTb_t\end{pmatrix}, \tag{3.21}$$

and

$$Var\begin{pmatrix}\gamma_{t+1}\\ y_{t+1}\end{pmatrix} = \begin{pmatrix}S_{t+1|t} & C_{t+1}\\ C_{t+1}' & D_{t+1}\end{pmatrix}. \tag{3.22}$$

Then the two-step ahead forecasts can be calculated, and so on.

$$E\begin{pmatrix}\gamma_{t+1}\\ y_{t+1}\end{pmatrix} = \begin{pmatrix}T & 0\\ xT & 0\end{pmatrix}\begin{pmatrix}Tb_t\\ xTb_t\end{pmatrix} = \begin{pmatrix}TTb_t\\ xTTb_t\end{pmatrix}, \tag{3.23}$$

and

$$Var\begin{pmatrix}\gamma_{t+2}\\ y_{t+2}\end{pmatrix} = \begin{pmatrix}T & 0\\ xT & 0\end{pmatrix}\begin{pmatrix}S_{t+1|t} & C_{t+1}\\ C_{t+1}' & D_{t+1}\end{pmatrix}\begin{pmatrix}T' & T'x'\\ 0 & 0\end{pmatrix}. \tag{3.24}$$

Alternative derivations of the Kalman Filter algorithm can be found extensively in the literature (Tsay 2010, Bishop 2006, Kitagawa & Gersch 1996, are just a few that we refer to). By implementing DLM, functional coefficients will enable the modeling of dynamic systems. By estimating a functional cointegrating vector, the relationship between two non-stationary series can be considered dynamic, and an optimal hedge can be generated when traditional models are not suitable.

4 Empirical Analysis & Testing Hedge Effectiveness

In this section, we discuss the applications of time-varying hedges and the data[2] on which it will be tested upon. Out-of sample testing is done on 3 pairs of assets from different markets. The equity market pair consists of the NASDAQ index tracking exchange-traded fund (QQQ) and Apple, Inc. stock (AAPL). With AAPL being a constituent of the QQQ itself, the two equity assets bear considerable correlation in their return series. For fixed income markets, exchange traded funds iShares iBoxx High Yield Corporate Bd (HYG) and iShares Core US Aggregate Bond (AGG) are used. A trader aiming to capture the nominal yield differential between the two bond portfolios, or more generally high-yield and investment-grade bonds, could go long HYG and hedge the market risk with AGG. From the commodities space we model the hedge ratio between West Texas Intermediate Crude Oil Front-Month Futures and U.S. Gulf Coast Jet Fuel Spot prices. WTI Crude futures, being one of the most liquid energy markets globally, provides ease of hedging against fluctuating Jet Fuel prices, and other petroleum-based products that may not have liquid futures markets, given that these products exhibit structural dependency.

A "Fast" Rolling-Window Regression (RWR) is implemented with a window size of $n = 2$. As such, the hedge ratio is effectively the slope between the two most recent observations. Empirically, this small window size outperforms longer window sizes. The Exponentially Weighted Moving Average

[2] AAPL & QQQ daily prices were collected from Yahoo! Finance, dating from 4/1/2005 to 4/1/2015. HYG & AGG daily prices were also taken from Yahoo! Finance, dating from 4/11/2007 to 4/1/2015. WTI Crude Oil Front Month Futures prices were taken from Quandl, Inc. and the Wiki Continuous Futures Database. U.S. Gulf Coast Jet Fuel spot prices are sourced from the U.S. Department of Energy. The two energy price series consist of daily data dating from 4/1/2005 to 4/1/2015.

model will hold a decay parameter $\lambda = .94$, which is the industry standard set by$RiskMetrics^{TM}$. As mentioned, the initial observation variance for the Dynamic Linear Model (DLM) is estimated using Maximum-Likelihood based optimization and the state variance is set to 1. The out-of sample testing is based on three scenarios: daily rebalancing, rebalancing every 2 days, and rebalancing every 5 days (or business week). No contemporaneous information is used to estimate hedge ratios to satisfy out-of-sample requirements such that the general model follows the equation

$$y_{t+k} = \hat{\gamma}_t x_{t+k} + z_{t+k}. \tag{4.1}$$

The hedge ratio $\hat{\gamma}$ is estimated recursively, using only data up to but not including the current value for the one-step ahead test, e.g. $k = \{1\}$. Similarly, the estimated $\hat{\gamma}$ uses data $k = \{1, 2\}$ when rebalancing every two days. Weekly rebalancing is emulated by estimating $\hat{\gamma}$ using $k = \{1, 2, 3, 4, 5\}$ such that the hedge ratio estimate is carried forward throughout the 5 day period before re-estimating and repeating. Why not rebalance daily and minimize the basis variance? Simply because of the practical costs of trading incurred with daily rebalancing. Rebalancing every 2 days would hypothetically increase the basis risk, though cost of hedging would be cut in half. Rebalancing weekly would further reduce costs of hedging. For these out-of-sample tests, the statistics for performance evaluation are the Root Mean Squared Error and Mean Absolute Deviation,

$$RMSE = \sqrt{\frac{1}{n}(\sum_{i=1}^{n} \hat{y}_t - y_t)^2}, \tag{4.2}$$

$$MAD = \frac{1}{n}\sum_{i=1}^{n} |\hat{y}_t - y_t|, \tag{4.3}$$

where $\hat{y}_t$ is the model fitted value at time t. We omit the first 500 observations from the out-of-sample analysis as a burn-in phase for the models, namely EWMA and DLM which estimate recursively. The remaining 8 years of daily observations are used for testing hedge effectiveness. Hedging effectiveness under the classic OLS / Spurious Regression framework is shown in Table 1. With AAPL trading in the \$100's, HYG trading in the \$50's and Jet Fuel trading in the \$2's, in-sample deviation statistics are roughly 10% across the

Table 1: Constant Estimates (OLS), in-sample RMSE/MAD

Pair	OLS Equation	RMSE	MAD
AAPL & QQQ	$AAPL = -36.56 + 1.459QQQ$	10.95	9.10
HYG & AGG	$HYG = -43.09 + 1.584AGG$	5.82	4.913
Jet Fuel & WTI	$Jet = 0.017 + 0.029WTI$	0.21	0.16

board. Tables 2 to 4 show hedge effectiveness for the pairs under a time-varying framework. The hedge ratios of AAPL/QQQ and HYG/AGG are characterized by heavy drift, suggesting that the dependence structure changes considerably over time. Jet Fuel/WTI, however, has a relatively stable time-varying hedge ratio, implying that the relationship between the two petroleum derivatives is structural and that the relationship could be modeled with a static model reliably (evidence that the two series are truly cointegrated).

A peculiar benefit of cointegration-based hedging with price series is that the $RMSE$ and MAD can be interpreted in dollar terms which lets us attach a hard value to the basis risk. All three models vastly outperform even the in-sample performance of the static OLS model, while the simple RWR performs surprisingly well and DLM outperforms under all rebalancing schemes. The resulting residual z_t based on time-varying models are all highly stationary, with ADF tests rejecting the null hypothesis of a unit-root in all cases ($k = 0, 1, 2, 5$) across all models.

Table 2: AAPL & QQQ

RMSE	$k = 1$	$k = 2$	$k = 5$	**MAD**	$k = 1$	$k = 2$	$k = 5$
RWR	0.83	0.98	1.37	RWR	0.52	0.61	0.85
EWMA	2.16	2.23	2.41	EWMA	1.51	1.55	1.68
DLM	0.74	0.94	1.34	DLM	0.46	0.57	0.81

Table 3: HYG & AGG

RMSE	$k = 1$	$k = 2$	$k = 5$	**MAD**	$k = 1$	$k = 2$	$k = 5$
RWR	0.49	0.57	0.77	RWR	0.34	0.40	0.54
EWMA	1.08	1.11	1.19	EWMA	0.86	0.88	0.95
DLM	0.44	0.54	0.75	DLM	0.31	0.37	0.52

Table 4: Jet Fuel & WTI Futures

RMSE	$k = 1$	$k = 2$	$k = 5$	**MAD**	$k = 1$	$k = 2$	$k = 5$
RWR	0.06	0.08	0.08	RWR	0.03	0.04	0.04
EWMA	0.10	0.10	0.11	EWMA	0.06	0.06	0.07
DLM	0.06	0.08	0.08	DLM	0.03	0.03	0.04

5 Simulation

To confirm the robustness of our out-of-sample performance, we refit and evaluate the RMSE and MAD statistics based on simulated sample distributions of the statistics. For each pair of securities, 10,000 bootstrapped samples were tested. Results were unanimously positive, confirming the reliability of the out-of-sample RMSE and MAD statistics reported in Section 3. Since time series data is subject to potential short-memory / autoregressive characteristics, we take a Stationary Block Bootstrap (Politis & Romano, 1994) approach. Traditional Monte Carlo bootstrapped simulation relies on the assumption that observations are independently and identically distributed, thus the data could be randomly sampled with replacement. Block Bootstrap is more appropriate for time series since the observations are split into blocks of a selected length, with the blocks rather than individual observations resampled. Stationary Block Bootstrap extends the Block Bootstrap in that rather than fixing the block length, it is allowed to vary such that the block length is random and generated from a geometric distribution with some mean number of observations per block - the specified mean block length for this study is 5. The process undertaken runs over the following steps, for each asset pair:

1. Transform price series pair to logged returns (for stationarity and removing long-memory).
2. Apply Stationary Block Bootstrap and generate a replicate, resampled with replacement, with an equal number of daily observations as original data (10 years).
3. Transform the multivariate series back to normalized price series, with initial value of 1 by taking cumulative products.
4. Re-scale price series to original values by multiplying series by price at time period 1.
5. Evaluate 1-day, 2-day, and 5-day out-of-sample RMSE and MAD statistics for each replicate, for all three models: RWR, EWMA, DLM.
6. Repeat steps 2 through 5 10,000 times.

Out of sample performance from Section 3 matches bootstrapped mean and median statistics closely, with the bootstrapped estimates being slightly more conservative. The bootstrapped statistics have the benefit of letting us observe the full sampling distribution of hedge effectiveness under simulated environments, and robustness of applying time-varying hedge models can be confirmed. Note that these figures can be interpreted in dollar terms. Though RWR performs second best, it is by far the slowest with respect to computation - taking nearly 10 times longer than EWMA, the fastest simulation. DLM computation takes about 3 times as long as EWMA. When speed is a necessity, the slight under performance of EWMA may be overlooked for its ease of computation[3]. Simulation results can be found in section 6, with histograms in the Appendix A.

6 Conclusion

Time-varying cointegration models for hedging provide unique insights and practical benefits in markets where cross-hedges are needed. We test and find that Dynamic Linear Models prove to be the best performing modeling scheme

[3]Computations and Simulations were done in the R Statistical Language.

in terms of stationarity in the residuals, root mean squared error and mean absolute deviation metrics. Rolling Window Regression and Exponentially Weighted Moving Average methods also performed well, with all three models vastly outperforming the static hedge benchmark out-of-sample. EWMA method, though under performed on a relative basis, has the fastest computation time. By treating price series as locally cointegrating, the application of relatively simple yet robust models enable the practitioner to meaningfully reduce basis risk, improving the practice of cross-hedging when traditional hedging derivatives are not available or not applicable.

7 Simulation Results

Table 5: **AAPL & QQQ**

[RWR]

Statistic	**Min**	**1st Qu.**	**Median**	**Mean**	**3rd Qu.**	**Max**
1-Day RMSE	0.03	0.34	0.64	1.00	1.21	21.84
1-Day MAD	0.02	0.21	0.37	0.55	0.68	11.48
2-Day RMSE	0.03	0.39	0.75	1.19	1.43	25.76
2-Day MAD	0.02	0.25	0.44	0.66	0.80	13.56
5-Day RMSE	0.05	0.54	1.02	1.61	1.94	36.42
5-Day MAD	0.03	0.34	0.61	0.90	1.12	19.70

[EWMA]

Statistic	**Min**	**1st Qu.**	**Median**	**Mean**	**3rd Qu.**	**Max**
1-Day RMSE	0.06	0.89	1.64	2.63	3.13	60.30
1-Day MAD	0.04	0.62	1.11	1.67	2.03	30.14
2-Day RMSE	0.06	0.92	1.67	2.71	3.22	62.12
2-Day MAD	0.05	0.64	1.15	1.72	2.09	31.04
5-Day RMSE	0.07	0.99	1.82	2.92	3.47	67.27
5-Day MAD	0.05	0.69	1.24	1.86	2.25	33.64

[DLM]

Statistic	**Min**	**1st Qu.**	**Median**	**Mean**	**3rd Qu.**	**Max**
1-Day RMSE	0.03	0.31	0.57	0.92	1.09	34.80
1-Day MAD	0.02	0.19	0.33	0.50	0.60	16.91
2-Day RMSE	0.03	0.38	0.70	1.13	1.34	37.42
2-Day MAD	0.02	0.23	0.41	0.61	0.74	17.86
5-Day RMSE	0.04	0.54	0.99	1.56	1.87	42.32
5-Day MAD	0.03	0.33	0.58	0.87	1.05	20.36

Table 6: **HYG & AGG**

[RWR]

Statistic	**Min**	**1st Qu.**	**Median**	**Mean**	**3rd Qu.**	**Max**
1-Day RMSE	0.30	0.59	0.71	0.75	0.87	2.54
1-Day MAD	0.18	0.36	0.44	0.45	0.53	1.35
2-Day RMSE	0.36	0.70	0.84	0.88	1.02	3.05
2-Day MAD	0.21	0.43	0.52	0.54	0.63	1.61
5-Day RMSE	0.48	0.94	1.013	1.18	1.37	4.32
5-Day MAD	0.28	0.59	0.71	0.74	0.86	2.30

[EWMA]

Statistic	**Min**	**1st Qu.**	**Median**	**Mean**	**3rd Qu.**	**Max**
1-Day RMSE	0.68	1.48	1.80	1.88	2.19	6.71
1-Day MAD	0.47	1.08	1.31	1.37	1.60	4.82
2-Day RMSE	0.70	1.52	1.85	1.93	2.25	6.90
2-Day MAD	0.48	1.11	1.35	1.41	1.64	4.96
5-Day RMSE	0.75	1.64	1.99	2.09	2.43	7.47
5-Day MAD	0.52	1.20	1.45	1.52	1.77	5.36

[DLM]

Statistic	**Min**	**1st Qu.**	**Median**	**Mean**	**3rd Qu.**	**Max**
1-Day RMSE	0.22	0.54	0.66	0.68	0.80	2.32
1-Day MAD	0.13	0.33	0.39	0.41	0.47	1.21
2-Day RMSE	0.27	0.66	0.80	0.83	0.96	2.80
2-Day MAD	0.16	0.39	0.48	0.49	0.57	1.47
5-Day RMSE	0.38	0.91	1.10	1.15	1.34	4.02
5-Day MAD	0.23	0.56	0.68	0.71	0.83	2.10

Table 7: **Jet Fuel & WTI Futures**

[RWR]

Statistic	Min	1st Qu.	Median	Mean	3rd Qu.	Max
1-Day RMSE	0.0015	0.0182	0.0300	0.0421	0.0504	1.7520
1-Day MAD	0.0006	0.0093	0.0155	0.0216	0.0263	0.6343
2-Day RMSE	0.0017	0.0213	0.0351	0.0492	0.0591	1.9760
2-Day MAD	0.0008	0.0109	0.0181	0.0253	0.0308	0.7380
5-Day RMSE	0.0023	0.0281	.0465	0.0653	0.0784	2.7520
5-Day MAD	0.0011	0.0147	0.0245	0.0342	0.0418	1.0700

[EWMA]

Statistic	Min	1st Qu.	Median	Mean	3rd Qu.	Max
1-Day RMSE	0.0036	0.0426	0.0695	0.0977	0.1177	3.304
1-Day MAD	0.0025	0.0267	0.0446	0.0620	0.0754	1.874
2-Day RMSE	0.0037	0.0437	0.0714	0.1005	0.1210	3.387
2-Day MAD	0.0025	0.0274	0.0459	0.0637	0.0775	1.925
5-Day RMSE	0.0040	0.0470	0.0769	0.1081	0.1300	3.625
5-Day MAD	0.0027	0.0295	0.0494	0.0686	0.0835	2.072

[DLM]

Statistic	Min	1st Qu.	Median	Mean	3rd Qu.	Max
1-Day RMSE	0.0013	0.0165	0.0272	0.0381	0.0456	1.5880
1-Day MAD	0.0005	0.0082	0.0137	0.0191	0.0233	0.5557
2-Day RMSE	0.0016	0.0199	0.0328	0.0461	0.0554	1.5870
2-Day MAD	0.0007	0.0100	0.0167	0.0233	0.0284	0.6400
5-Day RMSE	0.0023	0.0272	.0450	0.0633	0.0762	2.7410
5-Day MAD	0.0010	0.0140	0.0234	0.0327	0.0399	1.0150

Appendix A: Simulated RMSE / MAD Distributions

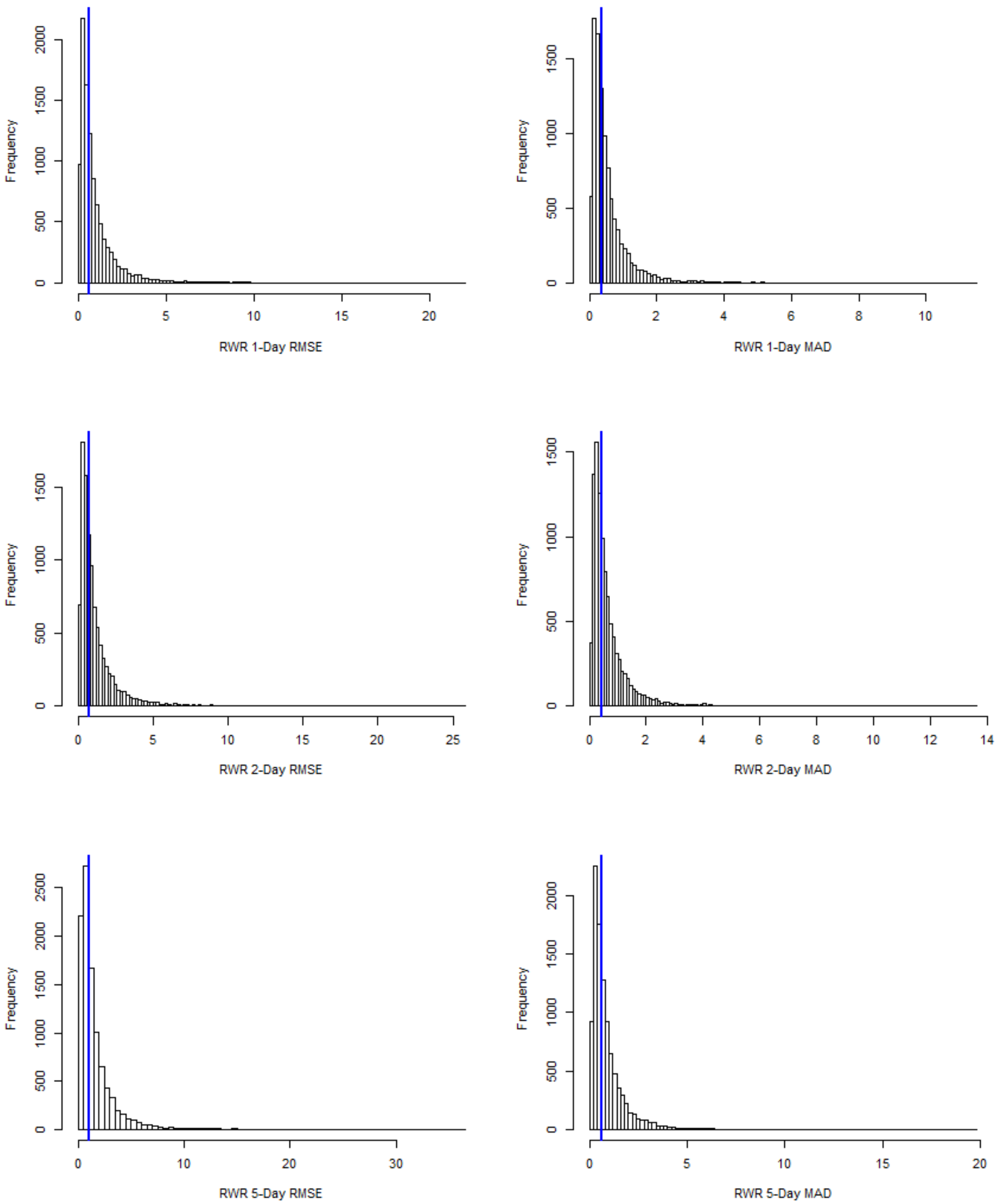

Figure 1: AAPL & QQQ : RWR

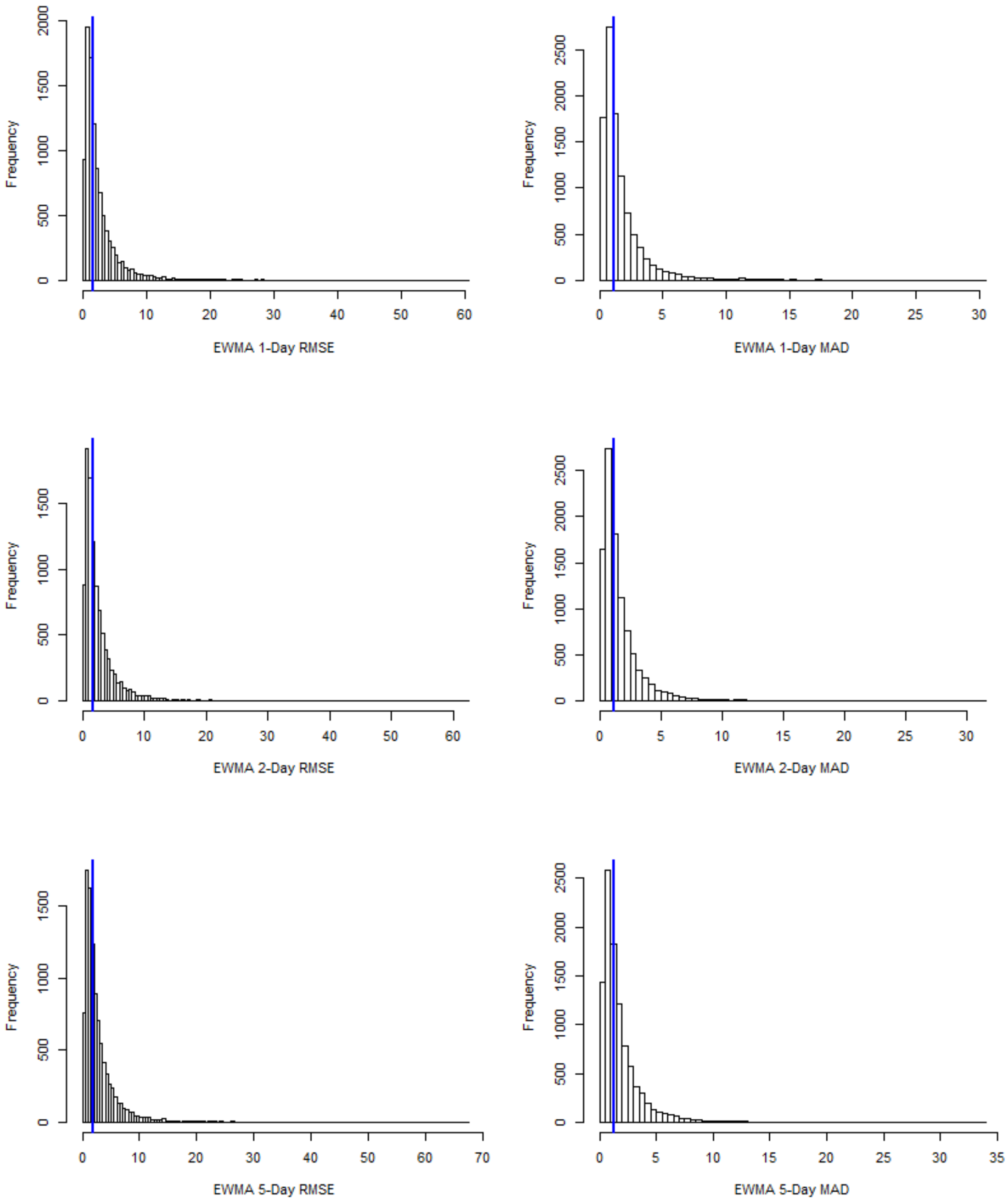

Figure 2: APPL & QQQ: EWMA

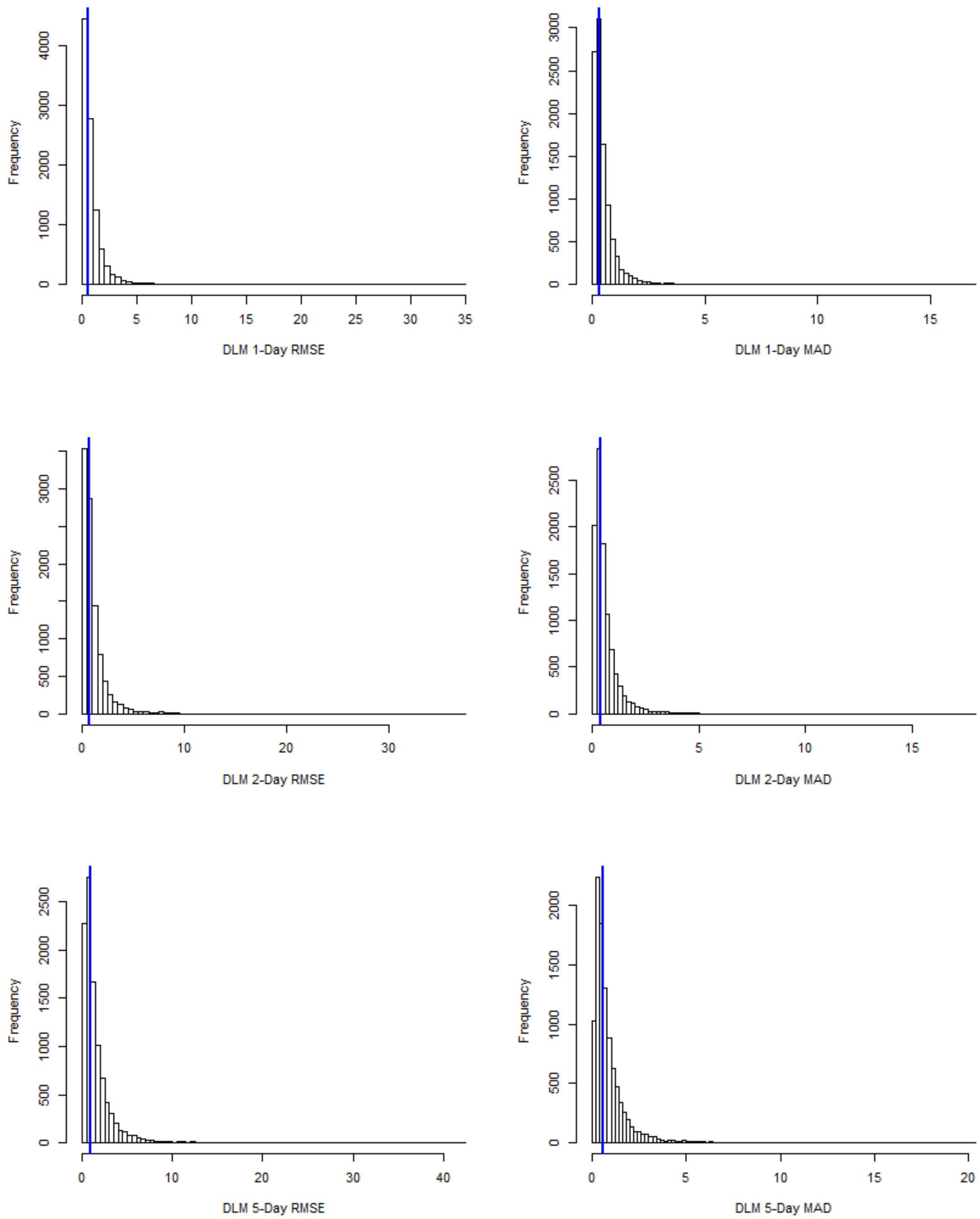

Figure 3: AAPL & QQQ: DLM

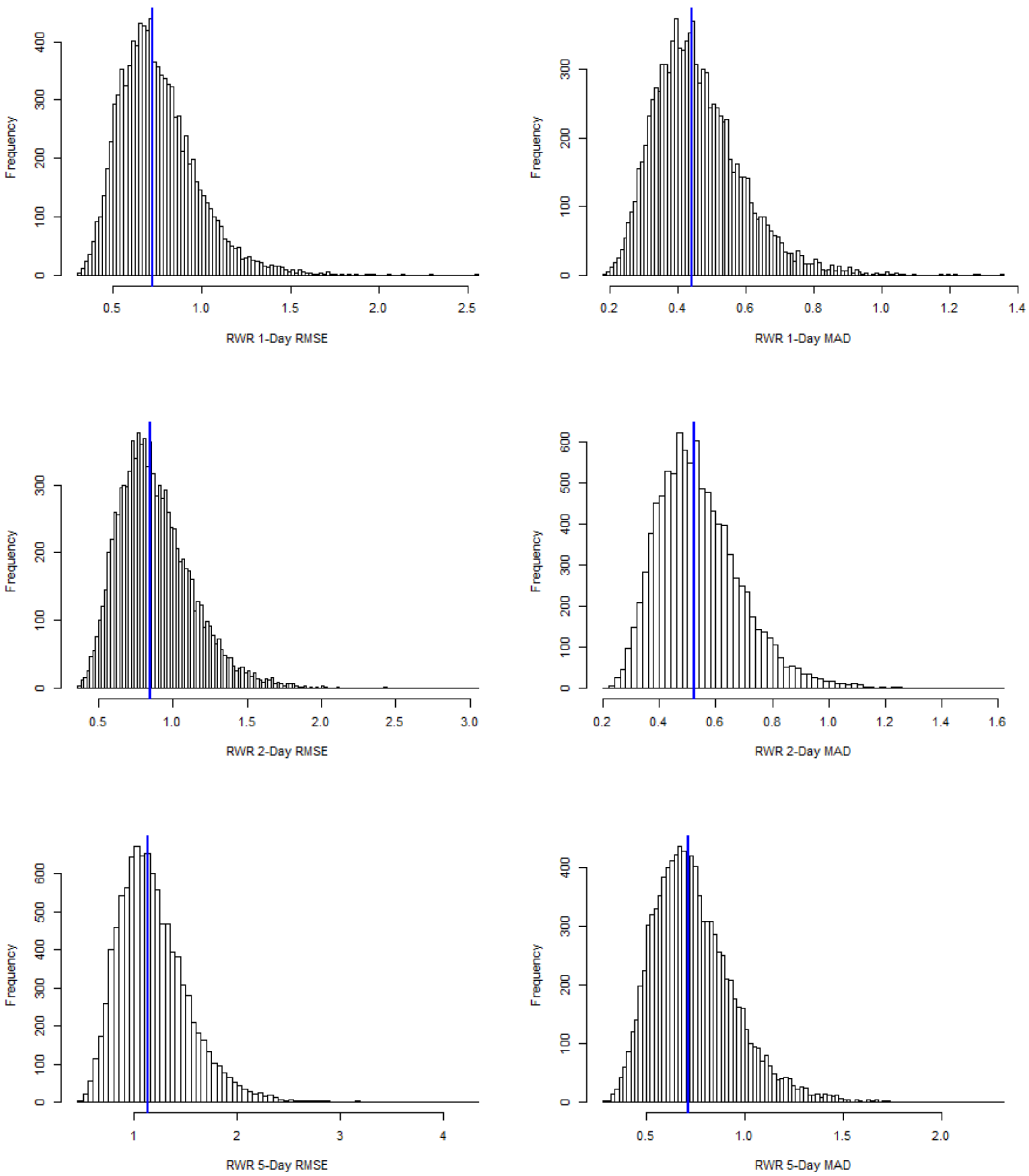

Figure 4: HYG & AGG: RWR

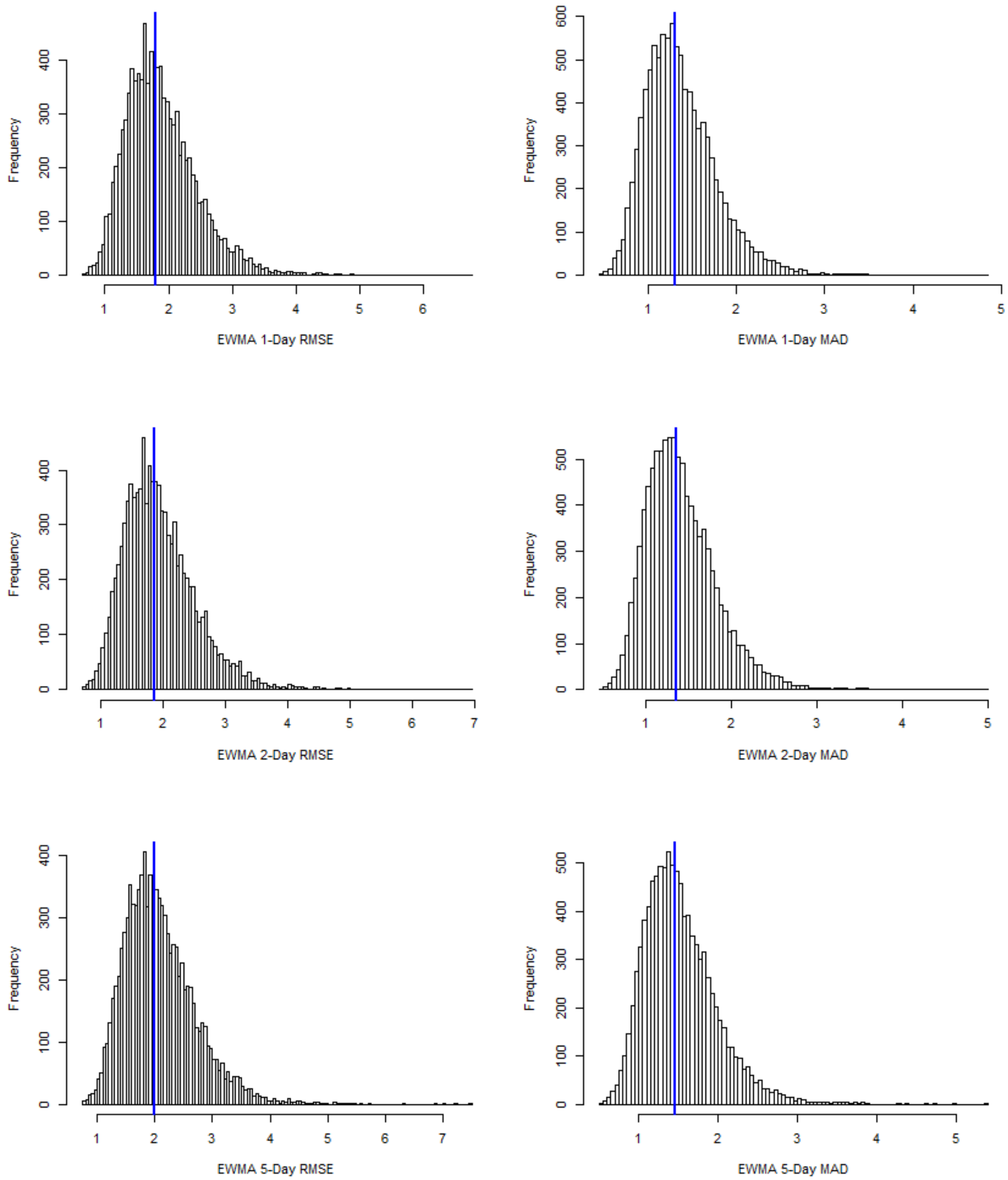

Figure 5: HYG & AGG: EWMA

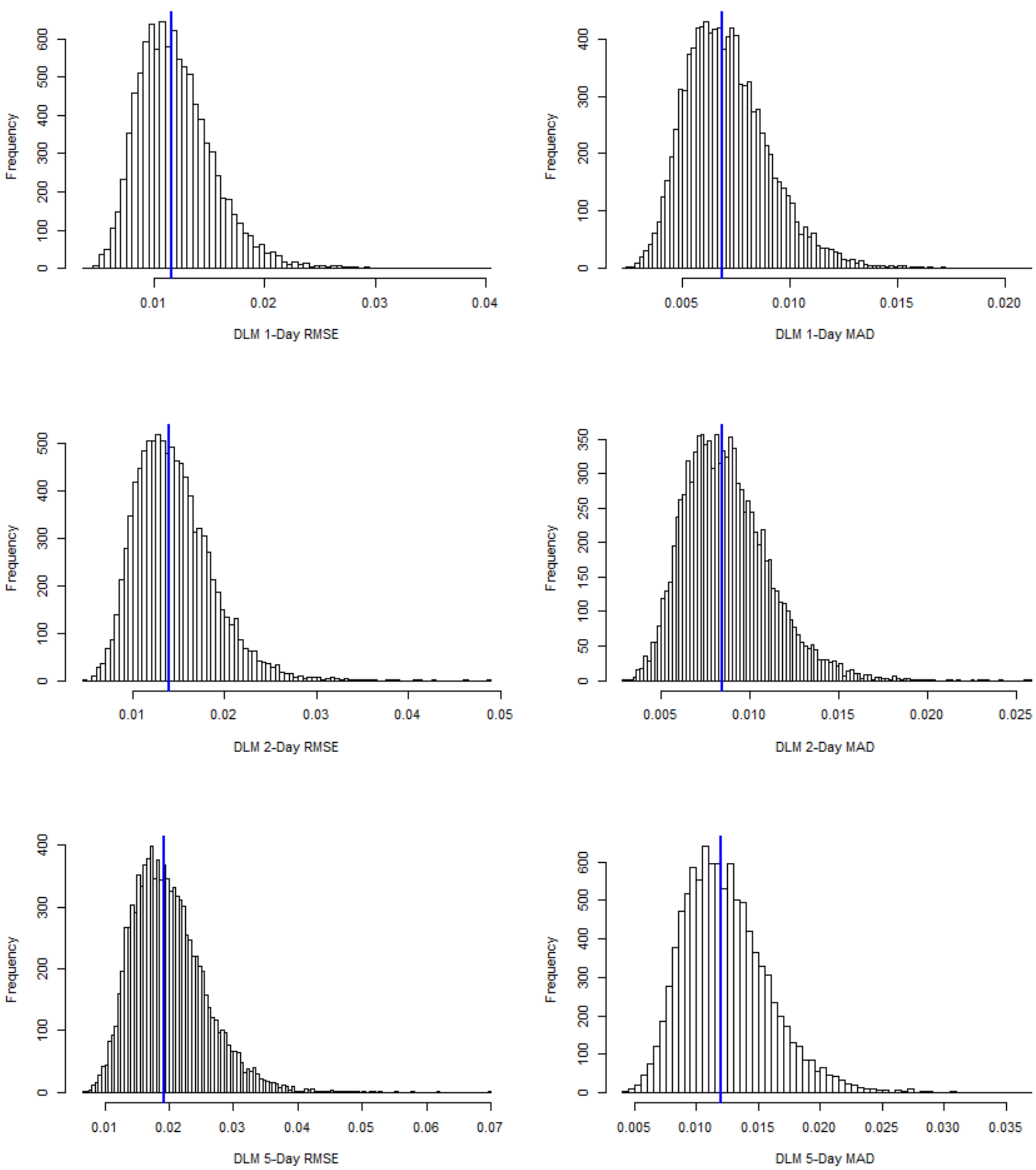

Figure 6: HYG & AGG: DLM

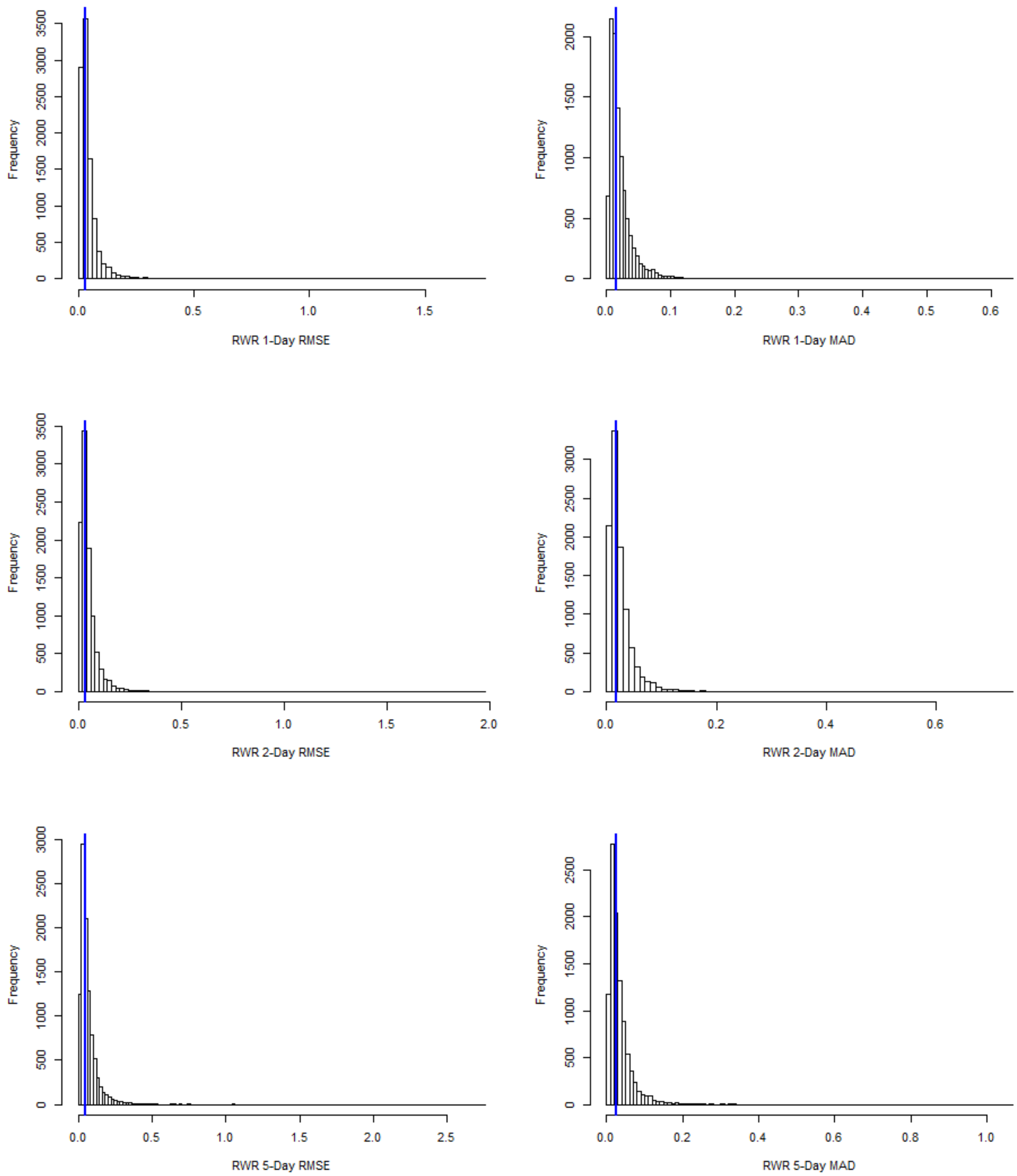

Figure 7: Jet Fuel & WTI Futures: RWR

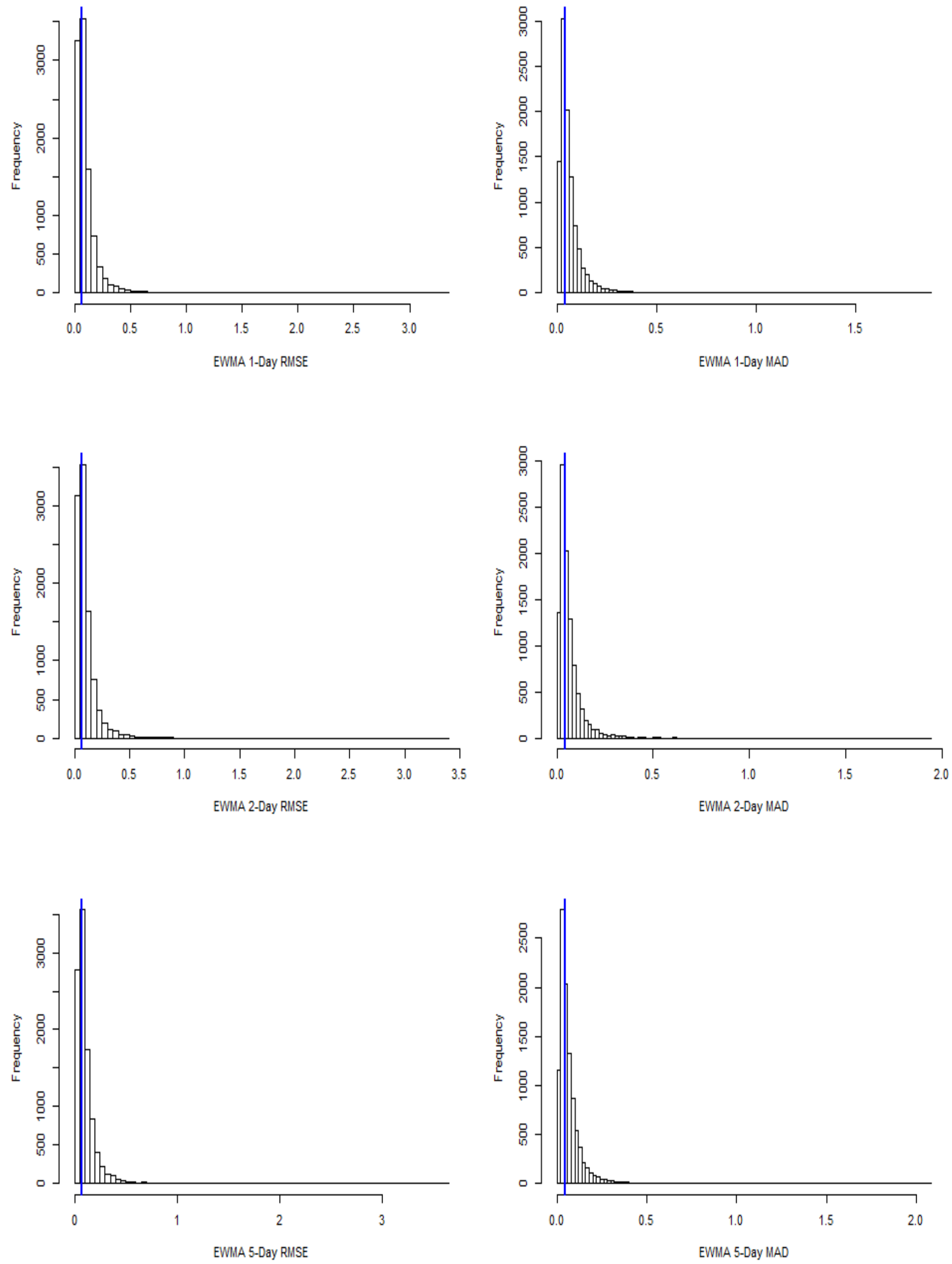

Figure 8: Jet Fuel & WTI Futures: EWMA

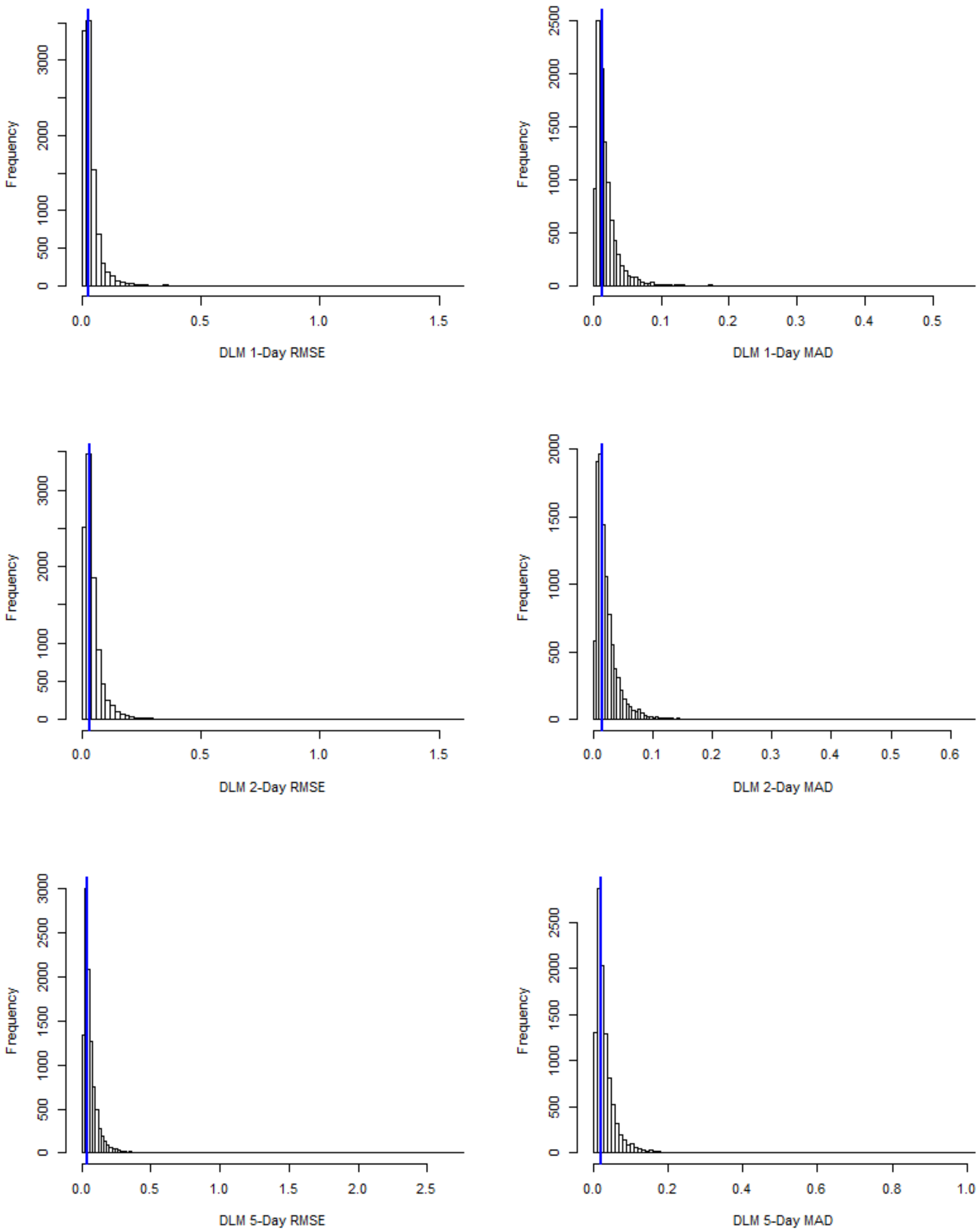

Figure 9: Jet Fuel & WTI Futures: DLM

Appendix B: Time-Varying Coefficient Estimates

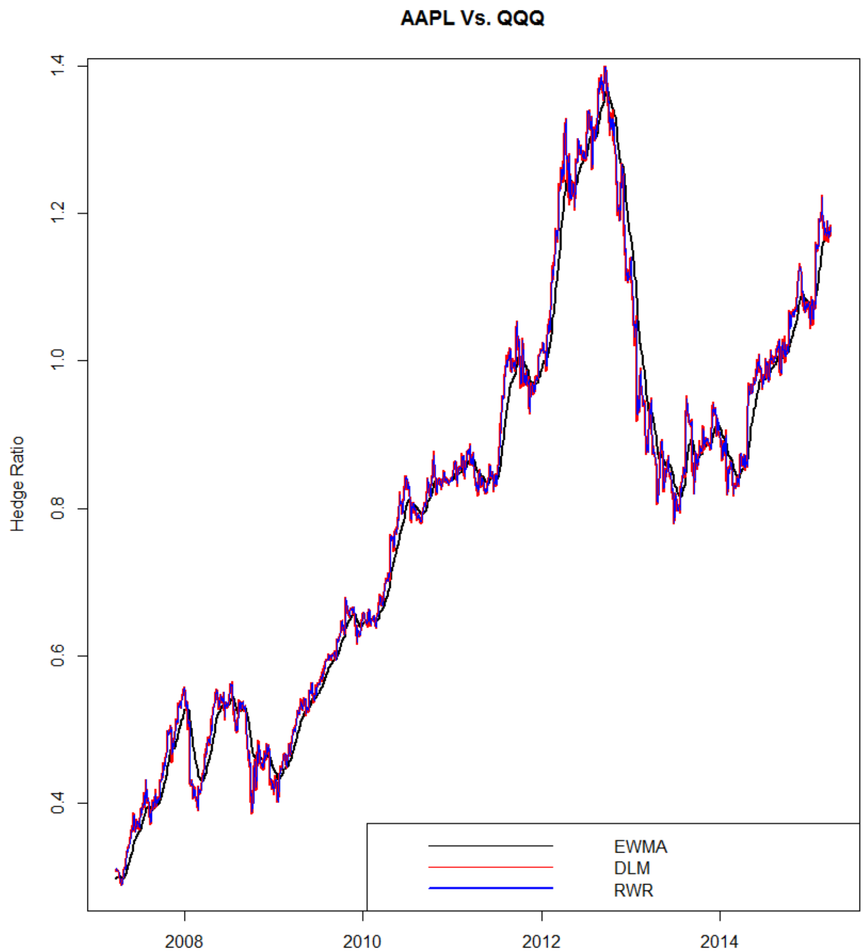

Figure 10: AAPL & QQQ

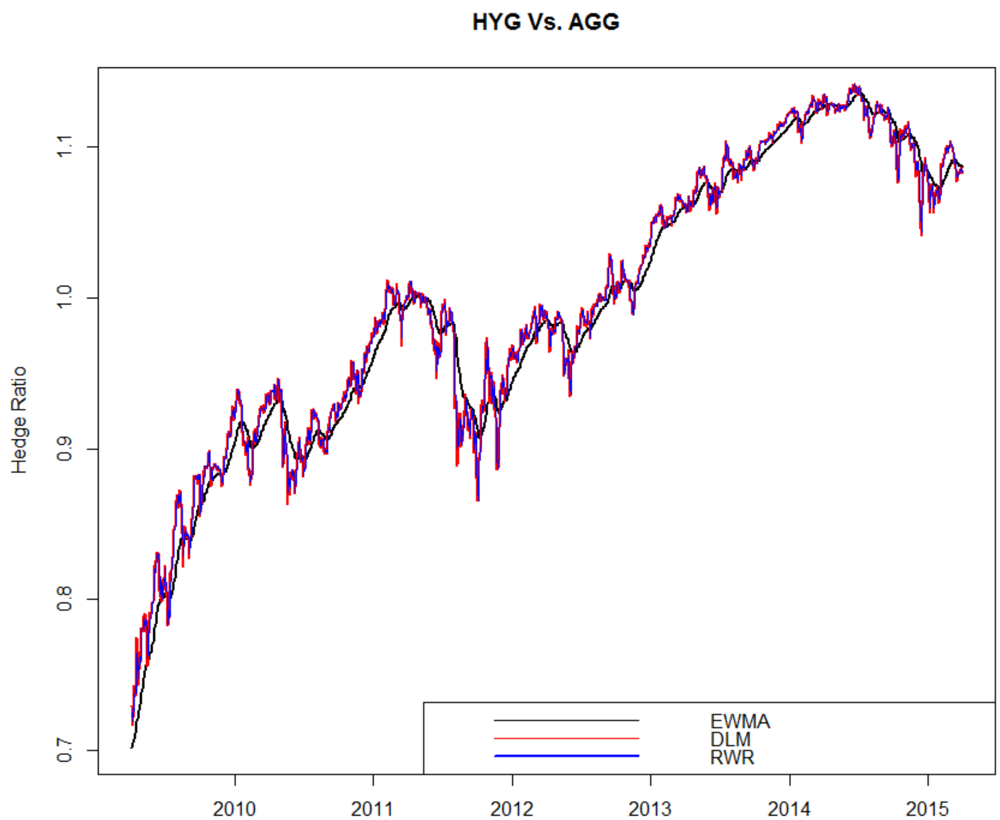

Figure 11: HYG & AGG

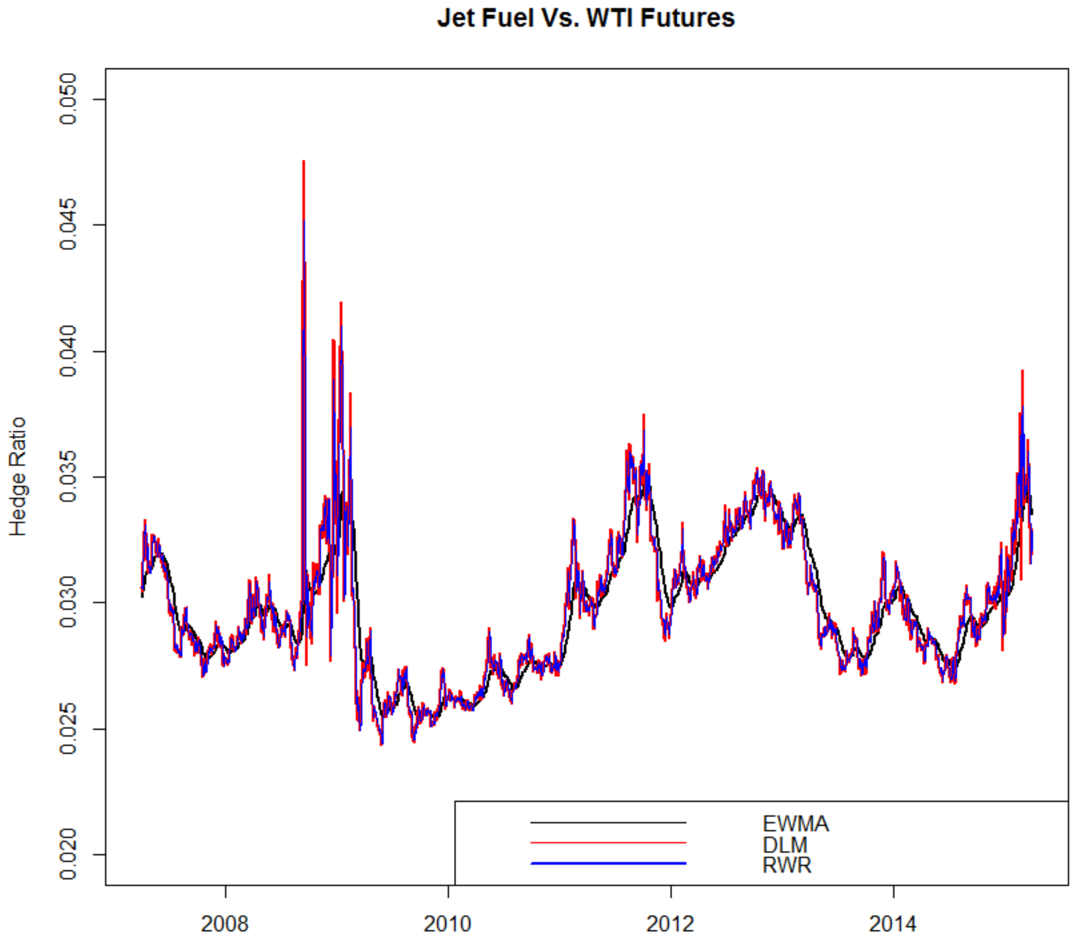

Figure 12: Jet Fuel & WTI Futures

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