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The Benford paradox

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Abstract

We consider Benford´s law, also called the first-digit law. Benford (*Proc. Amer. Phil. Soc., 78,* 1938:551-572) presented the law in 1938, but 57 years earlier Newcomb (*Amer. J. Math., 4,* 1881:39-40) made the same observation. The problem was identified when they used logarithm tables in performing numerical calculations. They noted that the earlier pages of the tables were more worn than later pages. Consistent with this observation, they noted using numbers starting with low digits more often than numbers with high. Benford considered different data sets and noted that for some this rule is valid while for others it is not. When he combined all data sets, the rule was satisfied. Benford was not the first to observe this curiosity, but Benford´s results aroused more attention. Consequently, in the literature the law was named Benford´s law.

This paradox has subsequently been established by other scientists, and it has been confirmed to hold under different circumstances. Benford´s law is a statistical tool of great interest for scientists both when they perform theoretical analyses or when they try to apply the law in empirical connections. There is an extensive literature concerning the use of Benford´s law in order to check data quality.

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1 Introduction

History of Benford´s law. The historical progress of Benford´s law follows two lines. The first describes its mathematical history and the other its empirical one. Benford [1] described his law in 1938, but already 57 years earlier Newcomb [2] had made the same observation, but Benford´s results aroused more attention in the literature than Newcomb´s.

To name a law, theorem or idea in science after a specific individual is associated with great risks. Stigler [3] introduced "*Stigler's Law of Eponymy*", briefly encapsulated as "*No scientific discovery is named after its original discoverer*". Stigler was convinced that also his law follows this rule. According to Atle Selberg, André Weil once stated that if some finding was named according to a specific individual this person often had very little to do with the discovery [4]. Weil gave his statement much before Stigler. I myself have in other situations observed the same phenomenon [5]. Arnold [6] presented the historical development of Benford´s law, but he ignored Benford completely when he described the law and its origin. When I asked Arnold about this, he responded that it was a conscious choice not to mention Benford. Also Block and Savits [7] noted that Benford was mistakenly attributed the law. Furthermore, they stated that this is a example of Stigler´s *Law of Eponymy*. In this study, we will, however, use the established name.

2 Mathematical Foundation of Benford´s Law

Bohl [8], Sierpinski [9] and Weyl [10], [11] laid the initial mathematical foundation. Arnold [6] describes this development and he presented the theorem in the following way. Let $[x]$ mean the greatest integer less or equal to *x*. Then $\langle x \rangle = x - [x]$ is the fractional part of *x*. If *x* is an irrational number, then the sequence of $\langle \mathrm{n}\langle \mathrm{x} \rangle \rangle$ is uniformly distributed over the interval $(0,1)$.

Diaconis [12] stated that the leading digit behaviour of a large class of arithmetic sequences is determined by using the results from the theory of uniform distribution mod *1*. He also defined the strong Benford sequence. Diaconis verified Benford´s conjecture that the distribution of digits in all places tends to be nearly uniform. In connection with Benford´s law, he noted the problem of a suitable definition for ”picking an integer at random”. Fu et al. [13] presented a generalised Benford´s law. They proposed to model the distribution of the first digit image models generated for digital image processes such as image filtering, coding and analysis. They showed that the Discrete Cosinus Transformation (DCT) follows Benford´s law and that the JPEG coefficients follow a generalised logarithmic law of Benford type. They observed that the distribution of first digits of the JPEG coefficients does not follow Benford´s law in its rigorous form. However, they noticed that the distributions still closely follow a logarithmic law. When Benford´s law is

$$P(D = x) = p(x) = log(1 + \frac{1}{x}),$$

then the generalised logarithmic model is

$$P(D = x) = p(x) = N\,log(1 + \frac{1}{s + x^q}).$$

The coefficient *N* is a normalising factor and *s* and *q* are model parameters describing distribution of images having different compression factors *Q*. When $s = 0$ and $q = 1$, Benford´s law is the special case.

Block and Savits [7] stressed that many authors have tried to obtain mathematical explanations for Benford´s law, but the results have been relatively fruitless. Methods involving probability have been somewhat more successful. They gave some reasons for this and also provided an example of a mixture of distributions that exactly satisfies Benford´s law.

3 Empirical Description of Benford´s Law

We have already stated that Benford [1] and before him Newcomb [2] found that the leading digit in large data sets with strong variation showed an unusual distribution with dominance of low numbers *1 - 3*. This observation followed from their study of numerical calculations based on the use of logarithm tables. They found that the first pages of a table of logarithms show more wear than the last pages, indicating that numbers beginning with the digit *1* are used relatively more than those beginning with the digit *9*. Both of them stated that the leading digit of the numbers in their studies was more often low than high. Benford studied different data sets and observed that sometimes they satisfied Benford´s law and sometimes not. When he combined all data sets, Benford´s law was satisfied [14]. Benford´s results received more attention than Newcomb´s, and therefore, the law was named after Benford despite the fact that he was not the original discoverer. A common idea among the empirical scientists who studied Benford´s law was that they looked for and identified specific data sets satisfying Benford´s law.

Example 1. Introduction to the Use of Logarithms. To explain Benford´s and Newcomb´s problem, we recall the basic knowledge of tables of Briggs´ logarithms and show their composition. The use of logarithms in calculations follows the following rules:

The logarithm of a product is the sum of the logarithms of the numbers being multiplied;

the logarithm of the ratio of two numbers is the difference of the logarithms;
the logarithm of the *p*-th power of a number is *p* times the logarithm of the number itself;
the logarithm of a *p*-th root is the logarithm of the number divided by *p*.
In Table 1a, we note that the logarithm contains two parts

- an integral part called the c*haracteristic* and
- a proper fraction called the *mantissa*.

The characteristic is one less than the number of digits before the decimal point. The mantissa is the same for the same order of digits in two different numbers, irrespective of where the decimal point is in the two numbers. The characteristic can easily be determined, but for the mantissa one needs logarithm tables. Table 1b shows the compositon of a table of logarithms with four decimals.

Table 1: Short description of Briggs´ logarithms.

Table 1a. Some Briggs´ logarithms with characteristic and mantissa.

x	*log(x)*
0.0004711	0.6731-4
0.004711	0.6731-3
0.04711	0.6731-2
0.4711	0.6731-1
4.711	0.6731
47.11	1.6731
471.1	2.6731
4711	3.6731
47110	4.6731
471100	5.6731

Table 1b. Excerpt of a table of logarithms with four decimals.

x	*log(x)*
1000	0.0000
1001	0.0004
1002	0.0009
1003	0.0013
9995	0.9998
9996	0.9998
9997	0.9999
9998	0.9999
9999	1.0000

Remark. The mantissa is also used in connection with the decimal part of floating point numbers [15].

4 Theoretical Benford Distribution

In Table 2, we present the theoretical Benford distribution. The mathematical formula is

$$P(D \le d) = log(d+1),$$

where D is the leading digit and d is the theoretical argument. Obviously, only the digits from 1 to 9 are of interest. Figure 1 presents the theoretical Benford distribution (in per cent) of the leading digit.

Table 2: Theoretical distribution of the leading digit (D) according to Benford´s law.

d	$P(D=d)$	%	$P(D \le d)$	%
1	$log(2)-log(1)$	30.10	$log(2)$	30.10
2	$log(3)-log(2)$	17.61	$log(3)$	47.71
3	$log(4)-log(3)$	12.49	$log(4)$	60.21
4	$log(5)-log(4)$	9.69	$log(5)$	69.90
5	$log(6)-log(5)$	7.92	$log(6)$	77.82
6	$log(7)-log(6)$	6.69	$log(7)$	84.51
7	$log(8)-log(7)$	5.80	$log(8)$	90.31
8	$log(9)-log(8)$	5.12	$log(9)$	95.42
9	$log(10)-log(9)$	4.58	$log(10)=1$	100.00
Total	1	100.00		

Note that $log(1)=0$ and that $log(10)=1$. Furthermore, note that more than 30% of the numbers start with 1, but less than 5% start with 9. The proportion of numbers starting with 1, 2, and 3 is 60%.

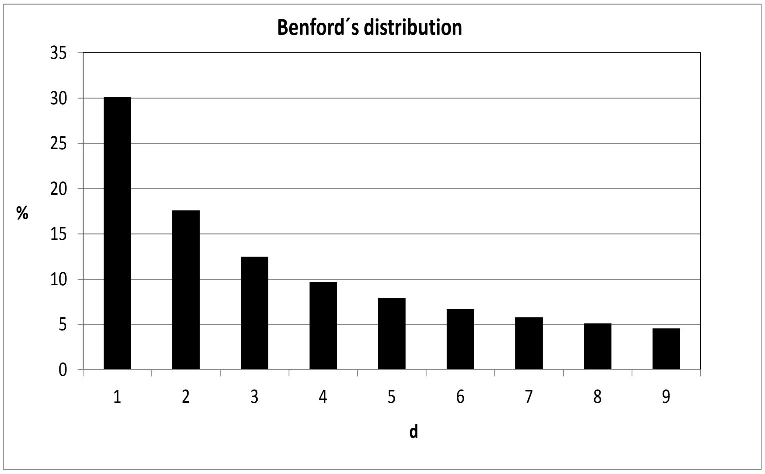

Figure 1: Theoretical Benford´s distribution (%) of the leading digit.

In the following, we introduce a small numerical data set. We assume that it is large enough and the variation in the numbers is sufficiently large that it can be applied for evaluation of the Benford´s law.

Example 2. Number of Inhabitants in Finnish Communes in 1982. In 1982, there were 461 communes in Finland and the number of citizens in the communes varied between 147 in Sottunga (Åland Islands) and 484 260 in the capital of Helsinki. The size of the data set and its variation seem to be sufficiently large for serve an example of Benford´s law. Our data set will be applied when we want to illustrate different properties of Benford´s law. In some examples, we also use the number of inhabitants in Finnish communes in 2013. In this later year, the number of inhabitants had increased, but the number of communes had decreased to 319.

Figure 2 presents the communes according to the leading digit of their numbers of inhabitants. Note that the distributions of both data sets are similar.

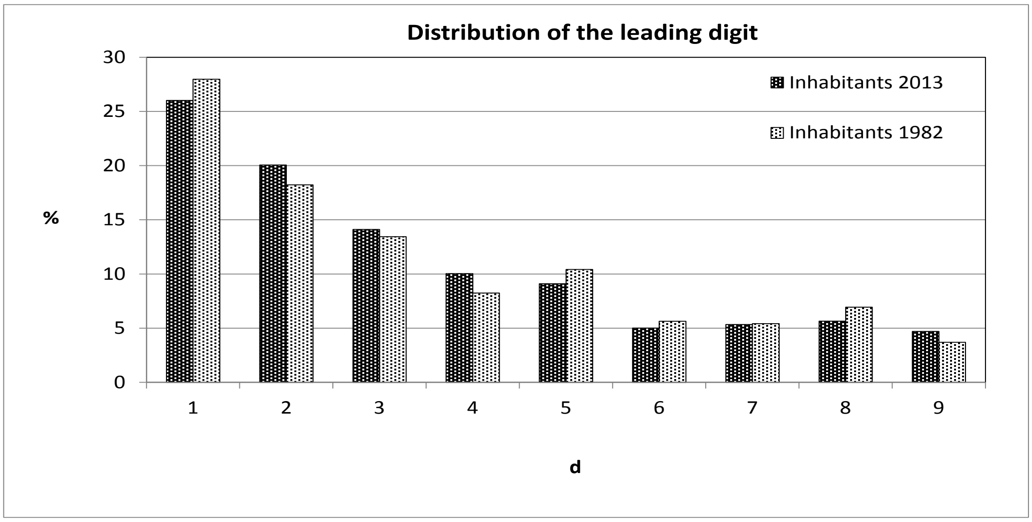

Figure 2: Number of inhabitants in Finnish communes distributed according to the leading digit. The discrepancy compared with a uniform distribution is obvious, and low digits are more common than high digits. Note that more than 25% of the communes had numbers starting with *1*, but less than 5% started with *9*.

In Figure 3, we compare our empirical distributions with Benford´s distribution. The agreement is high. When we test the discrepancy between Benford´s distribution and ours, we obtain $\chi^2 = 10.42$, with 8 degrees of freedom for the 1982 data and $\chi^2 = 5.76$ for the 2013 data. No statistically significant test results were obtained.

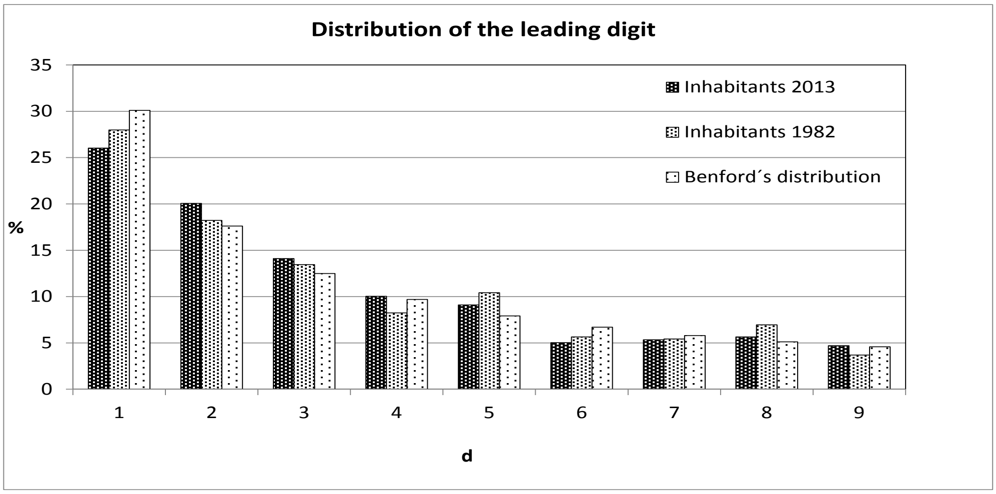

Figure 3: Comparison between the actual number of inhabitants in Finnish communes and Benford´s distribution. Good agreement was obtained (for details, see the text).

5 Distribution of the Mantissa of Briggs´ Logarithms

Consider an arbitrary number x having the leading digit *D*. We standardise the number to x_0 by division with a suitable power of 10 so that $D \le x_0 < D + 1$. This standardisation does not influence *D*. Now, $0 \le log(x_0) < 1$ and the characteristic is zero. Denote the mantissa *M*, and hence, $log(x_0) = M$.

If Benford´s rule holds, then Table 2 yields

$$P(x_0 \le d + 1) = P(D \le d) = log(d + 1).$$

Hence,

$$P(x_0 \le d + 1) = log(d + 1).$$

Using the logarithms, we obtain

$$log(d + 1) = P(x_0 \le d + 1) = P(log(x_0) \le log(d + 1)) = P(M \le log(d + 1)).$$

Denote $log(d + 1) = m$. The distribution of the mantissa is $P(M \le m) = m$. If the data set follows Benford´s law, then the mantissa is uniformly distributed over the interval $(0,1)$. This proof can be performed in the opposite direction. Hence, if the mantissa is uniformly distributed, then the initial data set follows Benford´s law. Already Newcomb [2] gave this result without a strict proof. A priori, one could expect that the leading digits are uniformly distributed, but this holds for the mantissa.

In Figure 4, we present the distributon of the mantissa of our commune data in 1982, and the agreement with a uniform distribution is excellent.

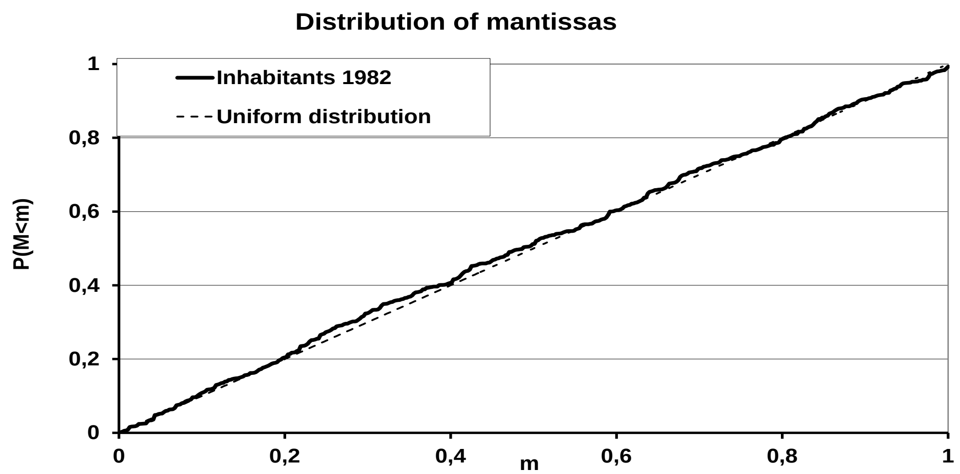

Figure 4: Distribution of the mantissas of the commune data set. The excellent agreement with a uniform distribution supports the assumption that our commune data follow Benford´s law.

6 Creating a Benford-distributed Variable

In the following way, one can generate a stochastic variable following Benford´s distribution. Assume a variable X having a given distribution $F_X(x)$. This can be a standardised Gaussian distribution $\Phi(x)$, which is easily created. Perform the transformation $Y = F_X(X)$. Y is defined over the interval $(0,1)$. Generate the data set $\{x\}$ with a distribution $F_X(x)$ and let $y = F_X(x)$. Hence, the distribution of Y is

$$F_Y(y) = P(Y \le y) = P(F_X(X) \le y) = P(F_X(X) \le F_X(x)) = P(X \le x) = F_X(x) = y.$$

Consequently, the variable Y is uniformly distributed over the interval $(0,1)$. Now we introduce the variable $Z = 10^Y$, that is $Y = log(Z)$. The variable Z has the range $(10^0, 10^1) = (1,10)$. Consequently, $Y = log(Z)$ is identical with the mantissa of Z. Hence, the mantissa is uniformly distributed over the interval $(0,1)$, indicating that Z follows Benford´s law.

7 When is Benford´s law satisfied?

Do all data sets satisfy Benford´s law? The answer is clearly no. Above, we noted that Benford stressed that different data sets showed different deviations to the law, but when the sets were combined an accurate fit was obtained. Consider the stature in centimetres of a cohort of recruits. The data set is huge, but the heterogeneity is restricted. The vast majority has the leading digit *1* and a very small minority has the leading digit *2*. Other possibilities cannot be found.

If one considers mathematical proofs and analyses presented in the literature, one observes that the data sets have to satisfy at least two basic and necessary conditions:

i. the data set must be large enough and

ii. the different numbers must show sufficiently large variation that different magnitudes can be observed.

Smith [16] stated that Benford´s law holds for distributions that are wide relatively to unit distance along the logarithmic scale. Likewise, the law is not followed by distributions that are narrow relative to unit distance. Alternative statements have also be made. Fewster [17] concluded that data from any distribution will tend to be “Benford” so long as the distribution spans several integers on the *log* scale

— several orders of magnitude on the original scale — and as long as the distribution is reasonably smooth. Aldous and Phan [18] improved the assumptions with the requirement that within a large data set of positive numerical data with a large spread on a logarithmic scale, the relative frequencies of leading digits will approximately follow Benford´s law.

8 Distribution of Subsequent Digits

Subsequent digits show simultaneously a quick convergence to a uniform distribution and an increasing complication in theoretical formulae [e.g. 19]. The formula for the second digit is

$$P(D=d)=\sum_{k=1}^{9} log\left(1+\frac{1}{10k+d}\right) \qquad d=0,1,2,...,9 .$$

Table 3: Distribution of D (%) for the consecutive digits $s=1, 2, 3,$ and 4. Note that the decreasing trend holds for all digits so that small digits are more probable that high ones. In addition, note the quick convergence to the uniform distribution [19].

	D										
s	0	1	2	3	4	5	6	7	8	9	Total
1		30.10	17.61	12.49	9.69	7.92	6.69	5.80	5.12	4.58	100.00
2	11.97	11.39	10.88	10.43	10.03	9.67	9.34	9.04	8.76	8.50	100.00
3	10.18	10.14	10.10	10.06	10.02	9.98	9.94	9.90	9.86	9.83	100.00
4	10.02	10.01	10.01	10.01	10.00	10.00	9.99	9.99	9.99	9.98	100.00

Note that zero has been included in the model. Already Newcomb [2] gave the numeric distribution of the second digit. We present in Table 3 and Figure 5 a comparison between the distributions of the first, second, third and fourth digits [19]. The digits are statistically dependent.

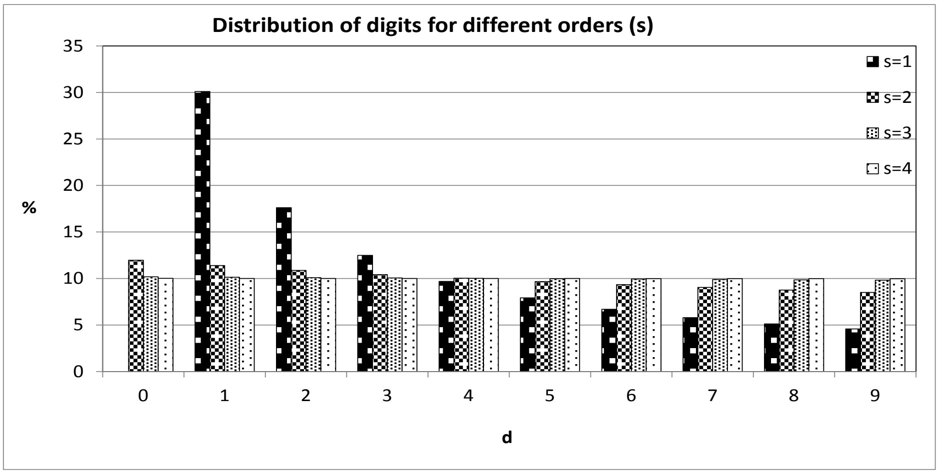

Figure 5: Graphical comparison between the distributions of consecutive digits (*s*) given in Table 3. Note for all digits an apparent decreasing trend and a quick convergence towards the uniform distribution.

In the following figures, we compare our empirical data set (1982) with the theoretical distributions for the successive digits.

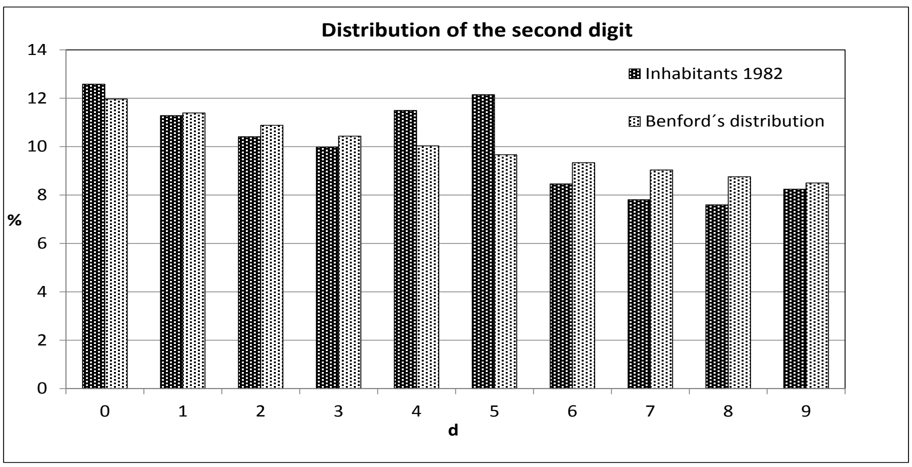

Figure 6: Comparison between the empirical distribution and the theoretical distribution for the second digit.

The agreement is good ($\chi^2 = 6.15$ with 9 degrees of freedom). Furthermore, we note that the empirical distribution is close to the uniform distribution. For the empirical distribution, the highest percentage is 12.6 and the lowest 7.6.

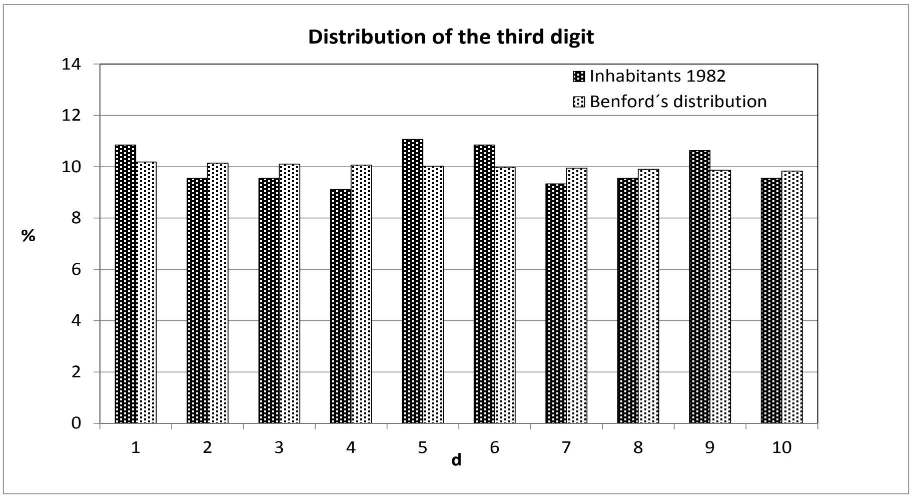

Figure 7: Comparison between the empirical distribution and the theoretical distribution for the third digit.

The agreement is good ($\chi^2 = 2.31$ with 9 degrees of freedom). Furthermore, we note that the empirical distribution is close to the uniform distribution. For the empirical distribution, the highest percentage is 11.1 and the lowest 9.3.

Furthermore, we note that the empirical distributions of the digits are close to the uniform distribution. For the empirical distribution of the second digit, the highest percentage is 12.6 and the lowest 7.6 and for the third digit, the highest percentage is 11.1 and the lowest 9.3.

9 Theoretical Proofs

In several theoretical papers, scientists have studied the conditions under which Benford´s law will hold.

Pinkham [21] started a theoretical discussion and asked why and how exactly does this unnormal law hold. Further, he stated that the only distribution that is invariant under scale change of the underlying distribution is $log(d + 1)$.

Contrary to suspicion, this is a non-trivial mathematical result, for the variable *n* is discrete. Also Hill [22] and Smith [16] showed that Benford´s law is equivalent to base and scale invariance of the underlying distribution.

Raimi [23] comprehensively surveyed the literature concerning Benford´s law. He especially paid attention to data sets not satisfying Benford´s law.

Earlier, all studies associated Benford´s law with data sets, but now scientists have started to combine Benford´s law and theoretical probability distributions. Hill [19] stated that it is an interesting and still at that time an unsolved problem to determine which common distributions and their combinations satisfy Benford´s law. He introduced a two-stage procedure, where he first selected distributions randomly and then among these random samples. This procedure resulted in Benford´s law. Using this process, he tendered the explanation that Benford´s analysis of different data set yielded fits of different goodness and that the compositon of the observations resulted in an acceptable adaptation of Benford´s law.

Leemis et al. [24] continued these investigations and their article quantified compliance with Benford´s law for several popular survival distributions. They also stated that the traditional analysis of Benford´s law considers its applicability to data sets. Block and Savitz [7] stressed that every combination of Benford´s distributions is a Benford´s distribution.

10 Applications

Nigrini and Woods [25] used Benford´s law when they studied the distribution of the population in USA according to the 1990 census. They obtained good agreement and stated that Benford´s law could be used for estimating future population statistics.

Ley [26] studied how well stock returns follow Benford´s law. Han considered *1*-day returns on the Dow-Jones Industrial Average Index and the Standard and Poor´s Index and found that they agree reasonably with Benford´s law.

Nigrini and Mittermaier [20] investigated whether the law could be used by auditors to detect fraud. The basic idea was that accounting data should follow Benford´s law. The data set is huge and the individual numbers show great variation. However, fraud may cause deviations from Benford´s law. They presented statistical tests according to which one can observe how well the data set adheres to Benford´s law.

Sandron [27] studied the populations in 198 different countries and found good agreement with Benford´s law. In addition, he investigated the distributions of surface areas and population densities and obtained similar results [cf. 20].

Sehity et al. [28] considered the pricing of goods. They assumed that the pricing based on psychology cannot follow Benford´s law. However, the Euro introduction in 2002, with its various exchange rates, distorted existing nominal price patterns, while simultaneously retaining real prices. They studied consumer prices before and after the introduction of the Euro by using Benford´s law as a benchmark for price adjustments. Results indicate the usefulness of this benchmark for detecting irregularities in prices and the clear trend towards psychological pricing after the nominal shock of the Euro introduction. In addition, the tendency towards psychological prices results in different inflation rates dependent on the price pattern.

Gonzalez-Garcia and Pastor [29] examined the usefulness of testing the conformity of macroeconomic data with Benford´s law. They noted that most macroeconomic data series tested conform to Benford´s law. However, the authors also noted that questions emerge on the reliability of such tests as indicators of data quality ratings included in the data module of ”*Reports on Observance of Standards and Codes (ROSCs)*”. The authors stated that interpreting the rejection of Benford´s law as a reliable indication of poor data quality is not supported by the analysis of the results. First, it is not possible to find a solid pattern of consistency between the results of goodness-of-fit tests and data quality ratings in data ROSCs. Second, rejections of Benford´s law may be unrelated to the quality of statistics. Rather, they found that economic variables showing marked structural shifts can result in rejection of Benford´s law regardless of the observance of best international practices.

Morrow [30] tested distributions in order to identify fraud. He started from the χ^2 test, that is

$$\chi^2 = N\sum_{i=1}^{9} \frac{(\hat{p}_i - p_i)^2}{p_i} .$$

But he discussed alternative tests. He especially followed Leemis et al. [24] and considered the alternative test

$$m = \underset{d\in\{1,2,3,\ldots,9\}}{Max} \left| \hat{p}_i - log(1+\frac{1}{d}) \right| .$$

Cho and Gaines [31] presented a similar test

$$D = \left\{ \sum_{i=1}^{9} \left| \hat{p}_i - log(1+\frac{1}{d}) \right|^2 \right\}^{\frac{1}{2}} .$$

These two tests do not have critical test values.

11 Discussion

Benford´s law is not only a curiosity, but it is a statistical tool of great interest for scientists both when they perform theoretical analyses and when they try to apply the law to different contexts. In different ways and under different assumptions, the researchers consider different distributions, and our data sets are adequate examples. One can state that irrespectively of whether the study is theoretical or empirical the basic interest is to state how general the distribution is and what the empirical data should satisfy in order to obtain Benford´s law.

Extensive literature exists concerning the use of Benford´s law for checking data quality. Studies have yielded good results, but the findings of Gonzalez-Garcia and Pastor [29] indicate that rejections of Benford´s law may be unrelated to the quality of statistics.

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