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Self-normalized laws of the iterated logarithm

Igor Zhdanov[1]

Abstract

Stronger versions of laws of the iterated logarithm for self-normalized sums of i.i.d. random variables are proved.

Keywords: Law of the iterated logarithm; stochastic compactness; self-normalization

1 Introduction

The law of the iterated logarithm is one of the fundamental laws of the classical probability theory. The reader will find various versions of the law of the iterated logarithm reviewed in [1]. In the last decade, analogues of the law of the iterated logarithm were proved for sequences of self-normalized sums of i.i.d. random variables. The results are available in articles [2] and [3] as well as sources referenced in these articles. This paper presents stronger versions

[1] Moscow State University named after M.V. Lomonosov, the Faculty of Computational Mathematics and Cybernetics. E-mail: zhdanov.i@gmail.com

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of some of the known statements regarding the law of the iterated logarithm for sequences of self-normalized sums of i.i.d. random variables.

Let i.i.d random variables $X_n, n \in \mathbb{N} = \{1, 2, \dots\}$, be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote $S_n = X_1 + \dots + X_n, V_n^2 = X_1^2 + \dots + X_n^2, n \in \mathbb{N}$. A self-normalized sum S_n/V_n is correctly defined on the set $\{V_n > 0\}$. For S_n/V_n to be defined on the entire set Ω, we put $S_n/V_n = 0$ on the set $\{V_n = 0\}$. Define a function $\lambda(n), n \in \mathbb{N}$, by putting $\lambda(n) = 1$ for $n = 1, \dots, 9$ and $\lambda(n) = [n - n/\ln\ln n]$ for $n \in \mathbb{N}, n \geq 10$. Square brackets stand for an integer part function of the number in the brackets. The nonnegative integer function $\lambda(n), n \in \mathbb{N}$, has the following properties: $\lambda(n) \leq \lambda(n+1)$ for all $n \in \mathbb{N}$ and $\lambda(n) < n$ for $n \in \mathbb{N}, n \geq 10$,

$$\lim_{n\to\infty} \frac{\lambda(n)}{n} = 1, \lim_{n\to\infty} (n - \lambda(n)) = \infty. \tag{1}$$

Theorem 1.1. *(i) If a sequence $\{S_n/V_n\}_{n\geq 1}$ is weakly compact, then*

$$\limsup_{n\to\infty} (2\log\log n)^{-1/2} \max_{\lambda(n)\leq k\leq n} \frac{|S_k|}{V_k} < \infty \ \textit{a.e.} \tag{2}$$

(ii) If random variables $X_n, n \in \mathbb{N}$, are symmetric, then

$$\limsup_{n\to\infty} (2\log\log n)^{-1/2} \max_{\lambda(n)\leq k\leq n} \frac{|S_k|}{V_k} \leq 1 \ \textit{a.e.} \tag{3}$$

As an immediate consequence, we get statements from the article [2].

Corollary 1.2. *(i) If a sequence $\{S_n/V_n\}_{n\geq 1}$ is weakly compact, then*

$$\limsup_{n\to\infty} \frac{|S_n|}{V_n\sqrt{2\log\log n}} < \infty \ \textit{a.e.}$$

(ii) If random variables $X_n, n \in \mathbb{N}$, are symmetric, then

$$\limsup_{n\to\infty} \frac{|S_n|}{V_n\sqrt{2\log\log n}} \leq 1 \ \textit{a.e.}$$

2 Supporting lemmas

Virtually all of the following lemmas are known to specialists. Let $\mathcal{F}_n = \sigma(S_k, V_k^2, k \geq n)$ denote a σ-algebra generated by random variables $S_k, V_k, k \geq n$.

Lemma 2.1. *If* $\mathbb{E}|X_1| < \infty$, *then for any* $n \in \mathbb{N}$ *and* $k = 1, \ldots, n+1$ *we have*

$$\mathbb{E}(S_k|\mathcal{F}_{n+1}) = \frac{k}{n+1} S_{n+1} \ a.e. \tag{4}$$

Proof. Equalities similar to (4) are offered as an exercise in some advanced textbooks on probability theory. It is also used in [2]. We were unable to find a source that has a proof of this equality. At the same time, the proof of our theorem is based on equality (4). For this reason we will give its proof here. It suffices to show the equality of conditional mathematical expectations $\mathbb{E}(X_k|\mathcal{F}_{n+1}) = \mathbb{E}(X_1|\mathcal{F}_{n+1})$ a.e. for any $k = 1, \ldots, n+1$. Then (4) follows from additivity of conditional mathematical expectation. Let $\mathcal{L}$ denote a class of events $A \in \mathcal{F}_{n+1}$, for which the following holds

$$\int_A X_k d\mathbb{P} = \int_A X_1 d\mathbb{P}. \tag{5}$$

Since X_1 X_k are i.i.d, equality (5) holds with $A = \Omega$. If equality (5) holds for $B, C \in \mathcal{F}_{n+1}$ and $B \subset C$, then it holds for $A = C \setminus B$ due to integral's property of linearity. If equality (5) holds for events $A_n, n \in \mathbb{N}$, and $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then it holds for $A = \cup_{n=1}^{\infty} A_n$ by the dominated convergence theorem. This implies that class $\mathcal{L}$ is a λ-class. Let $\mathcal{C}$ denote a class of events such as $A = \{S_{k_1} < c_1, , \ldots, S_{k_r} < c_r, V_{m_1}^2 < b_1, \ldots, V_{m_l}^2 < b_l\}$ for natural numbers $r, l, n+1 \le k_1 < \cdots < k_r, n+1 \le m_1 < \cdots < m_l$ and for real numbers $c_1, \ldots, c_r, b_1, \ldots, b_l$. It is easy to verify, using the integration measure change theorem, that equality (5) holds for any $A \in \mathcal{C}$. Intersection of any two events from class $\mathcal{C}$ is in $\mathcal{C}$. In other words, class $\mathcal{C}$ is a π-class. Clearly, π-class $\mathcal{C}$ generates σ-algebra $\mathcal{F}_{n+1}$. By Sierpinski's theorem ([4], p. 5), sigma-algebra $\sigma(\mathcal{C})$ generated by class $\mathcal{C}$ is in λ-class $\mathcal{L}$. Since $\mathcal{F}_{n+1} = \sigma(\mathcal{C}) \subseteq \mathcal{L} \subseteq \mathcal{F}_{n+1}$, it follows that $\mathcal{L} = \mathcal{F}_{n+1}$. Thus we proved that equality (5) holds for any $A \in \mathcal{F}_{n+1}$, which is equivalent to equality $\mathbb{E}(X_k|\mathcal{F}_{n+1}) = \mathbb{E}(X_1|\mathcal{F}_{n+1})$ a.e. This completes the proof of Lemma 1. $\square$

The following lemma demonstrates an important property of random variables

$$Y_k = \Big(\frac{n}{n+m-k} \frac{S_{n+m-k}}{V_{n+m-k}}\Big)^2, n, m \in \mathbb{N}, k = 1, \ldots, m.$$

Lemma 2.2. *Random variables* $Y_k, k = 1, \ldots, m$, *form a submartingale relative to filtration* $\mathcal{F}_{n+m-k}, k = 1, \ldots, m$.

Proof. In essence, a proof of this lemma is provided in [2]. We include it here for the reader's convenience. It is easily seen that for any $k = 1, \dots, m$ the random variable Y_k is measurable with respect to sigma-algebra $\mathcal{F}_{n+m-k}$, and $\mathbb{E}Y_k < \infty$. To complete the proof of the lemma, we have to verify that the submartingale condition $Y_k \le \mathbb{E}(Y_{k+1}|\mathcal{F}_{n+m-k})$ holds a.e. for any $1 \le k < m$. We will make an additional assumption $\mathbb{E}X_1^2 < \infty$. By equality (4), an obvious inequality $V_{n+m-k-1} \le V_{n+m-k}$ as well as Jensen's inequality for conditional mathematical expectations

$$(\mathbb{E}(S_{n+m-k-1}|\mathcal{F}_{n+m-k}))^2 \le \mathbb{E}(S^2_{n+m-k-1}|\mathcal{F}_{n+m-k}) \text{ a.e.},$$

we have

$$\begin{aligned}\mathbb{E}(Y_{k+1}|\mathcal{F}_{n+m-k}) &= \mathbb{E}\Big(\Big(\frac{n}{n+m-k-1}\frac{S_{n+m-k-1}}{V_{n+m-k-1}}\Big)^2|\mathcal{F}_{n+m-k}\Big)\\ &\ge \Big(\frac{n}{n+m-k-1}\frac{1}{V_{n+m-k}}\Big)^2\mathbb{E}(S^2_{n+m-k-1}|\mathcal{F}_{n+m-k})\\ &\ge \Big(\frac{n}{n+m-k-1}\frac{1}{V_{n+m-k}}\Big)^2(\mathbb{E}(S_{n+m-k-1}|\mathcal{F}_{n+m-k}))^2\\ &= \Big(\frac{n}{n+m-k-1}\frac{1}{V_{n+m-k}}\Big)^2\Big(\frac{n+m-k-1}{n+m-k}S_{n+m-k}\Big)^2\\ &= \Big(\frac{n}{n+m-k}\frac{S_{n+m-k}}{V_{n+m-k}}\Big)^2 = Y_k \text{ a.e.}\end{aligned}$$

We will now no longer assume that $\mathbb{E}X_1^2 < \infty$. By $\mathbb{1}_A$ denote the indicator function of an event $A \in \mathcal{F}$ and $X_k^{(r)} = X_k\mathbb{1}_{\{|X_k|\le r\}}$ for $k = 1, \dots, n$ and $r \in \mathbb{N}$. Define random variables $S_k^{(r)} = X_1^{(r)} + \dots + X_k^{(r)}$ and $Y_k^{(r)}$ similarly to S_k and Y_k. It is not difficult to see that

$$\lim_{r\to\infty} Y_k^{(r)} = Y_k \text{ a.e.}, \ \lim_{r\to\infty} \mathbb{E}|Y_k^{(r)} - Y_k| = 0. \tag{6}$$

Applying the arguments shown above, we have $Y_k^{(r)} \le \mathbb{E}(Y_{k+1}^{(r)}|\mathcal{F}_{n+m-k})$ a.e. Letting $r \to \infty$, we obtain inequality $Y_k \le \mathbb{E}(Y_{k+1}|\mathcal{F}_{n+m-k})$ a.e. It is well known that conditions (6) are sufficient to justify the limit operation on conditional mathematical expectations. This completes the proof of Lemma 2. □

3 Main Results

(i). Denote $n_r = [\exp\{r/(\log r)^2]$ for any $r \in \mathbb{N}, r \ge 2$ and $n_1 = 1$.

By Lemma 2 for $n = \lambda(n_r)$ and $m = n_{r+1} - \lambda(n_r), n_r > 10$, random variables $Y_k, k = 1, \dots, m$, form a submartingale relative to filtration $\mathcal{F}_{n+m-k}, k = 1, \dots, m$. For any $t > 0$ a convex function $\exp\{tx\}, x \geq 0$, is increasing. Therefore, a sequence $\exp\{tY_k\}, k = 1, \dots, m$, forms a submartingale relative to filtration $\mathcal{F}_{n+m-k}, k = 1, \dots, m$. Denote

$$A_r = \Big\{ \max_{\lambda(n_r) \leq k < n_{r+1}} \frac{\lambda(n_r)}{\lambda(n_r) + k} \frac{|S_k|}{V_k} > x(2\log r)^{1/2} \Big\}, r \in \mathbb{N}, n_r > 10, x > 0.$$

Event A_r can be written as

$$A_r = \Big\{ \max_{1 \leq k \leq m} \frac{n}{n+m-k} \frac{|S_{n+m-k}|}{V_{n+m-k}} > x(2\log r)^{1/2} \Big\} = \{ \max_{1 \leq k \leq m} \exp\{tY_k\} > \exp\{2x^2 t \log r\}\}$$

By the maximum inequality for submartingales ([4], p. 93) we have

$$\mathbb{P}\{A_r\} \leq \exp\{-2x^2 t \log r\} \mathbb{E} e^{tY_m} = \exp\{-2x^2 t \log r\} \mathbb{E} \exp\{t S^2_{\lambda(n_r)} / V^2_{\lambda(n_r)}\}. \tag{7}$$

In [5] inequality $\sup_{n \geq 1} \mathbb{E} \exp\{c S_n^2 / V_n^2\} \leq 2$ is proven for some constant $c > 0$. Putting $t = c$ and $x = \sqrt{2/c}$ in (7), we obtain $\mathbb{P}\{A_r\} \leq 2/r^2, n_r > 10$, and

$$\sum_{r=1}^{\infty} \mathbb{P}\{A_r\} \leq \sum_{r : n_r \leq 10} + \sum_{r : n_r > 10} \frac{2}{r^2} < \infty.$$

By the Borel-Cantelli lemma, we have

$$\limsup_{r \to \infty} (2\log r)^{-1/2} \max_{\lambda(n_r) \leq k < n_{r+1}} \frac{\lambda(n_r)}{\lambda(n_r) + k} \frac{|S_k|}{V_k} \leq x = 2\sqrt{2/c} \text{ a.e.}$$

It is not difficult to verify that $\lim_{r \to \infty} n_{r+1}/n_r = 1$ and $\lim_{r \to \infty} \log\log n_r / \log r = 1$. This along with the first statement in (1) implies that $\lim_{r \to \infty} \lambda(n_r)/n_{r+1} = 1$. From this, in turn, it follows that

$$\limsup_{r \to \infty} (2\log\log n_r)^{-1/2} \max_{\lambda(n_r) \leq k < n_{r+1}} \frac{|S_k|}{V_k} \leq 2\sqrt{2/c} \text{ a.e.} \tag{8}$$

For any $n \in \mathbb{N}$ there exists $r \in \mathbb{N}$ such that $n_r \leq n < n_{r+1}$ and, consequently, the following inequality holds

$$\max_{\lambda(n) \leq k \leq n} \frac{|S_k|}{V_k} \leq \max_{\lambda(n_r) \leq k < n_{r+1}} \frac{|S_k|}{V_k}.$$

This together with (8) proves (2).

(ii). If random variables $X_n, n \in \mathbb{N}$, are symmetric, then $\mathbb{E}\exp\{tS_n^2/V_n^2\} \leq 2/(1-2t)$ for any $t < 1/2$. This statement is proven in the article [2]. Now inequality (7) can be restated as follows

$$\mathbb{P}\{A_r\} \leq \exp\{-2x^2 t \log r\}\mathbb{E}e^{tY_m} = \exp\{-2x^2 t \log r\}\frac{2}{1-2t}, t < \frac{1}{2}, n_r > 10.$$

Put $x = 1+\varepsilon$, where ε is any positive number. There exists $t, 0 < t < 1/2$, such that $2(1+\varepsilon)^2 t > 1$. For such x and t the previous inequality implies that

$$\sum_{r=1}^{\infty}\mathbb{P}\{A_r\} \leq \sum_{r:n_r\leq 10}\mathbb{P}A_r\} + \frac{2}{1-2t}\sum_{r:n_r>10}\frac{1}{r^{2x^2t}} < \infty.$$

By the Borel-Cantelli lemma we have

$$\limsup_{r\to\infty}(2\log r)^{-1/2}\max_{\lambda(n_r)\leq k<n_{r+1}}\frac{\lambda(n_r)}{\lambda(n_r)+k}\frac{|S_k|}{V_k} \leq 1+\varepsilon \text{ a.e.}$$

Properties of numbers $n_r, r \in \mathbb{N}$, as well as the function $\lambda(n), n \in \mathbb{N}$, referred to in the proof of the first statement, imply that

$$\limsup_{r\to\infty}(2\log\log n_r)^{-1/2}\max_{\lambda(n_r)\leq k<n_{r+1}}\frac{|S_k|}{V_k} \leq 1+\varepsilon \text{ a.s.}$$

Using concluding arguments from the proof of the first statement, we can again verify that

$$\limsup_{n\to\infty}(2\log\log n)^{-1/2}\max_{\lambda(n)\leq k<n}\frac{|S_k|}{V_k} \leq 1+\varepsilon \text{ a.e.} \tag{9}$$

This inequality holds on an event Ω_ε of probability one. Inequality (9) holds on the event $\Omega' = \cap_{l=1}^{\infty}\Omega_{1/l}$ of probability one for any $\varepsilon = 1/l, l \in \mathbb{N}$. Letting $\varepsilon = 1/l \to 0$ in (9), we get (3). This completes the proof of the theorem.

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