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Efficient Point Estimation of the Sharpe Ratio

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Abstract

The Sharpe Ratio is defined as the mean excess return over the standard deviation of the excess returns for a given security market portfolio. Due in part to the dynamic nature of this measure and because of statistical issues, the sample estimation of this ratio is challenging and subject to substantial sampling error. As such, the purpose of this research was to develop and test an efficient point estimator of the Sharpe Ratio utilizing an approach that sought to explicitly reduce its associated sampling error through the minimization of the coefficient of variation (CV) and Mean Squared Error (MSE). An empirical simulation study was conducted to assess the potential gains of the novel method given stochastic variations present within time series of security price data, with results offering improvements across all specifications of sample sizes and population standard deviations. Overall, this work addressed a major limitation in the existing point estimate calculation of the Sharpe Ratio, particularly involving estimation error which is present even within large data sets.

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1 Introduction

The Sharpe Ratio is a frequently-used financial portfolio performance measure that provides an assessment of risk-adjusted performance (i.e., the mean excess return or risk premium divided by the standard deviation of the excess return), defined mathematically in its *ex ante* form as an expected value:

$$Sr = \frac{E\left(R_a - R_b\right)}{\sigma} = \frac{E\left(R_a - R_b\right)}{\sqrt{\operatorname{var}\left(R_a - R_b\right)}} \tag{1.1}$$

for portfolio $i = 1, 2, ..., n$ with R_a = asset return and R_b = risk-free rate of return or an index return.[1,2] While utilizing the same mathematical equation, the *ex-post* form of the Sharpe ratio incorporates realized returns rather than those that are expected.[1,2] Within the mean-variance Markowitz efficient frontier, by definition, the Sharpe Ratio is the slope of the capital market line.[3,4]

Even though an analyst may commonly rely upon the Sharpe Ratio to optimize portfolio choice, these values may be statistically biased due to inherent estimation errors, even within large data sets.[5,6] Christie (2007) directly commented that “A major limitation of Sharpe Ratios is that the ‘inputs’, namely expected returns and standard deviations, are measured with error,” being “an issue which the investment community practically ignores.”[5] As such, numerous authors have sought to develop generalizations of the Sharpe Ratio that correct for various statistical concerns, including autocorrelation, skew and kurtosis, and non-normality.[7-13] Skrepnek and Sahai (2011), to illustrate, developed a bootstrap resampling and Computational Intelligence approach to the point estimate and confidence interval for the Sharpe Ratio that offered improved estimation error correction relative to other measures.[7] Lo (2002) derived the statistical distribution of the Sharpe Ratio under numerous return distributions, though without explicitly considering the impact of sampling error upon statistical interpretation.[11] Furthermore, assuming multivariate normality, an approach was also developed by Jobson and Korkie (1981) though failing to achieve sufficient statistical power.[6]

Given the above, the purpose of this research was to develop and assess an efficient point estimate of the Sharpe Ratio utilizing an approach that sought to explicitly reduce its associated sampling error. Based upon information present within the given random sample, a novel method was developed that focuses upon minimizing the coefficient of variation (CV) and Mean Squared Error (MSE) to offer improved estimation for both the numerator and denominator of the Sharpe Ratio.

2 A Proposed Efficient Point Estimator of the Sharpe Ratio

The Sharpe Ratio for a given portfolio, $Sr = \frac{\mu}{\sigma}$, is typically estimated as $\widehat{Sr} = \frac{\bar{x}}{s} = Er\left(sr\right)$. In more detail, given that the population value of the Sharpe Ratio for portfolio *i* may be expressed as $Sr_i = \frac{\mu_i - z}{\sigma_i} = \frac{\mu_i - R_f}{\sigma_i}$ for $i = 1, 2, \ldots, n$, these population parameters are estimated by the sample counterparts including the sample mean and standard deviation based upon a random sample $X_1, X_2, \ldots, X_n$ as:

sample mean, $\bar{x} = \frac{\sum_{i=1}^{i=n} x_i}{n}$

sample standard deviation, $s = \sqrt{s^2}$; with $s^2 = \frac{\sum_{i=1}^{i=n} (x_i - \bar{x})^2}{(n-1)}$ (2.1)

Therefore, the typically-used point estimator for the Sharpe Ratio is $\widehat{Sr} = \frac{\bar{x}}{s} = Er\left(sr\right)$. Notably, $\frac{1}{s}$ may not necessarily be an efficient estimate of the population value $\frac{1}{\sigma}$, irrespective of large sample sizes, nor may $\bar{x}$ be an efficient estimate of μ in terms of an optimal coefficient of variation.[5,7] In the forthcoming, a proposed efficient estimator specifically focusing upon the numerator of this ratio (i.e., the population mean, μ) is offered. Following, the development of an efficient estimator of the denominator (i.e., the population standard deviation, σ) is undertaken. Finally, the proceeds are collectively incorporated into a novel, proposed point estimate of the Sharpe Ratio.

2.1 Efficient Estimation of Sharpe Ratio Numerator, μ

The proposed efficient estimator of the numerator of the Sharpe Ratio, μ, seeks to utilize information within the data more fully via incorporation of the sample coefficient of variation (CV), which is a normalized dispersion of a probability or frequency distribution that represents the extent of variability from a population mean, defined as a random variable's ratio of standard deviation to its expected value.[14] A lower CV reflects a smaller residual versus predicted value in a given model, suggesting improved goodness of fit. Quantitatively, the potential benefits of more efficient estimators also include, for example, a lower sample size requirement to achieve robust results.

In quantitative financial or econometric analyses, it may be common to encounter an analytic scenario wherein the sample estimate of the coefficient of variation of the sample mean (i.e., a more stable random variable than the original variable) will not be large. In those situations, the analyst may prefer a coefficient of variation as $CV\left(\bar{x}\right) < 1.0$, and more likely even markedly below 1.0. Hence, in such cases, the proposed alternative estimator of μ, denoted $t^{\otimes}$, versus the usual estimator which is denoted $\bar{x}$, is presented as:

$$t^{\otimes} = \bar{x} + \frac{\bar{x}}{n \cdot \left(\bar{x}^2\right) / s^2 - 1} \tag{2.2}$$

In percentage terms, the relative efficiency of this proposed estimator $t^{\otimes}$ with respect to the usual estimator $\bar{x}$ would be:

$$100/\eta = 100 \cdot \frac{E(\bar{x} - \mu)^2}{E(t^{\otimes} - \mu)^2} \tag{2.3}$$

An unbiased estimator of the efficiency ratio, η, as a function of ($\bar{x}$, s^2) alone is also required, as ($\bar{x}$, s^2) is jointly a complete sufficient statistic for (μ, σ^2). Being a function of a complete sufficient statistic, the unbiased estimator would be a Uniform Minimum Variance Unbiased Estimator (UMVUE), taken that:

$$\begin{aligned} \eta &= \frac{E\left(\bar{x} - \mu\right)^2}{E\left(t^{\otimes} - \mu\right)^2} \\ &= \left(\frac{n}{\sigma^2}\right) \cdot E\left[\left(\bar{x} - \mu\right) + \left(\frac{\bar{x}}{n \cdot \left(\bar{x}\right)^2 / s^2 - 1}\right)\right]^2 \\ &= 1 + 2A + B \end{aligned} \tag{2.4}$$

The following can be developed from (2.4), specifically concerning 'A':

$$\begin{aligned} A &= \left(\frac{n}{\sigma^2}\right) \cdot E\left[\frac{\bar{x} \cdot \left(\bar{x} - \mu\right)}{\left(n \cdot (\bar{x})^2 / s^2 - 1\right)}\right] \\ &= \left(\frac{n}{\sigma^2}\right) \cdot Es^2 \cdot E\bar{x} \cdot \left[\frac{\bar{x} \cdot \left(\bar{x} - \mu\right)}{n \cdot \left(x\right)^2 / s^2 - 1}\right] \end{aligned}$$

$$= \left(\frac{c \cdot n}{\sigma^2}\right) \cdot Es^2 \cdot \left[\int_{-\infty}^{+\infty} \frac{\bar{x}}{\left(n \cdot \left(\bar{x}\right)^2 \big/ s^2 - 1 \right)} \cdot \left(\bar{x} - \mu\right) \cdot \exp\left\{ \frac{-n \cdot \left(\bar{x} - \mu\right)^2}{2\sigma^2} \right\} d\bar{x} \right]$$

$$= \left(-c\right) \cdot Es^2 \cdot \left[\int_{-\infty}^{+\infty} \frac{\bar{x}}{\left(n \cdot \left(\bar{x}\right)^2 \big/ s^2 - 1 \right)} \cdot \left(d \big/ d\bar{x} \right) \cdot \exp\left\{ \frac{-n \cdot \left(\bar{x} - \mu\right)^2}{2\sigma^2} \right\} d\bar{x} \right]$$

as:

$$c = \left[\frac{n}{2\pi\sigma^2} \right]^{1/2} \tag{2.5}$$

Additionally, by applying integration by parts, the following proceedings are offered:

$$A = Es^2 \cdot E\bar{x}\left[a\right] = E\left[a\right]$$

with:

$$a = -\left[\left\{ n \cdot \left(\bar{x}\right)^2 \big/ s^2 + 1 \right\} \cdot \left\{ n \cdot \left(\bar{x}\right)^2 \big/ s^2 + 1 \right\}^{-2} \right]$$

$$= \left[\left(u+1\right) \cdot \left(u-1\right)^{-2} \right]$$

as:

$$u = n \cdot \left(\bar{x}\right)^2 \big/ s^2 \tag{2.6}$$

Pertaining specifically to '*B*' within (2.4), it should be recognized that independently of $\bar{x}$, $\frac{\left(n-1\right) \cdot s^2}{\sigma^2}$ is $\sim \chi^2$ distributed with $\left(n-1\right)$ degrees of freedom, yielding:

$$B = \left(\frac{n}{\sigma^2}\right) \cdot Es^2 \cdot E\bar{x} \cdot \left[\left(\bar{x}\right)^2 \cdot \left(n \cdot (\bar{x})^2 \big/ s^2 - 1 \right)^{-2} \right]$$

$$= \left(\frac{c^*}{\sigma^2}\right) \cdot E\bar{x} \cdot \left[\int_0^{+\infty} \left\{ \left(\frac{(\bar{\mathrm{x}})^2}{s^2} \right) \cdot \left(n \cdot (\bar{\mathrm{x}})^2 \Big/ s^2 - 1 \right)^{-2} \right\} \cdot \left((s^2)^{(n-1)/2} \right) \cdot \exp\left\{ \frac{-(n-1)\cdot s^2}{2\sigma^2} \right\} ds^2 \right]$$

where:

$$c^* = \left\{ \frac{(n-1)}{2\sigma^2} \right\}^{(n-1)/2} \cdot \left\{ \frac{1}{\Gamma \cdot (n-1)/2} \right\} \tag{2.7}$$

Importantly, the following is also noted:

$$d/ds^2 \cdot (s^2)^{(n-1)/2} \cdot \exp\left\{ \frac{-(n-1)\cdot s^2}{2\sigma^2} \right\}$$
$$= \left\{ \frac{(n-1)}{2\sigma^2} \right\} \cdot \sigma^2 \cdot (s^2)^{(n-3)/2} \cdot \exp\left\{ \frac{-(n-1)\cdot s^2}{2\sigma^2} \right\} - (s^2)^{(n-1)/2} \left\{ \frac{-(n-1)\cdot s^2}{2\sigma^2} \right\} \tag{2.8}$$

By applying (2.7) and (2.8):

$$B = \left(\frac{c^*}{n}\right) \cdot E\bar{x} \cdot \left[\int_0^{+\infty} \left\{ \left(\frac{n\cdot(\bar{\mathrm{x}})^2}{s^2} \right) \cdot \left(n \cdot (\bar{\mathrm{x}})^2 \Big/ s^2 - 1 \right)^{-2} \right\} \cdot \left((s^2)^{(n-3)/2} \right) \cdot \exp\left\{ \frac{-(n-1)\cdot s^2}{2\sigma^2} \right\} ds^2 \right]$$
$$- \left\{ \frac{2c^*}{n\cdot(n-1)} \right\} \cdot \bar{x} \left[\int_0^{+\infty} \left\{ \left(\frac{n\cdot(\bar{\mathrm{x}})^2}{s^2} \right) \cdot \left(n \cdot (\bar{\mathrm{x}})^2 \Big/ s^2 - 1 \right)^{-2} \cdot \left(\frac{d}{ds^2} \right) \cdot \left((s^2)^{(n-1)/2} \right) \cdot \exp\left\{ \frac{-(n-1)\cdot s^2}{2\sigma^2} \right\} ds^2 \right\} \right] \tag{2.9}$$

Through the application of integration by parts:

$$B = Es^2 \cdot E\bar{x}\left[\left(n\cdot(\bar{x})^2 \big/ s^2 \right)\cdot\left(n\cdot(\bar{x})^2 \big/ s^2 - 1 \right)^{-2}\right]$$

$$-\left\{\frac{2c^*}{n\cdot(n-1)}\right\}\cdot E\bar{x}\left[\int_0^{+\infty}\left\{\begin{array}{l}\left(\frac{n\cdot(\bar{x})^2}{s^2}\right)\cdot\left(n\cdot(\bar{x})^2 \big/ s^2 \; -1\right)^{-2}\cdot\left(\frac{d}{ds^2}\right)\\ \cdot\left(\left(s^2\right)^{(n-1)/2}\right)\cdot\exp\left\{\frac{-(n-1)\cdot s^2}{2\sigma^2}\right\}ds^2\end{array}\right\}\right]$$

$$= E\left[\left(n\cdot(\bar{x})^2 \big/ s^2 \right)\cdot\left(n\cdot(\bar{x})^2 \big/ s^2 - 1 \right)^{-2}\right]$$

$$-\left\{\frac{2}{(n-1)}\right\}\cdot E\left[\left(\frac{n\cdot(\bar{x})^2}{s^2}\right)\cdot\left(n\cdot(\bar{x})^2 \big/ s^2 \; +1\right)\cdot\left(n\cdot(\bar{x})^2 \big/ s^2 \; -1\right)^{-3}\right]$$

$$= E\left[\begin{array}{l}\left(n\cdot(\bar{x})^2 \big/ s^2 \right)\cdot\left(n\cdot(\bar{x})^2 \big/ s^2 - 1 \right)^{-2}\\ -\left\{\frac{2}{(n-1)}\right\}\\ \cdot\left[\left(\frac{n\cdot(\bar{x})^2}{s^2}\right)\cdot\left(n\cdot(\bar{x})^2 \big/ s^2 \; +1\right)\cdot\left(n\cdot(\bar{x})^2 \big/ s^2 \; -1\right)^{-3}\right]\end{array}\right] \qquad (2.10)$$

Alternatively, (2.10) may be expressed as $B = E(b)$, where:

$$b = u\cdot(u-1)^{-2} - \left\{\frac{2}{(n-1)}\right\}\cdot u\cdot(u+1)\cdot(u-1)^{-3}$$

$$= u\cdot(u-1)^{-3}\cdot\left\{(n-3) - \frac{u\cdot(n+1)}{(n-1)}\right\}$$

by setting the following from (2.6):

$$u = n\cdot\left(\bar{x}\right)^2 \big/ s^2 \qquad (2.11)$$

Overall, the UMVUE of the relative efficiency of $t^{\otimes}$ with respect to the usual estimator of $\bar{x}$ may be more appropriately expressed as $1+2a+b$ rather than $1+2A+B$. Per (2.6) and (2.8), this is:

$$1+2a+b=1-2\cdot\left[(u+1)\cdot(u-1)^{-2}\right]+\frac{u\cdot\left[(u-1)^{-3}\right]\cdot\left[(n-3)-u\cdot(n+1)\right]}{(n-1)}$$
$$=1-\left[\frac{(u-1)^{-3}\cdot\left\{u^2\cdot(n-3)+u\cdot(n-3)-2\cdot(n-1)\right\}}{(n-1)}\right] \tag{2.12}$$

with the following:

$$100\big/\eta=100\cdot\frac{E(\bar{x}-\mu)^2}{E(t^{\otimes}-\mu)^2}>100\% \tag{2.13}$$

if $0\prec\hat{\eta}\prec 1$, $0<1+2a+b<1$, or $0<\left\{u^2\cdot(n-3)+u\cdot(n-3)-2\cdot(n-1)\right\}$ per (2.12), as $u>1$ for all $\left(\bar{x}\right)^2\succ\frac{s^2}{n}$ and given that the coefficient of variation of $\bar{x}<1$ per (2.11), or if $u>\frac{(n+1)}{(n-3)}$.

Importantly, the aforementioned is consistent for all observed coefficients of variation for $\bar{x}$ in practice and, as such, the proposed alternative estimator $t^{\otimes}$ defined in (2.2) is a more efficient estimator of the normal mean μ rather than the usual estimator $\bar{x}$. Additionally, the proposed estimator $t^{\otimes}$ may also be expressed as a function of the square of the sample coefficient of variation, denoted v, as:

$$t^{\otimes}\cong\bar{x}^{*}(1+v)$$

with

$$v=\frac{s^2}{n^{*}\cdot\left(\bar{x}\right)^2} \tag{2.14}$$

2.2 Efficient Estimation of Sharpe Ratio Denominator, σ

In establishing the proposed approach for an improved Sharpe Ratio point estimation and focusing on the denominator, σ, an efficient estimator of the inverse of the normal standard deviation, $\frac{1}{\sigma}$, builds toward a proof of the following lemma.

Lemma. For a random sample $\left(X_1, X_2, X_3, ..., X_n\right)$ from a normal population $N\left(\mu, \sigma^2\right)$, $K^* \cdot \left(\frac{1}{s}\right)$ is the Minimum Mean Squared Error (MMSE) of the inverse of the normal population standard deviation, $\left(\frac{1}{\sigma}\right)$, wherein:

$$K^* = \frac{\sqrt{\frac{2}{(n-1)}} \cdot \Gamma\left(\frac{(n-2)}{2}\right)}{\Gamma\left(\frac{(n-3)}{2}\right)}. \tag{2.15}$$

Proof. K^* is the MMSE of $\left(\frac{1}{\sigma}\right)$ given that:

$$K^* = \frac{E\left(\frac{1}{s}\right) \cdot E\left(\frac{1}{\sigma}\right)}{E\left(\frac{1}{s^2}\right)}$$

with:

$$\left\{(n-1) \cdot \frac{s^2}{\sigma^2}\right\} \sim \chi^2_{n-1}$$

□

2.3 The Proposed Efficient Point Estimate of the Sharpe Ratio, $Sr = \frac{\mu}{\sigma}$

As often appears within the literature, the usual applied point estimate for the Sharpe Ratio is $Er\left(sr\right) = \frac{\bar{x}}{s}$, despite acknowledging that $\frac{1}{s}$ does not provide a good statistical estimate of $\frac{1}{\sigma}$ nor does $\bar{x}$ estimated from σ yield an optimal coefficient of variation.[5,7]

Building upon (2.14), the class of estimators $K \cdot sr$ is considered, wherein the MMSE estimator is developed as a more efficient point estimate of the Sharpe Ratio as:

$$\begin{aligned} Er^*\left(sr\right) &= K^* \cdot (1 + \text{v}) \cdot sr \\ &= K^* \cdot \frac{t^{\otimes}}{s} \end{aligned} \tag{2.16}$$

Given the aforementioned, K^* would be defined by minimizing the Mean Squared Error (MSE) of the estimator in the class of estimators $K \cdot sr$. Notably, the assessment of MSE within quantitative financial analyses is of importance primarily because the MSE is a predominant statistical approach used to assess the difference between values observed by an estimator versus the true value of the quantity being estimated.[14,15] The MSE is expressed mathematically as:

$$MSE\left(\hat{\theta}\right) = E\left[\left(\hat{\theta} - \theta\right)^2\right] \tag{2.17}$$

Marked differences reflected in the MSE may occur because an estimator poorly captures relevant information from the sample, therein producing an inaccurate estimate of the true value.

3 Empirical Simulation Study

3.1 Methodology

To assess the efficiency of the proposed point estimator of the Sharpe Ratio from (2.16), an empirical simulation study was developed utilizing Matlab 2010b [The Mathworks Inc., Natick, Massachusetts]. Comparisons were drawn for the proposed point estimator, denoted $Er^*(sr)$, versus the currently-utilized estimator for the Sharpe Ratio, denoted $Er(sr)$, across illustrative sample sizes of n = 6, 11, 21, 31, 41, 57, 71, 101, 202, and 303. The parent population was defined as normal with a population Sharpe Ratio, Sr = 0.5, and with varying population standard deviations of σ = 0.20, 0.25, 0.30, 0.35, 0.40, 0.45, and 0.50. The number of replications was 51,000.

The actual Mean Squared Error (MSE) of the usual estimator, $Er(sr)$, and the proposed estimator, $Er^*(sr)$, were calculated by averaging the squared deviation of the estimator's value from the population Sharpe Ratio (i.e., 0.5). As such, the Relative Efficiency, *RelEff*, of the proposed estimator compared to the usual estimator was calculated accordingly:

$$\text{RelEff}_{\%}\left\{Er^*\left(sr\right) versus\ Er\left(sr\right)\right\} = 100 \cdot \frac{MSE\left\{Er\left(sr\right)\right\}}{MSE\left\{Er^*\left(sr\right)\right\}} \tag{3.1}$$

3.2 Results

Presented in Table 1, the relative efficiency of the proposed point estimator, $Er^*(sr)$, ranged from a minimum of 137.022372 percent at the lower sample size and standard deviation (σ = 0.20, n = 11) to a maximum of 208.784860 percent with increasing standard deviations and sample sizes (σ = 0.35, n = 303). Variation in the Relative Efficiency of the proposed estimator was, however, predominantly observed to be a function of sample size rather than population standard deviation, with absolute percentage increases being approximately +35.17 percent from n = 11 to 21, +12.64 percent from n = 21 to 31, +6.50 percent from n = 31 to 41, +3.95 percent from n = 41 to 51, +4.57 percent from n = 51 to 71, +3.47 percent from n = 71 to 101, +4.10 percent from n = 101 to 202, and +1.37 percent from n = 202 to 303. Across every specification of sample size and population standard deviation, $Er^{*}(sr)$ yielded improvements in efficiency regarding MSEs vis-à-vis the approach commonly utilized in practice.

Table 1: Relative Efficiency of the Proposed Efficient Point Estimator of the Sharpe Ratio, $Er^{*}(sr)$, relative to the Usual Estimator, $Er(sr)$, in Percentage Terms

	Population Standard Deviation (σ)						
Sample Size (n)	**σ = 0.20**	**σ = 0.25**	**σ = 0.30**	**σ = 0.35**	**σ = 0.40**	**σ = 0.45**	**σ = 0.50**
n = 11	137.022372	137.027433	137.024652	137.021277	137.023239	137.029536	137.028980
n = 21	172.190248	172.187976	172.178787	172.182462	172.184202	172.190828	172.191347
n = 31	184.825329	184.830039	184.824943	184.825665	184.824585	184.823921	184.827177
n = 41	191.324570	191.326181	191.321978	191.325200	191.312896	191.321372	191.314357
n = 51	195.278201	195.274523	195.277076	195.280943	195.276298	195.270665	195.275535
n = 71	199.849984	199.847035	199.848930	199.847878	199.840840	199.846984	199.841017
n = 101	203.315631	203.307409	203.314405	203.314251	203.318790	203.304714	203.315814
n = 202	207.415527	207.415083	207.417526	207.414873	207.419385	207.412990	207.410603
n = 303	208.784029	208.781693	208.784297	208.784860	208.782720	208.782589	208.780899

4 Conclusion

By focusing upon the sample coefficient of variation (CV) and Minimum Mean Squared Error (MMSE), the proposed point estimator of the Sharpe Ratio yielded improvements across all specifications of sample sizes and population standard deviations to the existing method, with minimum relative efficiency increases beginning with 137 percent at lower sample sizes and ultimately exceeding 200 percent at sample sizes above 71. Overall, this work addressed a major limitation in the existing point estimate calculation of the Sharpe Ratio, particularly involving estimation error which is present even within large data sets.

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