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Ruin Probability in a generalized risk process out interest force with homogenous Markov chain premiums

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Abstract

In this paper, we consider generalized risk processes out interest force with assumption that sequence of premiums is homogenous markov chain, takes a finite number of possible integer values and claims are independent and identically distributed non – negative random variables with the same distributive function. The state space of premiums in this paper is finite, which it satisfies with cases of practice. The aim of this paper is to give recursive equations for finite time ruin probabilities and integral equation for ultimate ruin probability of generalized risk processes out interest force with homogenous markov chain premiums and it establish Generalized Lundberg inequalities for ruin probabilities. Generalized Lundberg inequalities for ruin probabilities are derived by using recursive technique. Theorem 2.1 give recursive equations for finite time ruin probabilities and integral equation for ultimate ruin probability. To establish probability inequalities for finite time ruin probabilities and ultimate ruin probability of this model, we built Lemma 3.1 to define a adjustment coefficient $R_o > 0$, this coefficient is belong to initial value of premiums. Using by Theorem 2.1 and Lemma 3.1, we establish Theorem 3.1, which it give probability inequality for ultimate ruin probability by an inductive approach. Exponential upper bounds for the finite time ruin probabilities and ultimate ruin probability were obtained in Theorem 3.1.

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1 Introduction

Ruin probability is a main area in risk theory (see [1]). Ruin probabilies in discrete time models have been considered in many papers. In classical risk model, no investment incomes were considered there. Recently, the models with stochastic interest rates have received increasingly a large amount of attention. Kalashnikov and Norberg (2002) assumed that the surplus of an insurance company was invested in a risk asset and obtained the upper bound and lower bound for ruin probability. Paulsen (1998) considered a diffusion risk models with stochastic investment incomes. Yang and Zhang (2003) extended the model in Browers et. al (1997) by using an autoregression process to model both the premiums and the claims, and they also included investment incomes in their model. Both exponential and non exponential upper bounds for the ruin probability were obtained. The usefulness of the upper bounds obtained in that article and the relationship between the parameters of the model and the ruin probability were illustrated by some numerical examples. Cai (2002a, 2002b) and Cai and Dickson (2004) considered the problems of ruin probabilies in discrete time models with random interest rates. In Cai (2002a) and Cai (2002b), the author assumed that the interest rates formed a sequence of independent and identically distributed random variables and an autoregressive time series models respectively. In Cai and Dickson (2004), interest rates followed a Markov chain.

In this paper, we study generalized risk processes out interest force. The surplus process $\{U_n\}_{n\geq 1}$ with initial u can be written as

$$U_n = u + \sum_{k=1}^{n} X_k - \sum_{k=1}^{n} Y_k \tag{1}$$

We assume that:

Assumption 1.1. $U_o = u > 0$,

Assumption 1.2. $X = \{X_n\}_{n\geq 0}$ is a homogeneous Markov chain, any n, X_n takes a finite set of possible integer numbers: $E = \{1, 2, ..., M\}$ with $X_o = i \in E$ and

$$p_{ij} = P\left[X_{n+1} = j \middle| X_n = i\right], (n \in N); i, j \in E \text{ where } \begin{cases} 0 \leq p_{ij} \leq 1 \\ \sum_{j\in E} p_{ij} = 1 \end{cases}.$$

Assumption 1.3. $Y = \{Y_n\}_{n\geq 0}$ is sequence of independent and identically distributed non – negative random variables with the same distributive function $F(y) = P(Y_0 \leq y)$.

Assumption 1.4. X and Y are assumed to be independent.

We define the finite time and ultimate ruin probabilities in model (1) with Assumption 1.1 to Assumption 1.4, respectively, by

$$\psi_n(u,i) = P\left(\bigcup_{k=1}^{n}(U_k < 0)\middle| U_o = u, X_o = i\right), \quad (2)$$

$$\psi(u,i) = \lim_{n\to\infty}\psi_n(u,i) = P\left(\bigcup_{k=1}^{\infty}(U_k < 0)\middle| U_o = u, X_o = i\right) \quad (3)$$

In this paper, we derive probability inequalities for $\psi(u,i)$. In section 2, we first give recursive equation for $\psi_n(u,i)$ and integral equation for $\psi(u,i)$. We the derive probability inequalities for $\psi_n(u,i)$ and $\psi(u,i)$ in section 3 by an inductive approach. Finally, we conclude our paper in section 4.

2 Integral equation for ruin probabilities

Throughout this paper, denote the tail of any distribution function B by $\overline{B}(x) = 1 - B(x)$. We first give recursive equations for $\psi_n(u,i)$ and an integral equation for $\psi(u,i)$.

Theorem 2.1. Let model (1) satisfies Asumption 1.1 to Asumption 1.4 then, for $n = 1,2,...$

$$\psi_{n+1}(u,i) = \sum_{j\in E} p_{ij}\left\{\int_0^{u+j}\psi_n(u+j-y, j)dF(y) + \overline{F}(u+j)\right\}, \quad (4)$$

and

$$\psi(u,i) = \sum_{j\in E} p_{ij}\left\{\int_0^{u+j}\psi(u+j-y, j)dF(y) + \overline{F}(u+j)\right\}. \quad (5)$$

Proof.

Let $X_1 = j \in E; Y_1 = y \in R$, from (1), we have

$$U_1 = u + X_1 - Y_1 = u + j - y$$

Thus, if $u + j < y$ then

$$P\left(U_1 < 0 \middle| U_o = u, X_1 = j, X_o = i, Y_1 = y\right) = 1$$

$$\Rightarrow P\left(\bigcup_{k=1}^{n+1}(U_k < 0)\middle| U_o = u, X_1 = j, X_o = i, Y_1 = y\right) = 1, \quad (6)$$

while, if $0 \le y \le u + j$ then

$$P\left(U_1 < 0 \middle| U_o = u, X_1 = j, X_o = i, Y_1 = y\right) = 0. \quad (7)$$

Let $\{\tilde{X}_n\}_{n\ge0}, \{\tilde{Y}_n\}_{n\ge0}$ be independent copies of $\{X_n\}_{n\ge0}$ và $\{Y_n\}_{n\ge0}$ respectively with $\tilde{X}_o = X_1 = j, \tilde{Y}_o = Y_1 = y$

Thus, (2) and (7) imply that for $0 \le y \le u + j$

$$P\left(\bigcup_{k=1}^{n+1}(U_k < 0)\middle| U_o = u, X_1 = j, X_o = i, Y_1 = y\right)$$

$$= P\left(\bigcup_{k=2}^{n+1}(U_k < 0)\middle| U_o = u, X_1 = j, X_o = i, Y_1 = y\right)$$

$$= P\left(\bigcup_{k=2}^{n+1}(u + X_1 - Y_1 + \sum_{j=2}^{k} X_j - \sum_{j=2}^{k} Y_j < 0)\middle| U_o = u, X_1 = j, X_o = i, Y_1 = y\right)$$

$$= P\left(\bigcup_{k=2}^{n+1}(u + j - y + \sum_{j=2}^{k} X_j - \sum_{j=2}^{k} Y_j < 0)\middle| U_o = u, X_1 = j, X_o = i, Y_1 = y\right)$$

$$= P\left(\bigcup_{k=1}^{n}(u + j - y + \sum_{j=1}^{k} \tilde{X}_j - \sum_{j=1}^{k} \tilde{Y}_j < 0)\middle| \tilde{U}_o = u + j - y, \tilde{X}_o = j\right)$$

$$= \psi_n(u + j - y, j) \quad (8)$$

That, from (1), we have $\psi_{n+1}(u,i) = P\left\{\bigcup_{k=1}^{n+1}(U_k < 0)\middle| U_o = u, X_o = i\right\}$

Let $A = \{U_o = u, X_1 = j, X_o = i, Y_1 = y\}$. Thus, we have

$$\psi_{n+1}(u,i) = \sum_{j\in E} p_{ij} \int_0^{+\infty} P\left\{\bigcup_{k=1}^{n+1}(U_k < 0)\middle| A\right\} dF(y)$$

$$= \sum_{j\in E} p_{ij} \left\{\int_0^{u+j} P\left\{\bigcup_{k=1}^{n+1}(U_k < 0)\middle| A\right\} dF(y) + \int_{u+j}^{+\infty} P\left\{\bigcup_{k=1}^{n+1}(U_k < 0)\middle| A\right\} dF(y)\right\} \quad (9)$$

Thus, from (6), (8) and (9), we have

$$\psi_{n+1}(u,i) = \sum_{j\in E} p_{ij} \left\{\int_0^{u+j} \psi_n(u + j - y, j) dF(y) + \int_{u+j}^{+\infty} dF(y)\right\} \quad (10)$$

Thus, from the dominated convergence theorem, the integaral equation for $\psi(u,i)$ in Theorem 2.1 follows immediately by letting $n \to \infty$ in (10).

This completes the proof. □ □

Next, we establish probability inequalities for ruin probabilities of model (1).

3 Probability inequalities for ruin probabilities

To establish probability inequalities for ruin probabilities of model (1), we first proof the following Lemma.

Lemma 3.1. Let model (1) satisfies Asumption 1.1 to Asumption 1.4.
Any $i \in E$, if

$$E(Y_1) < E\left(X_1 \middle| X_o = i\right) \text{ and } P\left((Y_1 - X_1) > 0 \middle| X_o = i\right) > 0 \quad (11)$$

Then, there exists a unique positive constant R_i satisfying:

$$E\left(e^{R_i(Y_1 - X_1)} \middle| X_o = i\right) = 1 \quad (12)$$

Proof.
Define

$$f(r_i) = E\left\{e^{r_i(Y_1 - X_1)} \middle| X_o = i\right\} - 1; r_i \in (0, +\infty).$$

Then

$$f'(r_i) = E\left\{(Y_1 - X_1)e^{r_i(Y_1 - X_1)} \middle| X_o = i\right\}$$

$$f''(r_i) = E\left\{\left(Y_1 - X_1\right)^2 e^{r_i(Y_1 - X_1)} \middle| X_o = i\right\} \geq 0.$$

Which implies that

$$f(r_i) \text{ is a convex function with } f(0) = 0 \quad (13)$$

and

$$f'(0) = E\left\{(Y_1 - X_1) \middle| X_o = i\right\} = EY_1 - E(X_1 | X_o = i) < 0. \quad (14)$$

By $P\left((Y_1 - X_1) > 0 \middle| X_o = i\right) > 0$, we can find some constant $\delta > 0$ such that

$$P\left((Y_1 - X_1) > \delta > 0 \middle| X_o = i\right) > 0$$

Then, we can get that

$$f(r_i) = E\left\{e^{r_i(Y_1 - X_1)} \middle| X_o = i\right\} - 1 \geq E\left\{e^{r_i(Y_1 - X_1)} \middle| X_o = i\right\}.1_{\left\{(Y_1 - X_1) > \delta \middle| X_o = i\right\}} - 1$$

$$\geq e^{r_i \delta}.P\left((Y - X_1) > \delta \middle| X_o = i\right) - 1.$$

Imply

$$\lim_{r_i \to +\infty} f(r_i) = +\infty. \quad (15)$$

From (13), (14) and (15) suy ra there exists a unique positive constant R_i satisfying (12).
This completes the proof. □ □

Let: $R_o = \min\left\{R_i > 0 : E\left(e^{R_i(Y_1 - X_1)} \middle| X_o = i\right) = 1, (i \in E)\right\}$

Use Lemma 3.1 and Theorem 2.1, we now obtain a probability inequality for $\psi(u, i)$ by an inductive approach.

Theorem 3.1. Let model (1) satisfies Asumption 1.1 to Asumption 1.4 and (11). For any $u>0$ and $i\in E$, we have

$$\psi(u,i)\le \beta_1.e^{-R_o u} \tag{16}$$

where

$$\beta_1^{-1}=\inf_{t\ge 0}\frac{\int_t^{+\infty} e^{R_o y}dF(y)}{e^{R_o t}.\overline{F}(t)} \tag{17}$$

Proof.
Firstly, we have

$$\beta_1^{-1}=\inf_{t\ge 0}\frac{\int_t^{+\infty} e^{R_o y}dF(y)}{e^{R_o t}.\overline{F}(t)}\ge \inf_{t\ge 0}\frac{\int_t^{+\infty} e^{R_o t}dF(y)}{e^{R_o t}.\overline{F}(t)}=\inf_{t\ge 0}\frac{\int_t^{+\infty} dF(y)}{\overline{F}(t)}=1\Rightarrow \frac{1}{\beta_1}\ge 1\Rightarrow \beta_1\le 1.$$

For any $t\ge 0$, we have

$$\overline{F}(t)=\left[\frac{\int_t^{+\infty} e^{R_o y}dF(y)}{e^{R_o t}.\overline{F}(t)}\right]^{-1}.e^{-R_o t}.\int_t^{+\infty} e^{R_o y}dF(y)$$

$$\le \beta_1.e^{-R_o t}.\int_t^{+\infty} e^{R_o y}dF(y) \tag{18}$$

$$\le \beta_1.e^{-R_o t}.\int_0^{+\infty} e^{R_o y}dF(y)=\beta_1.e^{-R_o t}.E\left[e^{R_o Y_1}\right]. \tag{19}$$

Then, for $u>0$ and $i\in E$,

$$\psi_1(u,i)=P(U_1>0\,|\,U_o=u,X_o=i)=\sum_{j\in E}p_{ij}\overline{F}(u+j) \tag{20}$$

Thus, from (19) and (20), we have

$$\psi_1(u,i)=\sum_{j\in E}p_{ij}\overline{F}(u+j)$$

$$\le \beta_1\sum_{j\in E}p_{ij}e^{-R_o(u+j)}.\int_{u+j}^{+\infty} e^{R_o y}dF(y)$$

$$\le \beta_1\sum_{j\in E}p_{ij}e^{-R_o(u+j)}.\int_0^{+\infty} e^{R_o y}dF(y)$$

$$\le \beta_1.E\left[e^{R_o Y_1}\right]\sum_{j\in E}p_{ij}e^{-R_o(u+j)}=\beta_1.E\left[e^{R_o(Y_1-X_1)}\,\middle|\,X_o=i\right].e^{-R_o u}=\beta_1.e^{-R_o u}. \tag{21}$$

Under an inductive hypothesis, we assume for any $u>0$ and $i\in E$,

$$\psi_n(u,i) \le \beta_1.e^{-R_o u} \qquad (22)$$

From (21) implies (22) holds with n = 1.
For $j \in E$, $u + j - y > 0$, we have

$$\psi_n(u+j-y, j) \le \beta_1^*.e^{-R_o^*(u+j-y)}$$

Where

$$\beta_1^{*-1} = \inf_{t\ge 0} \frac{\int_t^{+\infty} e^{R_o^* y} dF(y)}{e^{R_o^* t}.\overline{F}(t)}, E\left(e^{R_o^*(Y_1 - X_1)} \middle| X_o = j\right) = 1 \text{ and } R_o^* \ge R_o > 0.$$

Any $t \ge 0$: $\dfrac{\int_t^{+\infty} e^{R_o y} dF(y)}{e^{R_o t}.\overline{F}(t)} = \dfrac{\int_t^{+\infty} e^{R_o(y-t)} dF(y)}{\overline{F}(t)} \le \dfrac{\int_t^{+\infty} e^{R_o^*(y-t)} dF(y)}{\overline{F}(t)} = \dfrac{\int_t^{+\infty} e^{R_o^* y} dF(y)}{e^{R_o^* t}.\overline{F}(t)}$ then

$$\beta_1^{-1} = \inf_{t\ge 0} \frac{\int_t^{+\infty} e^{R_o y} dF(y)}{e^{R_o t}.\overline{F}(t)} \le \beta_1^{*-1} = \inf_{t\ge 0} \frac{\int_t^{+\infty} e^{R_o^* y} dF(y)}{e^{R_o^* t}.\overline{F}(t)} \Leftrightarrow \frac{1}{\beta_1} \le \frac{1}{\beta_1^*} \Leftrightarrow \beta_1^* \le \beta_1$$

That

$$R_o^*\left(u+j-y\right) \ge R_o\left(u+j-y\right) > 0,$$

then

$$\psi_n(u+j-y, j) \le \beta_1.e^{-R_o(u+j-y)} \qquad (23)$$

Therefore, by (4), Lemma 3.1, (18) and (23), we get

$$\psi_{n+1}(u,i) = \sum_{j\in E} p_{ij} \left\{ \int_0^{u+j} \psi_n(u+j-y, j) dF(y) + \overline{F}(u+j) \right\}$$

$$\le \sum_{j\in E} p_{ij} \left\{ \beta_1. \int_0^{u+j} e^{-R_o(u+j-y)} dF(y) + \overline{F}(u+j) \right\}$$

$$\le \sum_{j\in E} p_{ij} \left\{ \beta_1. \int_0^{u+j} e^{-R_o(u+j)}.e^{R_o y} dF(y) + \beta_1.e^{-R_o(u+j)}. \int_{u+j}^{+\infty} e^{R_o y} dF(y) \right\}$$

$$= \beta_1.\sum_{j\in E} p_{ij} e^{-R_o(u+j)}. \int_0^{+\infty} e^{R_o y} dF(y) = \beta_1.e^{-R_o u}.E\left[e^{R_o(Y_1 - X_1)} \middle| X_o = i\right] = \beta_1.e^{-R_o u}.$$

Hence, for any $n = 1, 2, \ldots$ (22) holds. Therefore, (16) follows by letting $n \to \infty$ in (22). □ □

4 Concluding

Our main results in this paper, Theorem 2.1 give recursive equations for $\psi_n(u,i)$ and an integral equation for $\psi(u,i)$, Theorem 3.1 give probability inequalities for $\psi_n(u,i)$ and $\psi(u,i)$ by an inductive approach.

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