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# On Two-stage LAO Testing of Multiple Hypotheses for a Pair of Families of Probability Distributions

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**Abstract**

A multiple hypotheses two-stage testing for an object characterized by two separated families of hypothetical probability distributions is considered. We introduce two versions of procedure of multiple hypotheses testing in a pair of stages such that in the first stage we determine one family of distributions and then at the second stage of test we indicate the object's distribution within the selected family. The matrix of reliabilities (error probability exponents) of logarithmically asymptotically optimal (LAO) hypothesis testing by a pair of stages is investigated and compared with the analogous matrix of one-stage procedure of testing designed in earlier works. Advantages of the two-stage LAO testing are revealed.

**Mathematics Subject Classification:** 62F03, 94A15

**Keywords:** Logarithmically asymptotically optimal test, multiple hypotheses testing, multistage tests, error probability exponent

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# 1 Introduction

In this paper, the process of hypotheses logarithmically asymptotically optimal (LAO) testing for a model with two families of distributions is studied. The hypotheses are probabilities distributions hypothetical (PDs) which characterize the studied object. The list of $S$ hypothetical PDs is given. The statistical problem is to detect actual PD from this list using a sample of $N$ experiments' outcomes. Notice that before testing all the hypotheses are equally possible, and no additional data is known about their a priori PDs. The overwhelming majority of published works are dedicated to the case of two hypotheses [13]. The LAO testing of two hypotheses is the procedure ensuring the best exponential decrease with growing $N$ of the error probability of one hypothesis subject to the given exponent of the error probability of the other hypotheses.

In case of multiple hypotheses we examine the matrix of error probability exponents, which we call reliabilities. The elements of this matrix are reliabilities $E_{l|s}$ corresponding to the case of acceptance of $l$-th hypothesis given that $s$-th is correct, $l, s = \overline{1, S}$, $l \neq s$. $E_{s|s}$ is the reliability of rejection of the correct $s$, $s = \overline{1, S}$.

Hoeffding [11] and later Csiszár and Longo [5], Tusnady [14], Birge [2] and others investigated LAO tests in the case of two hypotheses. The LAO testing for multiple hypotheses first was considered in [8]. In [1], [7] and [8] some results on multiple hypotheses testing and identification are presented. The inquired here problem was discussed briefly in [12].

We consider multiple hypothesis two-stage LAO testing for an object characterized by a pair of disjoint families of PDs. The two-stage tests have become popular in applications, especially in the field of clinical trials and genomics [7].

The organization of this paper is as follows. Section 2 is dedicated to necessary notions and properties. Section 3 introduces the problem of multiple hypothesis LAO testing and exposes the one-stage test from [8]. Section 4 presents the two-stage LAO test by one sample. Section 5 explores the two-stage LAO test by a pair of samples. In Section 6 we compare procedures of calculations and matrices of reliabilities of three methods: of the one-stage and the two-stage LAO hypotheses testing with one sample and a pair of samples.

# 2 Preliminaries

Random variable (RV) $X$ characterizing the studied object takes values in the finite set $\mathcal{X}$, and $\mathcal{P}(\mathcal{X})$ is the space of all distributions on $\mathcal{X}$. $S$ hypothetical probability distributions (PDs) of $X$ are given. They are partitioned in two disjoint families of distributions. The first family includes $R$ hypotheses $P_1, P_2, ..., P_R$ and the second family consists of $S-R$ hypotheses $P_{R+1}, P_{R+2}, ..., P_S$.

Let sample $\mathrm{x} = (x_1, x_2, \dots, x_N)$, be a vector of results of $N$ independent observations of the RV $X$. The purpose of the test is to detect the actual distribution from the given list using sample x.

The entropy of RV $X$ with PD $Q$ and the divergence (Kullback-Leibler distance) of PDs $Q$ and $P$, are defined [3, 4, 6, 9] as follows:

$$\mathrm{H}_Q(X) \stackrel{\triangle}{=} -\sum_{x\in\mathcal{X}} Q(x)\log Q(x),$$

$$\mathrm{D}\left(Q \parallel P\right) \stackrel{\triangle}{=} \sum_{x\in\mathcal{X}} Q(x)\log\frac{Q(x)}{P(x)}.$$

The method of types is a base for our proofs, so it is worthy to bring some definitions and estimates [3, 4]. $N(x|\mathrm{x})$ is the number of repetitions of the element $x \in \mathcal{X}$ in the vector $\mathrm{x} \in \mathcal{X}^N$, and

$$Q\left(\mathrm{x}\right) \stackrel{\triangle}{=} \{\frac{N\left(x|\mathrm{x}\right)}{N}\ ,\quad x\in\mathcal{X}\},$$

is the PD, called in statistics *the empirical probability distribution* of the sample x, but we prefer a shorter term from information theory: *the type* of x.

Let $\mathcal{P}^N(\mathcal{X})$ be the set of all possible types on $\mathcal{X}^N$ for $N$ observations. We denote by $\mathcal{T}_Q^N$ the set of all vectors x of the type $Q \in \mathcal{P}^N(\mathcal{X})$.

We will use the following well known properties of types:

$$|\mathcal{P}^N(\mathcal{X})| \le (N+1)^{|\mathcal{X}|},$$

$$(N+1)^{-|\mathcal{X}|}.\exp\{N\mathrm{H}_Q(X)\} \le |\mathcal{T}_Q^N| \le \exp\{N\mathrm{H}_Q(X)\},$$

$$P^N(\mathrm{x}) = \exp\{-N(\mathrm{H}_Q(X) + \mathrm{D}(Q||P))\},\quad \text{for}\ \ \mathrm{x}\in\mathcal{T}_Q^N.$$

# 3 One-stage LAO Test for Multiple Hypothesis Testing

We call the procedure of making decision on the base of $N$-sample the test, and denote it by $\phi^N$. For detecting the actual PD amongst $S$ PDs $P_s,\ s=\overline{1,S}$, the test $\phi^N$ can be designed by partitioning the sample space $\mathcal{X}^N$ to $S$ disjoint subsets $\mathcal{G}_s^{(N)},\ s=\overline{1,S}$, such that the set $\mathcal{G}_s^{(N)}$ consists of all samples x, for which $s$-th PD is adopted

$$\mathcal{G}_s^{(N)} \triangleq \{\mathrm{x} : \phi^N(\mathrm{x}) = s\}, \quad s=\overline{1,S},$$

$$\bigcup_{s=1}^{S} \mathcal{G}_s^{(N)} = \mathcal{X}^N, \quad \mathcal{G}_l^{(N)} \bigcap \mathcal{G}_s^{(N)} = \emptyset, \quad l \neq s, \quad l,s=\overline{1,S}.$$

Let $\alpha_{l|s}$ be the probability of the erroneous detection of PD $P_l$ provided $P_s$ is true

$$\alpha_{l|s}(\phi^N) \triangleq P_s^N\left(\mathcal{G}_l^{(N)}\right), \quad l,s=\overline{1,S}, \quad l\neq s. \tag{1}$$

The probability to reject $P_s$, when it is true, is

$$\alpha_{s|s}(\phi^N) \triangleq P_s^N\left(\overline{\mathcal{G}}_s^{(N)}\right) = \sum_{l\neq s} \alpha_{l|s}(\phi^N), \quad l,s=\overline{1,S}. \tag{2}$$

We denote the infinite sequences of tests by $\phi$ (for brevity below for such sequence we again apply term test), corresponding to error probability exponents, named *reliabilities*, defined as follows:

$$E_{l|s}(\phi) \triangleq \liminf_{N\to\infty} \left\{-\frac{1}{N}\log \alpha_{l|s}\left(\phi^N\right)\right\}, \ l,s=\overline{1,S}. \tag{3}$$

The following matrix is the *matrix of reliabilities* of the test $\phi$:

$$\mathbf{E}(\phi) = \begin{bmatrix} E_{1|1} & \dots & E_{R|1} & E_{R+1|1} & \dots & E_{S|1} \\ E_{1|2} & \dots & E_{R|2} & E_{R+1|2} & \dots & E_{S|2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ E_{1|R} & \dots & E_{R|R} & E_{R+1|R} & \dots & E_{S|R} \\ E_{1|R+1} & \dots & E_{R|R+1} & E_{R+1|R+1} & \dots & E_{S|R+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ E_{1|S} & \dots & E_{R|S} & E_{R+1|S} & \dots & E_{S|S} \end{bmatrix}.$$

From (1)–(3) it follows that [1]

$$E_{s|s} = \min_{l\neq s} E_{l|s}, \quad s=\overline{1,S}. \tag{4}$$

The sequence of tests $\phi^*$ is LAO if it provides maximal values for elements of the matrix $\mathbf{E}(\phi^*)$, provided $S-1$ diagonal elements of it are given.

The following theorem gives a solution of the problem of LAO test $\phi^*$ construction and contains conditions of existence of the test all elements of matrix $\mathbf{E}(\phi^*)$ which are positive.

**Theorem 3.1.** *[8] Consider an object with $S$ different hypotheses $P_s$, $s = \overline{1,S}$. For given positive numbers $E_{1|1}, E_{2|2}, ..., E_{S-1|S-1}$ let us introduce the regions:*

$$\mathcal{R}_s = \left\{Q : \mathrm{D}\left(Q \parallel P_s\right) \leq E_{s|s}\right\}, \quad s = \overline{1, S-1},$$

$$\mathcal{R}_S = \left\{Q : \mathrm{D}\left(Q \parallel P_s\right) > E_{s|s}, \quad s = \overline{1, S-1}\right\},$$

*and the following values for elements of the resultant matrix of reliabilities $\mathbf{E}(\phi^*)$ of the LAO test sequence $\phi^*$:*

$$E^*_{s|s} = E_{s|s}, \quad s = \overline{1, S-1},$$

$$E^*_{l|s} = \inf_{Q \in \mathcal{R}_l} \mathrm{D}\left(Q \parallel P_s\right), \quad l, s = \overline{1,S}, \ l \neq s,$$

$$E^*_{S|S} = \min_{l \neq S} E^*_{l|S}.$$

*If the following compatibility conditions take place*

$$0 < E_{1|1} < \min_{s=\overline{2,S}} \mathrm{D}(P_s \parallel P_1),$$

$$0 < E_{s|s} < \min[\min_{1 \leq l < s} E^*_{l|s}, \min_{s < l \leq S} \mathrm{D}(P_l \parallel P_s)], \quad 2 \leq s \leq S-1,$$

*then there exists a LAO sequence of tests $\phi^*$ with matrix of reliabilities $\mathbf{E}(\phi^*) = \{E^*_{l|s}, \ l, s = \overline{1,S}\}$.*

*Even if one of the compatibility conditions is violated, the matrix of reliabilities of that test will contain at least one element equal to zero.*

This Theorem is proved by Haroutunian in [8]. The following approaches are generalized for the composite hypotheses.

**Remark 3.2.** *In [10] it is proven that in case of LAO test*

$$E^*_{s|s} = E^*_{S|s}, \quad s = \overline{1,S}.$$

# 4 The Two-stage LAO Testing using One Sample

Let us consider two sets of indexes $\mathcal{D}_1 = \{\overline{1, R}\}$ and $\mathcal{D}_2 = \{\overline{R+1, S}\}$ and a pair of disjoint families of PDs $\mathcal{P}_1$ and $\mathcal{P}_2$

$$\mathcal{P}_1 = \{P_s, \quad s \in \mathcal{D}_1\}, \qquad \mathcal{P}_2 = \{P_s, \quad s \in \mathcal{D}_2\}.$$

We denote the two-stage test on the base of $N$-sample x by $\Phi_1^N$. Such test may be realized by a pair of tests $\varphi_1^N$ and $\varphi_2^N$ for the two consecutive stages, and we write $\Phi_1^N = (\varphi_1^N, \varphi_2^N)$. The first stage is for choice of a family of PDs, which is executed with a non-randomized test $\varphi_1^N(\mathrm{x})$ using sample x. The next stage is for making decision in the determined family of PDs, which is accomplished with a non-randomized test $\varphi_2^N(\mathrm{x}|\varphi_1 = i),\ i = 1, 2$, based on the sample x and on the result 1 or 2 of the test $\varphi_1^N$.

## 4.1 First Stage of Two-stage Test applying One Sample

The first stage of decision making consists in using sample x for selection of one family from two by a test $\varphi_1^N(\mathrm{x})$. It can be defined through partitioning the sample space $\mathcal{X}^N$ into the pair of disjoint subsets:

$$\mathcal{A}_i^N \triangleq \{\mathrm{x} : \varphi_1^N(\mathrm{x}) = i\}, \quad i = 1, 2, \quad \mathcal{A}_1^N \cup \mathcal{A}_2^N = \mathcal{X}^N.$$

The set $\mathcal{A}_i^N$ embraces all vectors x for which $i$-th family of PDs must be adopted.

The test $\varphi_1^N(\mathrm{x})$ can have two kinds of errors for the pair of hypotheses $\mathcal{P}_i,\ i = 1, 2$. Let $\alpha'_{2|1}(\varphi_1^N)$ be the probability of the erroneous acceptance of the second family of PDs provided the first family of PDs is true (that is the correct PD is in the first family) and $\alpha'_{1|2}(\varphi_1^N)$ be the probability of the erroneous adoption of the first family of PDs provided the second family $\mathcal{P}_2$ of PDs is correct. Let $\alpha'_{1|1}(\varphi_1^N)$ be the probability of rejection of the first family of PDs when it is right. We define

$$\alpha'_{2|1}(\varphi_1^N) \triangleq \alpha'_{1|1}(\varphi_1^N) \triangleq \max_{s \in \mathcal{D}_1} P_s^N(\mathcal{A}_2^N), \tag{5}$$

$$\alpha'_{1|2}(\varphi_1^N) \triangleq \alpha'_{2|2}(\varphi_1^N) \triangleq \max_{s \in \mathcal{D}_2} P_s^N(\mathcal{A}_1^N). \tag{6}$$

We consider reliabilities of the infinite sequence of tests $\varphi_1$:

$$E'_{i|j}(\varphi_1) \stackrel{\triangle}{=} \liminf_{N\to\infty} \left\{-\frac{1}{N}\log \alpha'_{i|j}(\varphi_1^N)\right\}, \quad i,j=1,2. \tag{7}$$

The matrix of reliabilities for the first stage is the following

$$\mathbf{E}'(\varphi_1) = \left[\begin{array}{cc} E'_{1|1} & E'_{2|1} \\ E'_{1|2} & E'_{2|2} \end{array}\right],$$

and one can see from (5)-(7) that there are only two different elements in it, namely

$$E'_{1|1} = E'_{2|1}, \quad E'_{1|2} = E'_{2|2}.$$

The test sequence $\varphi_1$ is LAO if for given $E'_{1|1}$ it provides the largest value to $E'_{2|2}$.

For given $E'^{*}_{1|1}$ we can determine LAO test $\varphi_1^{*N}$ by division of $\mathcal{X}^N$ into the following two disjoint subsets

$$\mathcal{A}_1^{*N} = \bigcup_{Q:\,\min\limits_{s\in\mathcal{D}_1} \mathrm{D}(Q||P_s)\leq E'^{*}_{1|1}} \mathcal{T}_Q^N, \quad \text{and} \quad \mathcal{A}_2^{*N} = \mathcal{X}^N \setminus \mathcal{A}_1^{*N}, \quad Q\in\mathcal{P}^N(\mathcal{X}).$$

Applying the properties of types to estimation of error probabilities and using the definition of the reliability $E'^{*}_{1|1}$ we will obtain dependence $E'^{*}_{2|2}(E'^{*}_{1|1})$.

We can estimate $\alpha'_{1|1}(\varphi_1^{*N})$ as follows:

$$\begin{aligned}
\alpha'_{1|1}(\varphi_1^{*N}) &= \max_{s\in\mathcal{D}_1} P_s^N\left(\mathcal{A}_2^{*N}\right) \\
&= \max_{s\in\mathcal{D}_1} P_s^N\Big(\bigcup_{Q:\,\min\limits_{l\in\mathcal{D}_1} \mathrm{D}(Q||P_l)> E'^{*}_{1|1}} \mathcal{T}_Q^N\Big) \\
&\leq \max_{s\in\mathcal{D}_1} (N+1)^{|\mathcal{X}|} \sup_{Q:\,\min\limits_{l\in\mathcal{D}_1} \mathrm{D}(Q||P_l)> E'^{*}_{1|1}} P_s^N\left(\mathcal{T}_Q^N\right) \\
&\leq \max_{s\in\mathcal{D}_1} (N+1)^{|\mathcal{X}|} \sup_{Q:\,\min\limits_{l\in\mathcal{D}_1} \mathrm{D}(Q||P_l)> E'^{*}_{1|1}} \exp\left\{-N\mathrm{D}(Q||P_s)\right\} \\
&= \exp\Big\{-N\big[\min_{s\in\mathcal{D}_1} \inf_{Q:\,\min\limits_{l\in\mathcal{D}_1} \mathrm{D}(Q||P_l)> E'^{*}_{1|1}} \mathrm{D}(Q||P_s) - o_N(1)\big]\Big\} \\
&\leq \exp\left\{-N\{E'^{*}_{1|1} - o_N(1)\}\right\}.
\end{aligned}$$

We estimate the other error probability similarly:

$$\begin{aligned}
\alpha'_{2|2}(\varphi_1^{*N}) &= \max_{s\in\mathcal{D}_2} P_s^N\left(\mathcal{A}_1^{*N}\right) \\
&= \max_{s\in\mathcal{D}_2} P_s^N\Big(\bigcup_{Q:\min\limits_{l\in\mathcal{D}_1}\mathrm{D}(Q||P_l)\le E'^{*}_{1|1}} \mathcal{T}_Q^N\Big) \\
&\le \max_{s\in\mathcal{D}_2} (N+1)^{|\mathcal{X}|} \sup_{Q:\min\limits_{l\in\mathcal{D}_1}\mathrm{D}(Q||P_l)\le E'^{*}_{1|1}} P_s^N\left(\mathcal{T}_Q^N\right) \\
&\le \max_{s\in\mathcal{D}_2} (N+1)^{|\mathcal{X}|} \sup_{Q:\min\limits_{l\in\mathcal{D}_1}\mathrm{D}(Q||P_l)\le E'^{*}_{1|1}} \exp\{-N\mathrm{D}(Q||P_s)\} \\
&= \exp\Big\{-N\big[\min_{s\in\mathcal{D}_2} \inf_{Q:\min\limits_{l\in\mathcal{D}_1}\mathrm{D}(Q||P_l)\le E'^{*}_{1|1}} \mathrm{D}(Q||P_s) - o_N(1)\big]\Big\}. \qquad (8)
\end{aligned}$$

Now let us prove the inverse inequality

$$\begin{aligned}
\alpha'_{2|2}(\varphi_1^{*N}) &= \max_{s\in\mathcal{D}_2} P_s^N\left(\mathcal{A}_1^{*N}\right) \\
&= \max_{s\in\mathcal{D}_2} P_s^N\Big(\bigcup_{Q:\min\limits_{l\in\mathcal{D}_1}\mathrm{D}(Q||P_l)\le E'^{*}_{1|1}} \mathcal{T}_Q^N\Big) \\
&\ge \max_{s\in\mathcal{D}_2} \sup_{Q:\min\limits_{l\in\mathcal{D}_1}\mathrm{D}(Q||P_l)\le E'^{*}_{1|1}} P_s^N\left(\mathcal{T}_Q^N\right) \\
&\ge \max_{s\in\mathcal{D}_2} (N+1)^{-|\mathcal{X}|} \sup_{Q_{\mathbf{x}}:\min\limits_{l\in\mathcal{D}_1}\mathrm{D}(Q||P_l)\le E'^{*}_{1|1}} \exp\{-N\mathrm{D}(Q||P_s)\} \\
&= \exp\Big\{-N\big[\min_{s\in\mathcal{D}_2} \inf_{Q:\min\limits_{l\in\mathcal{D}_1}\mathrm{D}(Q||P_l)\le E'^{*}_{1|1}} \mathrm{D}(Q||P_s) + o_N(1)\big]\Big\}. \qquad (9)
\end{aligned}$$

According to the definition of the reliability $E'^{*}_{2|2}$ from (8) and (9) we deduce

$$E'^{*}_{2|2}(E'^{*}_{1|1}) = \min_{s\in\mathcal{D}_2} \inf_{Q:\min\limits_{l\in\mathcal{D}_1}\mathrm{D}(Q||P_l)\le E'^{*}_{1|1}} \mathrm{D}(Q||P_s), \qquad (10)$$

where $Q$ is arbitrary PD on $\mathcal{X}$.

**Theorem 4.1.** *If all distributions $P_s$, $s=\overline{1,S}$, are different and $E'^{*}_{1|1}$ is such a positive number that the following inequality holds*

$$E'^{*}_{1|1} < \min_{s\in\mathcal{D}_2,\ l\in\mathcal{D}_1} \mathrm{D}(P_s||P_l),$$

*then there exists a LAO sequence of tests $\varphi_1^*$ such that reliability $E'^{*}_{2|2}(E'^{*}_{1|1})$ is positive and is defined in (10).*

**Corollary 4.2.** *If $R = 1$, $S = 2$ we possess hypotheses $P_1$ and $P_2$ Then we need only the one-stage test and Theorem 4.1 in this case is equivalent to Hoeffding's Theorem* [11], *where for $E'^{*}_{1|1} < \mathrm{D}(P_2||P_1)$,*

$$E'^{*}_{2|2}(E'^{*}_{1|1}) = \inf_{Q:\mathrm{D}(Q||P_1)\leq E'^{*}_{1|1}} D(Q||P_2).$$

Application of Theorem 4.1 also gives solution of the problem of LAO identification for the model with one family of $S$ hypotheses. The LAO statistical identification, which was considered in [1, 9], gives the answer to the question: whether $r$-th PD holds, or not. There are two error probabilities for each $r$: $\alpha^N_{l\neq r|s=r}$, $r, s = \overline{1, S}$, is the error probability that $P_r$ is correct but has been rejected, and $\alpha^N_{l=r|s\neq r}$ is the error probability that $P_r$ is selected but it is not true. The reliability approach to identification aims to determine the optimal dependence of the reliability $E_{l=r|s\neq r}$ upon given reliability $E_{l\neq r|s=r}$.

**Corollary 4.3.** *When we consider the sets $\mathcal{D}_1 = \{P_r\}$* and *$\mathcal{D}_2 = \{P_s : s \neq r, \quad s = \overline{1, S}\}$, Theorem 4.1 gives the result of* [1], *that is for*

$$E_{l\neq r|s=r} < \min_{s\neq r} \mathrm{D}(P_s\|P_r),$$

*the solution of the problem of LAO identification is:*

$$E_{l=r|s\neq r}(E_{l\neq r|s=r}) = \min_{s\neq r} \inf_{Q:\mathrm{D}(Q\|P_r)\leq E_{l\neq r|s=r}} \mathrm{D}(Q\|P_s).$$

**Example 4.4.** Suppose $\mathcal{X} = \{a, b\}$. The first family of PDs has only one PD $P_1 = (0.1, 0.9)$, and the second family of PDs consists of three PDs: $P_2 = (0.4, 0.6)$, $P_3 = (0.5, 0.5)$ and $P_4 = (0.7, 0.3)$. We calculate

$$0 < E'^{*}_{1|1} < \min_{s=\overline{2,4}} \mathrm{D}(P_s||P_1) = 0.135.$$

So by Theorem 4.1

$$E'^{*}_{2|2}(E'^{*}_{1|1}) = \min_{s=\overline{2,4}} \inf_{Q:\mathrm{D}(Q||P_1)\leq E'^{*}_{1|1}} \mathrm{D}(Q||P_s),$$

then we see that for $0 < E'^{*}_{1|1} < 0.135$, we gain $0.098 > E'^{*}_{2|2} > 0$. At Figure 1 we present the relation between $E'^{*}_{1|1}$ and $E'^{*}_{2|2}$.

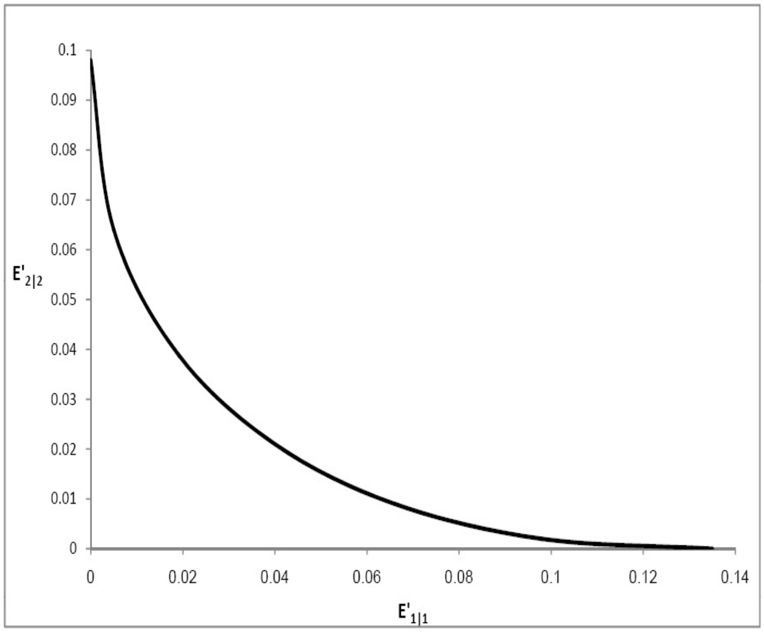


Figure 1: Error probability exponents for the first stage of test for the given example of 4 PDs with $R = 1$ and $S = 4$

### 4.2 Second Stage of the Two-stage Test with One Sample

Since a family of PDs has been selected, it is necessary to discover one PD in this family. If the first family of PDs is accepted, then we consider test $\varphi_2^N(\mathrm{x}|\varphi_1 = 1)$, which can be defined by partitioning the sample space $\mathcal{A}_1^{*N}$ to $R$ distinct subsets

$$\mathcal{B}_s^N \triangleq \{\mathrm{x} : \varphi_2^N(\mathrm{x}) = s\}, \quad s \in \mathcal{D}_1.$$

Let $\alpha''_{l|s}\left(\varphi_2^N|\varphi_1^N = 1\right)$ be the probability acceptance of PD $P_l$ instead of true PD $P_s$ at the second stage of test,

$$\alpha''_{l|s}\left(\varphi_2^N|\varphi_1^N = 1\right) \triangleq P_s^N\left(\mathcal{B}_l^N\right), \ l \in \mathcal{D}_1, \ s = \overline{1, S}, \ l \neq s.$$

The probability to reject $P_s$, when it is true, is

$$\alpha''_{s|s}\left(\varphi_2^N|\varphi_1^N = 1\right) \triangleq P_s^N\left(\overline{\mathcal{B}}_s^N\right) = \sum_{l \neq s, l=1}^{R} \alpha''_{l|s}\left(\varphi_2^N|\varphi_1^N = 1\right) + P_s^N(\mathcal{A}_2^{*N}), \quad s \in \mathcal{D}_1. \tag{11}$$

Corresponding reliabilities for the second stage of test, are defined as

$$E''_{l|s}(\varphi_2|\varphi_1=1) \triangleq \liminf_{N\to\infty}\left\{-\frac{1}{N}\log\alpha''_{l|s}\left(\varphi_2^N|\varphi_1^N=1\right)\right\}, \quad l\in\mathcal{D}_1,\ s=\overline{1,S}. \tag{12}$$

Using properties of types we get the following equalities and brief notations:

$$\lim_{N\to\infty}\left\{-\frac{1}{N}\log P_s^N(\mathcal{A}_2^{*N})\right\} = \inf_{Q:\min\limits_{l\in D_1} D(Q||P_l)>E'^{*}_{1|1}} D(Q||P_s) \triangleq E^I_{2|s}, \quad s\in\mathcal{D}_1. \tag{13}$$

From (11)–(13) it follows that similarly to (4)

$$E''_{s|s}(\varphi_2|\varphi_1=1) = \min[\min_{l\neq s,\ l\in\mathcal{D}_1} E''_{l|s}(\varphi_2|\varphi_1=1),\, E^I_{2|s}], \quad s\in\mathcal{D}_1.$$

If at the first stage the first family of PDs is accepted, the part $\mathbf{E}''(\varphi_2|\varphi_1=1)$ of the matrix of reliabilities for the second stage of test will be the following:

$$\mathbf{E}''(\varphi_2|\varphi_1=1) = \begin{bmatrix} E''_{1|1} & E''_{2|1} & \dots & E''_{R|1} \\ E''_{1|2} & E''_{2|2} & \dots & E''_{R|2} \\ \dots & \dots & \dots & \dots \\ E''_{1|R} & E''_{2|R} & \dots & E''_{R|R} \\ E''_{1|R+1} & E''_{2|R+1} & \dots & E''_{R|R+1} \\ \dots & \dots & \dots & \dots \\ E''_{1|S} & E''_{2|S} & \dots & E''_{R|S} \end{bmatrix}.$$

**Theorem 4.5.** *If at the first stage of test the first family of PDs is accepted, then for the given positive values $E''_{s|s}$, $s=\overline{1,R-1}$ of the matrix of reliabilities $\mathbf{E}''(\varphi_2|\varphi_1=1)$ let us consider the regions:*

$$\mathcal{R}''_s = \left\{Q:\min_{l\in\mathcal{D}_1}\mathrm{D}(Q||P_l)\le E'^{*}_{1|1},\quad \mathrm{D}\left(Q\parallel P_s\right)\le E''_{s|s}\right\}, \quad s=\overline{1,R-1},$$

$$\mathcal{R}''_R = \left\{Q:\min_{l\in\mathcal{D}_1}\mathrm{D}(Q||P_l)\le E'^{*}_{1|1},\quad \mathrm{D}\left(Q\parallel P_s\right)> E''_{s|s},\quad s=\overline{1,R-1}\right\},$$

*and the following values of elements of the future matrix of reliabilities $\mathbf{E}''(\varphi_2^*|\varphi_1=1)$ of the LAO tests sequence:*

$$E''^{*}_{s|s} = E''_{s|s}, \quad s=\overline{1,R-1},$$

$$E''^{*}_{l|s} = \inf_{Q\in\mathcal{R}''_l}\mathrm{D}\left(Q\parallel P_s\right), \quad l\in\mathcal{D}_1,\ s=\overline{1,S},\ l\neq s,$$

$$E''^{*}_{R|R} = \min[\min_{l:l<R} E''^{*}_{l|R},\ E^{I}_{2|R}].$$

*If the following compatibility conditions are valid*

$$E''_{1|1} < \min[\min_{s=\overline{2,R}} \mathrm{D}(P_s \parallel P_1), E^{I}_{2|1}],$$

$$E''_{s|s} < \min[\min_{1\leq l<s} E''^{*}_{l|s}, \min_{s<l\leq R} \mathrm{D}(P_l \parallel P_s),\ E^{I}_{2|s}], \quad 2 \leq s \leq R-1,$$

*then there exists a LAO sequence of tests* $\varphi_2^*$, *elements* $E''^{*}_{l|s}$ *of matrix of reliabilities* $\mathbf{E}''(\varphi_2^*|\varphi_1 = 1)$ *of which are defined above and are positive.*

*If one of the compatibility conditions is violated, then at least one element of the matrix* $\mathbf{E}''(\varphi_2^*|\varphi_1 = 1)$ *is equal to zero.*

Proof of Theorem 4.3 is similar to proof of Theorem 4.1. But following comments are relevant. For construction of LAO test of second stage, the set $\mathcal{A}_1^{*N}$ is partitioned into the following subsets:

$$\mathcal{B}_s^{*N} = \bigcup_{Q_{\mathbf{x}}:Q_{\mathbf{x}}\in\mathcal{R}_s''^{(N)}} \mathcal{T}^{N}_{Q_{\mathbf{x}}}.$$

Conditions $E''_{s|s} < E^{I}_{2|s},\ s = \overline{1,R}$, reveal the dependence of reliabilities of the second stage on the reliability of the first stage $E'^{*}_{1|1}$, since $E^{I}_{2|s}$ is a function of reliability $E'^{*}_{1|1}$ (see (13)). These conditions are necessary to ensure that subsets $\mathcal{B}_s^{*N}$ do not intersect with $\mathcal{A}_2^{*N}$.

When the second family of PDs is accepted, then the test $\varphi_2^N(\mathrm{x}|\varphi_1 = 2)$ is realized by partitioning the sample space $\mathcal{A}_2^{*N}$ to $S-R$ distinct subsets

$$\mathcal{B}_s^N \triangleq \{\mathrm{x} : \varphi_2^N(\mathrm{x}) = s\}, \quad s \in \mathcal{D}_2.$$

As was defined above the probability of the wrong decision at the second stage of test, when PD $P_l$ is accepted but $P_s$ is true, is

$$\alpha''_{l|s}\left(\varphi_2^N|\varphi_1^N = 2\right) \triangleq P_s^N\left(\mathcal{B}_l^N\right),\ l \in \mathcal{D}_2,\ s = \overline{1,S},\ l \neq s.$$

The probability to reject $P_s$, when it is true, is

$$\alpha''_{s|s}\left(\varphi_2^N|\varphi_1^N = 2\right) \triangleq P_s^N\left(\overline{\mathcal{B}}_s^N\right) = \sum_{l\neq s, l=R+1}^{S} \alpha''_{l|s}\left(\varphi_2^N|\varphi_1^N = 2\right) + P_s^N(\mathcal{A}_1^{*N}),\ \ s \in \mathcal{D}_2. \tag{14}$$

Corresponding reliabilities for the second stage of test are specified as follows:

$$E''_{l|s}(\varphi_2|\varphi_1 = 2) \triangleq \liminf_{N\to\infty} \left\{ -\frac{1}{N} \log \alpha''_{l|s} \left( \varphi_2^N | \varphi_1^N = 2 \right) \right\}, \quad l \in \mathcal{D}_2, \ s = \overline{1, S}. \tag{15}$$

Using properties of types, we obtain the following equalities:

$$\lim_{N\to\infty} \left\{ -\frac{1}{N} \log P_s^N(\mathcal{A}_1^{*N}) \right\} = \inf_{Q:\min\limits_{l\in D_1} D(Q||P_l)\leq E'^*_{1|1}} D(Q||P_s) \triangleq E^I_{1|s}, \quad s \in \mathcal{D}_2. \tag{16}$$

From (14)–(16) it follows that

$$E''_{s|s}(\varphi_2|\varphi_1 = 2) = \min[\min_{l\neq s,\ l\in\mathcal{D}_2} E''_{l|s}(\varphi_2|\varphi_1 = 2),\ E^I_{1|s}], \quad s \in \mathcal{D}_2.$$

When at the first stage the second family of PDs is accepted, the matrix of reliabilities for the second stage of test $\mathbf{E}''(\varphi_2|\varphi_1 = 2)$ will be the following:

$$\mathbf{E}''(\varphi_2|\varphi_1 = 2) = \begin{bmatrix} E''_{R+1|1} & E''_{R+2|1} & \dots & E''_{S|1} \\ E''_{R+1|2} & E''_{R+2|2} & \dots & E''_{S|2} \\ \dots & \dots & \dots & \dots \\ E''_{R+1|R} & E''_{R+2|R} & \dots & E''_{S|R} \\ \dots & \dots & \dots & \dots \\ E''_{R+1|S} & E''_{R+2|S} & \dots & E''_{S|S} \end{bmatrix}.$$

**Theorem 4.6.** *If at the first stage of test, the second family of PDs is accepted, then for the given positive values $E''_{s|s}$, $s = \overline{R+1, S-1}$ of the matrix of reliabilities $\mathbf{E}''(\varphi_2|\varphi_1 = 2)$ let us introduce the regions*

$$\mathcal{R}''_s = \left\{ Q : \min_{l\in\mathcal{D}_1} \mathrm{D}(Q||P_l) > E'^*_{1|1}, \quad \mathrm{D}\left(Q \parallel P_s\right) \leq E''_{s|s} \right\}, \quad s = \overline{R+1, S-1},$$

$$\mathcal{R}''_S = \left\{ Q : \min_{l\in\mathcal{D}_1} \mathrm{D}(Q||P_l) > E'^*_{1|1}, \quad \mathrm{D}\left(Q \parallel P_s\right) > E''_{s|s}, \quad s = \overline{R+1, S-1} \right\},$$

*and the following values of elements of the future matrix of reliabilities $\mathbf{E}''(\varphi_2^*|\varphi_1 = 2)$ of the LAO tests sequence:*

$$E''^*_{s|s} = E''_{s|s}, \quad s = \overline{R+1, S-1},$$

$$E''^*_{l|s} = \inf_{Q\in\mathcal{R}''_l} \mathrm{D}\left(Q \parallel P_s\right), \quad l \in \mathcal{D}_2,\ s = \overline{1, S},\ l \neq s$$

$$E''^*_{S|S} = \min[\min_{R<l<S} E''^*_{l|R},\ E^I_{1|S}].$$

*If the following compatibility conditions are fulfilled*

$$E''_{R+1|R+1} < \min[\min_{s=\overline{R+2,S}} \mathrm{D}(P_s \parallel P_{R+1}), E^I_{1|R+1}],$$

$$E''_{s|s} < \min[\min_{R+1 \leq l < s} E''^{*}_{l|s}, \min_{s<l \leq S} \mathrm{D}(P_l \parallel P_s),\, E^I_{1|s}], \quad R+2 \leq s \leq S-1,$$

*then there exists a LAO sequence of tests $\varphi_2^*$, elements of matrix of reliabilities $\mathbf{E}''(\varphi_2^*|\varphi_1 = 2)$ of which are defined above and are positive.*

*When one of the compatibility conditions is violated, then at least one element of the matrix $\mathbf{E}''(\varphi_2^*|\varphi_1 = 2)$ is equal to zero.*

The proof and the comments for Theorem 4.4 are similar to those of Theorem 4.3.

The answer to the question: which is the best value of $E'^{*}_{1|1}$ giving the best values to reliabilities $E''^{*}_{l|s}$, is in the following.

**Theorem 4.7.** *For the given positive values $E'^{*}_{1|1}$ and $E''_{s|s}$, $\; s = \overline{1, R-1} \bigcup \overline{R+1, S-1}$, the following inequality takes place*

$$\max_{s=\overline{1,R-1}} E''_{s|s} \leq E'^{*}_{1|1} \leq \min_{s \in \mathcal{D}_2, l \in \mathcal{D}_1} \mathrm{D}\left(P_s \parallel P_l\right), \tag{17}$$

*and the best value for $E'^{*}_{1|1}$ is $\max_{s=\overline{1,R-1}} E''_{s|s}$.*

*Proof.* From Theorem 3.1 and Theorem 4.3 for $s = \overline{1, R-1}$, we have

$$\mathcal{R}_s = \left\{Q : \mathrm{D}\left(Q \parallel P_s\right) \leq E_{s|s}\right\},$$

$$\mathcal{R}''_s = \left\{Q : \min_{l \in \mathcal{D}_1} \mathrm{D}(Q||P_l) \leq E'^{*}_{1|1}, \;\; \mathrm{D}\left(Q \parallel P_s\right) \leq E''_{s|s}\right\}.$$

Let $E_{s|s} = E''_{s|s}$ and $E'^{*}_{1|1} < E''_{s|s}$ for $s = \overline{1, R-1}$, then we will have $\mathcal{R}''_s \subset \mathcal{R}_s$. Since $\mathcal{R}''_s$, $s = \overline{1, R}$ are disjoint and $\mathcal{R}_l \cap \mathcal{R}_s = \emptyset$, $l \neq s$, we can find $Q^* \in \mathcal{R}_s \backslash \mathcal{R}''_s$ such that $Q^* \notin \mathcal{R}''_l$, $l \neq s$, $l = \overline{1, R-1}$, and we get

$$E^I_{2|s} = \inf_{Q: \min\limits_{l \in D_1} D(Q||P_l) > E'^{*}_{1|1}} D(Q||P_s) \leq D(Q^*||P_s) < E_{s|s} = E''_{s|s},$$

then we obtain $E^I_{2|s} < E''_{s|s}$, and that contradicts to compatibility conditions of Theorem 4.3. So we have $E'^{*}_{1|1} \geq E''_{s|s}$ for $s = \overline{1, R-1}$. As a result we have

$$\max_{s=\overline{1,R-1}} E''_{s|s} \leq E'^{*}_{1|1},$$

and in this case we have $\mathcal{R}''_s = \mathcal{R}_s$ for $s = \overline{1, R-1}$. The upper bound for $E'^{*}_{1|1}$ follows from Theorem 4.1

$$E'^{*}_{1|1} \leq \min_{s\in\mathcal{D}_2, l\in\mathcal{D}_1} \mathrm{D}\left(P_s \parallel P_l\right).$$

We suggest the lower bound in (17) for the best values of $E'^{*}_{1|1}$, since for all $\varepsilon > 0$ and $s \in \mathcal{D}_2$

$$\begin{aligned}
E'''^{*}_{R|s} &= \inf_{Q\in\mathcal{R}'''^{*}_R} \mathrm{D}\left(Q \parallel P_s\right) \\
&= \inf_{Q:\{\mathrm{D}(Q\|P_s)>E''_{s|s},\quad s=\overline{1,R-1},\quad \min\limits_{s\in\mathcal{D}_1} \mathrm{D}(Q\|P_s)\leq E'^{*}_{1|1}\}} \mathrm{D}\left(Q \parallel P_s\right) \\
&= \inf_{Q:\{\mathrm{D}(Q\|P_s)>E''_{s|s},\quad s=\overline{1,R-1},\quad \min\limits_{s\in\mathcal{D}_1} \mathrm{D}(Q\|P_s)\leq \max\limits_{s=\overline{1,R-1}} E_{s|s}\}} \mathrm{D}\left(Q \parallel P_s\right) \\
&\geq \inf_{Q:\{\mathrm{D}(Q\|P_s)>E''_{s|s},\quad s=\overline{1,R-1},\quad \min\limits_{s\in\mathcal{D}_1} \mathrm{D}(Q\|P_s)\leq \max\limits_{s=\overline{1,R-1}} E_{s|s}+\varepsilon\}} \mathrm{D}\left(Q \parallel P_s\right) \\
&= \inf_{Q\in\mathcal{R}'''_R} \mathrm{D}\left(Q \parallel P_s\right) = E'''_{R|s}.
\end{aligned}$$

Therefore, for the lower bound we get the best reliabilities in $R$-th column and the proof of Theorem is accomplished. □

### 4.3 Reliabilities of Two-stage Test using One Sample

The test on the base of $N$ independent observations, denoted by $\Phi^{*N}_1 = (\varphi^{*N}_1, \varphi^{*N}_2)$, is formed by a pair of LAO tests $\varphi^{*N}_1$ and $\varphi^{*N}_2$. In two-stage decision making the test $\Phi^{*N}_1$ can be realized by partitioning the sample space $\mathcal{X}^N$ to disjoint subsets if at the first stage of LAO test the $i$-th family of PDs is accepted:

$$\mathcal{C}^{*N}_s \triangleq \mathcal{A}^{*N}_i \cap \mathcal{B}^{*N}_s, \quad s \in \mathcal{D}_i, \quad i = 1, 2,$$

So we have $S$ disjoint subsets $\mathcal{C}^{*N}_s$, $s = \overline{1, S}$, such that $\bigcup_{s=1}^{S} \mathcal{C}^{*N}_s = \mathcal{X}^N$.

Let $\alpha'''_{l|s}$ be the probability of the false acceptance by two-stage test of PD $P_l$ when $P_s$ is true:

$$\alpha'''_{l|s}(\Phi^{*N}_1) \triangleq P^N_s(\mathcal{C}^{*N}_l), \quad l, s = \overline{1, S}, \quad l \neq s.$$

And the probability to reject $P_s$, when it is right, is

$$\alpha'''_{s|s}\left(\Phi^{*N}_1\right) \triangleq P^N_s\left(\overline{\mathcal{C}}^{*N}_s\right), \quad s = \overline{1, S}. \tag{18}$$

From (18) we can show that $\alpha'''_{s|s}(\Phi_1^{*N}) = \alpha''_{s|s}(\varphi_2^{*N})$.

If $l$ and $s$ are from the same set $\mathcal{D}_i$, $i = \overline{1,2}$, we do not have error in the first stage, but sample $\mathbf{x}$ is from $\mathcal{B}_l^{*N} \subset \mathcal{A}_i^{*N}$, that is $\alpha'''_{l|s} = P_s^N\left(\mathcal{B}_l^{*N}\right) = \alpha''_{l|s}$. When $l$ and $s$ are from different sets: $l \in \mathcal{D}_i$, $s \in \mathcal{D}_j$, $i \neq j$, $i,j = \overline{1,2}$, our wrong decision came from the first stage. It means that in the first stage sample $\mathbf{x}$ belongs to $\mathcal{A}_i^{*N}$, and at the second stage $\mathbf{x} \in \mathcal{B}_l^{*N} \subset \mathcal{A}_i^{*N}$. That is again $\alpha'''_{l|s} = P_s^N\left(\mathcal{B}_l^{*N}\right) = \alpha''_{l|s}$.

According to this discourse, we obtain the following relationships between error probabilities for the two-stage test with one sample and the first and the second stages of LAO tests:

a) if $l \in \mathcal{D}_1$, $s = \overline{1,S}$, then

$$\alpha'''_{l|s}(\Phi_1^{*N}) = P_s^N(\mathcal{A}_1^{*N} \cap \mathcal{B}_l^{*N}) = P_s^N(\mathcal{B}_l^{*N}) = \alpha''_{l|s}(\varphi_2^{*N}|\varphi_1^{*N} = 1) \qquad (19)$$

b) if $l \in \mathcal{D}_2$, $s = \overline{1,S}$, then

$$\alpha'''_{l|s}(\Phi_1^{*N}) = P_s^N(\mathcal{A}_2^{*N} \cap \mathcal{B}_l^{*N}) = P_s^N(\mathcal{B}_l^{*N}) = \alpha''_{l|s}(\varphi_2^{*N}|\varphi_1^{*N} = 2) \qquad (20)$$

We define reliabilities:

$$E'''_{l|s}(\Phi_1^*) \triangleq \liminf_{N\to\infty} \left\{-\frac{1}{N}\log \alpha'''_{l|s}\left(\Phi_1^{*N}\right)\right\}, \quad l,s = \overline{1,S}. \qquad (21)$$

According to (19)–(21), we get

$$E'''_{l|s}(\Phi_1^*) = E''_{l|s}(\varphi_2^*|\varphi_1^*), \quad l,s = \overline{1,S}. \qquad (22)$$

**Theorem 4.8.** *If all distributions $P_s$, $s = \overline{1,S}$, are different and positive values $E'_{1|1}$ and $E''_{r|r}$, $r = \overline{1,R-1} \bigcup \overline{R+1,S-1}$, satisfy compatibility conditions of, correspondingly, Theorems (4.1)–(4.4), then elements of matrix of reliabilities $\mathbf{E}'''(\Phi_1^*)$ of the two-stage test by one sample $\Phi_1^*$ can be found by (22).*

*When one of the compatibility conditions is violated, then at least one element of the matrix $\mathbf{E}'''(\Phi_1^*)$ is equal to zero.*

# 5 The Two-stage LAO Testing applying a Pair of Samples

Now we will discuss another version of testing. Suppose $N = N_1 + N_2$ be such that:

$$N_1 = \lceil \psi N \rceil, \quad N_2 = [(1-\psi)N], \quad 0 < \psi < 1,$$

$$\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2), \ \mathrm{x} \in \mathcal{X}^N,$$

$$\mathrm{x}_1 = (x_1, x_2, \dots, x_{N_1}), \quad \mathrm{x}_1 \in \mathcal{X}^{N_1},$$

$$\mathrm{x}_2 = (x_{N_1+1}, x_{N_1+2}, \dots, x_N), \quad \mathrm{x}_2 \in \mathcal{X}^{N_2},$$

$$\mathcal{X}^N = \mathcal{X}^{N_1} \times \mathcal{X}^{N_2}.$$

We denote the two-stage test with the pair of samples $\mathrm{x}_1$ and $\mathrm{x}_2$ on the base of $N$ observations by $\Phi_2^N$. The test $\Phi_2^N$ may be constructed by the pair of tests $\varphi_1^{N_1}$ and $\varphi_2^{N_2}$ for two consecutive stages and we write $\Phi_2^N = (\varphi_1^{N_1}, \varphi_2^{N_2})$. The first stage for selection of a family of PDs is a non-randomized test $\varphi_1^{N_1}(\mathrm{x}_1)$ using the sample $\mathrm{x}_1$. The next stage is for making decision in the determined family of PDs, it is a non-randomized test $\varphi_2^{N_2}(\mathrm{x}_2, \mathrm{x}_1)$ based on sample $\mathrm{x}_2$ and the outcome of test $\varphi_1^{N_1}(\mathrm{x}_1)$.

## 5.1 First Stage of Two-stage Test with a Pair of Samples

The first stage of decision making for choice of a family of PDs by a test $\varphi_1^{N_1}(\mathrm{x}_1)$ can be defined by partitioning the sample space $\mathcal{X}^{N_1}$ into a pair of distinct subsets

$$\mathcal{A}_i^{N_1} \triangleq \{\mathrm{x}_1 : \varphi_1^{N_1}(\mathrm{x}_1) = i\}, \quad i = 1, 2.$$

The test $\varphi_1^{N_1}(\mathrm{x}_1)$ can possess two kinds of errors for the pair of hypotheses $P \in \mathcal{P}_i, \ i = 1, 2$, analogous to definitions (5) and (6),

$$\alpha'_{i|i}(\varphi_1^{N_1}) \triangleq \max_{s \in \mathcal{D}_i} P_s^{N_1}(\mathcal{A}_{3-i}^{N_1}), \quad i = 1, 2.$$

We consider reliabilities of the infinite sequence of tests $\varphi_1$:

$$E'_{i|i}(\varphi_1) \triangleq \liminf_{N_1 \to \infty} \left\{ -\frac{1}{N_1} \log \alpha'_{i|i}(\varphi_1^{N_1}) \right\}, \quad i = 1, 2.$$

For given $E'^{*}_{1|1}$ we define the test $\varphi_1^{*N_1}$ by division of $\mathcal{X}^{N_1}$ into the following disjoint subsets

$$\mathcal{A}_1^{*N_1} = \bigcup_{Q_{\mathbf{x}_1}:\min\limits_{s\in\mathcal{D}_1} \mathrm{D}(Q_{\mathbf{x}_1}||P_s)\leq E'^{*}_{1|1}} \mathcal{T}_{Q_{\mathbf{x}_1}}^{N_1}, \quad \text{and} \quad \mathcal{A}_2^{*N_1} = \mathcal{X}^{N_1} \setminus \mathcal{A}_1^{*N_1}.$$

Applying the properties of types for estimation of error probabilities and the definition of the reliability $E'^{*}_{1|1}$, we obtain dependence $E'^{*}_{2|2}(E'^{*}_{1|1})$ as in Theorem 4.1. Then, similarly, for

$$E'^{*}_{1|1} < \min_{s\in\mathcal{D}_2,\ l:l\in\mathcal{D}_1} \mathrm{D}(P_s||P_l),$$

we construct a LAO sequence of tests $\varphi_1^{*}$ with the following positive reliability

$$E'^{*}_{2|2}(E'^{*}_{1|1}) = \min_{s\in\mathcal{D}_2} \inf_{Q:\min\limits_{l:l\in\mathcal{D}_1} \mathrm{D}(Q||P_l)\leq E'^{*}_{1|1}} \mathrm{D}(Q||P_s).$$

**Example 5.1.** Suppose $\mathcal{X} = \{a, b, c\}$, the first family of PDs contains two PDs: $P_1 = (0.1, 0.1, 0.8)$ and $P_2 = (0.2, 0.2, 0.6)$ and the second family owns three PDs: $P_3 = (0.4, 0.4, 0.2)$, $P_4 = (0.5, 0.4, 0.1)$ and $P_5 = (0.6, 0.2, 0.2)$. In table 1 we present values of divergences of all pairs of PDs.

Table 1: Values of $\mathrm{D}(P_s\|P_l)$

| No. | $l = 1$ | $l = 2$ |
|---|---|---|
| $s = 3$ | 0.3612 | 0.1454 |
| $s = 4$ | 0.5 | 0.2415 |
| $s = 5$ | 0.4067 | 0.1908 |

Applying Theorem 4.1 we see that

$$0 < E'^{*}_{1|1} < \min_{s=\overline{3,5},\ l=\overline{1,2}} \mathrm{D}(P_s||P_l) = 0.1454,$$

and consequently

$$E'^{*}_{2|2}(E'^{*}_{1|1}) = \min_{s=\overline{3,5}} \inf_{\min\limits_{l=\overline{1,2}} Q:\mathrm{D}(Q||P_l)\leq E'^{*}_{1|1}} \mathrm{D}(Q||P_s),$$

then we observe that for $0 < E'^{*}_{1|1} < 0.1454$, we have $0.1658 > E'^{*}_{2|2} > 0$. In Figure 2 we show the graph of relationship of $E'^{*}_{2|2}$ and $E'^{*}_{1|1}$.

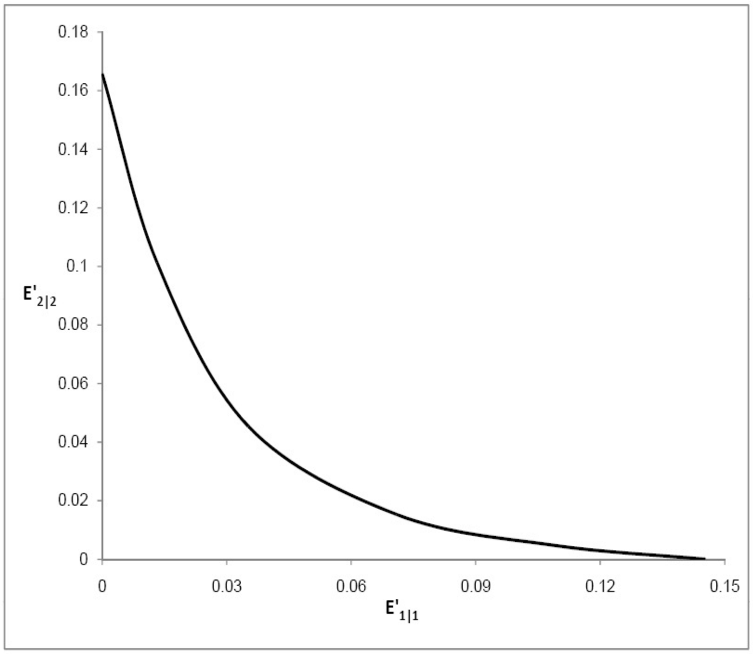


Figure 2: Error probability exponents for the first stage of test with $R = 2$ and $S = 5$

## 5.2 Second Stage of the Two-stage Test with a Pair of Samples

After selecting a family of PDs, it is necessary to detect one PD in this family by test $\varphi_2^{N_2}(\mathrm{x}_2, \mathrm{x}_1)$ which can be defined by partitioning the sample space $\mathcal{X}^{N_2}$ into $R$ (or $S - R$) distinct subsets. If the first family of PDs is accepted, then

$$\mathcal{B}_s^{N_2} \stackrel{\triangle}{=} \{\mathrm{x}_2 : \varphi_2^{N_2}(\mathrm{x}_2, \mathrm{x}_1) = s\}, \quad s \in \mathcal{D}_1,$$

And if the second family of PDs is accepted, then $s \in \mathcal{D}_2$.

At the second stage of test the probability of the fallacious acceptance of PD $P_l$, when $P_s$ is correct, is

$$\alpha''_{l|s}(\varphi_2^{N_2}|\varphi_1^{N_1} = i) \stackrel{\triangle}{=} P_s^{N_2}(\mathcal{B}_l^{N_2}),\ l \in \mathcal{D}_i,\ i = 1, 2,\ l \neq s,\ s = \overline{1, S}.$$

The probability to reject $P_s$, when it is true and $i$-th family of PDs is accepted, is

$$\alpha''_{s|s}(\varphi_2^{N_2}|\varphi_1^{N_1} = i) \stackrel{\triangle}{=} P_s^{N_2}(\overline{\mathcal{B}}_s^{N_2}) = \sum_{l \neq s,\ l \in \mathcal{D}_i} \alpha''_{l|s}(\varphi_2^{N_2}|\varphi_1^{N_1} = i), \quad s \in \mathcal{D}_i,\ i = 1, 2. \tag{23}$$

Corresponding reliabilities for the second stage of test are

$$E''_{l|s}(\varphi_2|\varphi_1=i) \triangleq \liminf_{N_2\to\infty}\left\{-\frac{1}{N_2}\log\alpha''_{l|s}(\varphi_2^{N_2}|\varphi_1^{N_1}=i)\right\}, \quad l,s=\overline{1,S}. \tag{24}$$

It follows from (23) and (24) that

$$E''_{s|s}(\varphi_2|\varphi_1=i)=\min_{l\neq s}E''_{l|s}(\varphi_2|\varphi_1=i), \quad l,s=\overline{1,S}.$$

If at the first stage the first family of PDs $\mathcal{P}_1$ is accepted correctly, the matrix of reliabilities for the second stage of test $\mathbf{E}''^{(1|1)}(\varphi_2|\varphi_1=1)$ will be the next

$$\mathbf{E}''^{(1|1)}(\varphi_2|\varphi_1=1)=\begin{bmatrix} E''_{1|1} & E''_{2|1} & \dots & E''_{R|1}\\ E''_{1|2} & E''_{2|2} & \dots & E''_{R|2}\\ \dots & \dots & \dots & \dots\\ E''_{1|R} & E''_{2|R} & \dots & E''_{R|R}\end{bmatrix},$$

and if at the first stage the first family of PDs $\mathcal{P}_1$ is accepted erroneously, the matrix of reliabilities $\mathbf{E}''^{(1|2)}(\varphi_2|\varphi_1=1)$ for the second stage will be

$$\mathbf{E}''^{(1|2)}(\varphi_2|\varphi_1=1)=\begin{bmatrix} E''_{1|R+1} & E''_{2|R+1} & \dots & E''_{R|R+1}\\ E''_{1|R+2} & E''_{2|R+2} & \dots & E''_{R|R+2}\\ \dots & \dots & \dots & \dots\\ E''_{1|S} & E''_{2|S} & \dots & E''_{R|S}\end{bmatrix}.$$

If at the first stage of test the second family of PDs $\mathcal{P}_2$ is correctly accepted, the matrix of reliabilities for the second stage of test $\mathbf{E}''^{(2|2)}(\varphi_2|\varphi_1=2)$ will be the following

$$\mathbf{E}''^{(2|2)}(\varphi_2|\varphi_1=2)=\begin{bmatrix} E''_{R+1|R+1} & E''_{R+2|R+1} & \dots & E''_{S|R+1}\\ E''_{R+1|R+2} & E''_{R+2|R+2} & \dots & E''_{S|R+2}\\ \dots & \dots & \dots & \dots\\ E''_{R+1|S} & E''_{R+2|S} & \dots & E''_{S|S}\end{bmatrix},$$

and when at the first stage the second family of PDs $\mathcal{P}_2$ is incorrectly accepted, the matrix of reliabilities of test $\mathbf{E}''^{(2|1)}(\varphi_2|\varphi_1=2)$ for the second stage will be

$$\mathbf{E}''^{(2|1)}(\varphi_2|\varphi_1=2)=\begin{bmatrix} E''_{R+1|1} & E''_{R+2|1} & \dots & E''_{S|1}\\ E''_{R+1|2} & E''_{R+2|2} & \dots & E''_{S|2}\\ \dots & \dots & \dots & \dots\\ E''_{R+1|R} & E''_{R+2|R} & \dots & E''_{S|R}\end{bmatrix}.$$

If at the first stage the first (or the second) family of PDs is accepted, then the second stage of test will be LAO, if for the given positive values of $R-1$(or $S-R-1$) diagonal elements of the matrix $\mathbf{E}''^{(i|i)}(\varphi_2|\varphi_1 = i),\ i = 1, 2$, the procedure gives maximal values for all other elements of the matrix.

**Theorem 5.2.** *If at the first stage of test the first family of PDs is accepted, then for given positive and finite values $E''_{r|r},\ r = \overline{1, R-1}$, of the matrix of reliabilities $\mathbf{E}''^{(1|1)}(\varphi_2|\varphi_1 = 1)$, let us introduce the regions:*

$$\mathcal{R}''_s = \left\{Q : \min_{l\in\mathcal{D}_1} \mathrm{D}(Q||P_l) \leq E'^{*}_{1|1},\quad \mathrm{D}\left(Q \parallel P_s\right) \leq E''_{s|s}\right\},\quad s = \overline{1, R-1},$$

$$\mathcal{R}''_R = \left\{Q : \min_{l\in\mathcal{D}_1} \mathrm{D}(Q||P_l) \leq E'^{*}_{1|1},\quad \mathrm{D}\left(Q \parallel P_s\right) > E''_{s|s},\quad s = \overline{1, R-1}\right\},$$

*and then the following values of elements of the future matrix of reliabilities $\mathbf{E}''^{(1|1)}(\varphi^*_2|\varphi^*_1 = 1)$ of the LAO tests sequence:*

$$E''^{*}_{s|s} = E''_{s|s},\quad s = \overline{1, R-1},$$

$$E''^{*}_{l|s} = \inf_{Q\in\mathcal{R}''_l} \mathrm{D}\left(Q \parallel P_s\right),\quad l, s = \overline{1, R},\ l \neq s,$$

$$E''^{*}_{R|R} = \min_{l\neq R} E''^{*}_{l|R}.$$

*When the following compatibility conditions are valid*

$$E''_{1|1} < \min_{s=\overline{2,R}} \mathrm{D}(P_s \parallel P_1),$$

$$E''_{s|s} < \min[\min_{1\leq l<s} E''^{*}_{l|s}, \min_{s<l\leq R} \mathrm{D}(P_l \parallel P_s)],\quad 2 \leq s \leq R-1,$$

*then there exists a LAO sequence of tests $\varphi^*_2$, elements of matrix of reliabilities $\mathbf{E}''^{(1|1)}(\varphi^*_2|\varphi^*_1 = 1) = \{E''^{*}_{l|s},\ l, s \in \mathcal{D}_1\}$ of which are defined above and are positive.*

*Even if one of the compatibility conditions is violated, then the matrix of reliabilities of such test will contain at least one element equal to zero.*

If at the first stage of test, the second family of PDs is accepted, then for $S-R-1$ given positive and finite values $E''_{r|r},\ r = \overline{R+1, S-1}$ of matrix of reliabilities $\mathbf{E}''^{(2|2)}(\varphi^*_2)$, the procedure is analogous.

## 5.3 Reliabilities of the Two-stage Testing with a Pair of Samples

The tool of making decision according to $N$ independent observations denoted by $\Phi_2^{*N} = (\varphi_1^{*N_1}, \varphi_2^{*N_2})$ is organized by a pair of LAO tests $\varphi_1^{*N_1}$ and $\varphi_2^{*N_2}$. In the two-stage decision making, the test $\Phi_2^{*N}$ can be defined by partitioning the sample space $\mathcal{X}^N$ into separate subsets as follows, if at the first stage of LAO test the first family of PDs is accepted, then

$$\mathcal{C}_s^{*N} \triangleq \mathcal{A}_1^{*N_1} \times \mathcal{B}_s^{*N_2}, \quad s \in \mathcal{D}_1,$$

$$\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2) \in \mathcal{C}_s^{*N} \quad \text{when} \quad \mathrm{x}_1 \in \mathcal{A}_1^{*N_1}, \quad \mathrm{x}_2 \in \mathcal{B}_s^{*N_2}, \ s \in \mathcal{D}_1,$$

and if at the first stage of LAO test the second family of PDs is accepted, then

$$\mathcal{C}_s^{*N} \triangleq \mathcal{A}_2^{*N_1} \times \mathcal{B}_s^{*N_2}, \quad s \in \mathcal{D}_2,$$

$$\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2) \in \mathcal{C}_s^{*N} \quad \text{when} \quad \mathrm{x}_1 \in \mathcal{A}_2^{*N_1}, \quad \mathrm{x}_2 \in \mathcal{B}_s^{*N_2}, \ s \in \mathcal{D}_2.$$

So we have $S$ disjoint subsets $\mathcal{C}_s^{*N}$ such that $\bigcup_{s=1}^{S} \mathcal{C}_s^{*N} = \mathcal{X}^N$ and the set $\mathcal{C}_s^{*N}$ comprises all vectors x for which $s$-th PD is adopted at two-stage test.

Let $\alpha_{l|s}'''$ be the probability of the erroneous acceptance by two-stage test with the pair of samples of PD $P_l$ when $P_s$ is true:

$$\alpha_{l|s}'''(\Phi_2^{*N}) \triangleq P_s^N(\mathcal{C}_l^{*N}), \quad l, s = \overline{1, S}, \quad l \neq s.$$

And the probability to reject $P_s$ in two-stage test by the pair of samples, when it is correct, is

$$\alpha_{s|s}''' \left(\Phi_2^{*N}\right) \triangleq P_s^N \left(\overline{\mathcal{C}}_s^{*N}\right) = \sum_{l \neq s} \alpha_{l|s}''' \left(\Phi_2^{*N}\right), \quad s = \overline{1, S}.$$

We denote by $\Phi_2^* = (\varphi_1^*, \varphi_2^*)$ the infinite sequences of tests and define reliabilities:

$$E_{l|s}'''(\Phi_2^*) \triangleq \liminf_{N \to \infty} \left\{ -\frac{1}{N} \log \alpha_{l|s}''' \left(\Phi_2^{*N}\right) \right\}, \quad l, s = \overline{1, S}.$$

We want to determine the relationship between reliabilities of the two-stage test by the pair of samples and reliabilities of the first and the second stages of LAO tests. For that we consider error probabilities as follows
a) if $l$, $s \in \mathcal{D}_1$ then

$$\alpha_{l|s}'''(\Phi_2^{*N}) = P_s^{N_1}(\mathcal{A}_1^{*N_1}) \cdot P_s^{N_2}(\mathcal{B}_l^{*N_2}) \tag{25}$$

b) if $l,\ s \in \mathcal{D}_2$ then

$$\alpha'''_{l|s}(\Phi_2^{*N}) = P_s^{N_1}(\mathcal{A}_2^{*N_1}) \cdot P_s^{N_2}(\mathcal{B}_l^{*N_2}) \tag{26}$$

c) if $s \in \mathcal{D}_1$ and $l \in \mathcal{D}_2$ then

$$\alpha'''_{l|s}(\Phi_2^{*N}) = P_s^{N_1}(\mathcal{A}_2^{*N_1}) \cdot P_s^{N_2}(\mathcal{B}_l^{*N_2}) \tag{27}$$

d) if $s \in \mathcal{D}_2$ and $l \in \mathcal{D}_1$ then

$$\alpha'''_{l|s}(\Phi_2^{*N}) = P_s^{N_1}(\mathcal{A}_1^{*N_1}) \cdot P_s^{N_2}(\mathcal{B}_l^{*N_2}) \tag{28}$$

Using properties of types we get the following equalities:

$$\lim_{N_1\to\infty}\left\{-\frac{1}{N_1}\log P_s^{N_1}(\mathcal{A}_1^{*N_1})\right\} = \inf_{Q:\min\limits_{l\in D_1} D(Q||P_l)\leq E'^{*}_{1|1}} D(Q||P_s) \stackrel{\triangle}{=} E^I_{1|s}, \quad s \in \mathcal{D}_2, \tag{29}$$

$$\lim_{N_1\to\infty}\left\{-\frac{1}{N_1}\log P_s^{N_1}(\mathcal{A}_2^{*N_1})\right\} = \inf_{Q:\min\limits_{l\in D_1} D(Q||P_l)> E'^{*}_{1|1}} D(Q||P_s) \stackrel{\triangle}{=} E^I_{2|s}, \quad s \in \mathcal{D}_1. \tag{30}$$

According to (25)–(30) and definition of reliabilities we obtain
a) if $l,\ s \in \mathcal{D}_1$, then

$$E'''_{l|s}(\Phi_2^*) = (1-\psi)E''^*_{l|s}, \tag{31}$$

b) if $l,\ s \in \mathcal{D}_2$, then

$$E'''_{l|s}(\Phi_2^*) = (1-\psi)E''^*_{l|s}, \tag{32}$$

c) if $s \in \mathcal{D}_1$ and $l \in \mathcal{D}_2$, then

$$E'''_{l|s}(\Phi_2^*) = \psi E^I_{2|s} + (1-\psi)E''^*_{l|s}, \tag{33}$$

d) if $s \in \mathcal{D}_2$ and $l \in \mathcal{D}_1$, then

$$E'''_{l|s}(\Phi_2^*) = \psi E^I_{1|s} + (1-\psi)E''^*_{l|s}, \tag{34}$$

and
e) if $s \in \mathcal{D}_i,\ i = 1, 2$, then

$$E'''_{s|s}(\Phi_2^*) = \min_{l\neq s} E'''_{l|s}(\Phi_2^*). \tag{35}$$

**Theorem 5.3.** *If all distributions $P_s$, $s=\overline{1,S}$, are different and positive numbers $E'^{*}_{1|1}$ and $E''_{r|r}$, $r=\overline{1,R-1}\bigcup\overline{R+1,S-1}$, satisfy compatibility conditions similar to those of Theorems 4.1 and 4.6, then elements of matrix of reliabilities $\mathbf{E}'''(\Phi_2^*)$, of the two-stage test by the pair of samples $\Phi_2^*$ are defined in (31)–(35).*

*When one of the compatibility conditions is violated, then at least one element of the matrix $\mathbf{E}'''(\Phi_2^*)$ will be equal to zero.*

# 6 Comparison of Matrices of Reliabilities of Three Procedures

We compare two matrices of reliabilities obtained for the one-stage and the two-stage tests, described in Theorems 3.1 and 4.6. For comparison we will give the same diagonal elements $E_{s|s}=E'''_{s|s}$, $s=\overline{1,S-1}$, to the matrices of reliabilities. For the one-stage test the elements of each column $s$, $s=\overline{1,S-1}$ are functions of the diagonal element of the same column. For the two-stage test the element of column $s$, $s=\overline{1,R-1}\bigcup\overline{R+1,S-1}$ are also functions of diagonal elements of corresponding column, the elements of column $s=\overline{1,R-1}$ $\bigcup\overline{R+1,S-1}$ of two matrices are equal and the elements of columns $R$ and $S$ of the matrices can be different. Since for $s\in\{\overline{1,R-1}\bigcup\overline{R+1,S-1}\}$ the sets for both cases are equal

$$\mathcal{R}'''_s=\left\{Q:\min_{l\in\mathcal{D}_1}\mathrm{D}(Q||P_l)\le E'^{*}_{1|1},\quad \mathrm{D}\left(Q\parallel P_s\right)\le E'''_{s|s}\right\}$$

$$=\left\{Q:\min_{l\in\mathcal{D}_1}\mathrm{D}(Q||P_l)\le E'^{*}_{1|1},\quad \mathrm{D}\left(Q\parallel P_s\right)\le E_{s|s}\right\}=\mathcal{R}_s,$$

reliabilities are also equal

$$E'''_{s|l}=\inf_{Q\in\mathcal{R}'''_s}\mathrm{D}\left(Q\parallel P_l\right)=\inf_{Q\in\mathcal{R}_s}\mathrm{D}\left(Q\parallel P_l\right)=E_{s|l}.$$

For $S$-th column we have

$$\mathcal{R}'''_S=\left\{Q:\mathrm{D}\left(Q\parallel P_s\right)>E_{s|s},\quad s=\overline{R+1,S-1},\quad \min_{s\in\mathcal{D}_1}\mathrm{D}(Q||P_s)>E'^{*}_{1|1}\right\},$$

and get that

$$E'''_{S|l} = \inf_{Q \in \mathcal{R}'''_S} \mathrm{D}\left(Q \parallel P_l\right),$$

from where it follows that

$$E'''_{S|l} = E_{l|l}, \quad \text{if} \quad l = \overline{R+1, S-1},$$

$$E'''_{S|l} \geq E'^{*}_{1|1}, \quad \text{if} \quad l = \overline{1, R}.$$

**Theorem 6.1.** *If all distributions $P_s$, $s = \overline{1,S}$, are different and positive values of diagonal elements $E_{s|s} = E'''_{s|s}$, $s = \overline{1, S-1}$ of the reliabilities matrices of one-stage and two-stage cases satisfy compatibility conditions shown in Theorems (4.1)–(4.6), then for columns $s = \overline{1, R-1} \bigcup \overline{R+1, S-1}$ reliabilities of two matrices are equal, but for $R$-th and $S$-th columns reliabilities can be different.*

**Example 6.2.** Suppose $\mathcal{X} = \{a, b, c\}$, the first family of PDs includes three PDs: $P_1 = (0.1, 0.4, 0.5)$, $P_2 = (0.2, 0.5, 0.3)$ and $P_3 = (0.3, 0.6, 0.1)$ and the second family consists of five PDs: $P_4 = (0.4, 0.3, 0.3)$, $P_5 = (0.5, 0.4, 0.1)$. $P_6 = (0.6, 0.2, 0.2)$, $P_7 = (0.7, 0.2, 0.1)$ and $P_8 = (0.8, 0.1, 0.1)$.

In Table 2 values of divergences of all pairs of PDs are presented.

Table 2: Values of $\mathrm{D}(P_s \| P_l)$

| No. | $l=1$ | $l=2$ | $l=3$ |
|---|---|---|---|
| $s=4$ | 0.1368 | 0.0539 | 0.1028 |
| $s=5$ | 0.2796 | 0.1125 | **0.0405** |
| $s=6$ | 0.3271 | 0.1715 | 0.1454 |
| $s=7$ | 0.4615 | 0.2535 | 0.1622 |
| $s=8$ | 0.5924 | 0.3640 | 0.2630 |

In Figure 3 the graph of relationship of $E'^{*}_{2|2}$ and $E'^{*}_{1|1}$ is depicted.

The reliabilities matrices of the one-stage test $\mathbf{E}(\phi^*)$ and the two-stage test $\mathbf{E}'''(\Phi^*_1)$ with $E'^{*}_{1|1} = \max\limits_{s=\overline{1,2}} E_{s|s} = 0.01$ and the same diagonal elements are as follows

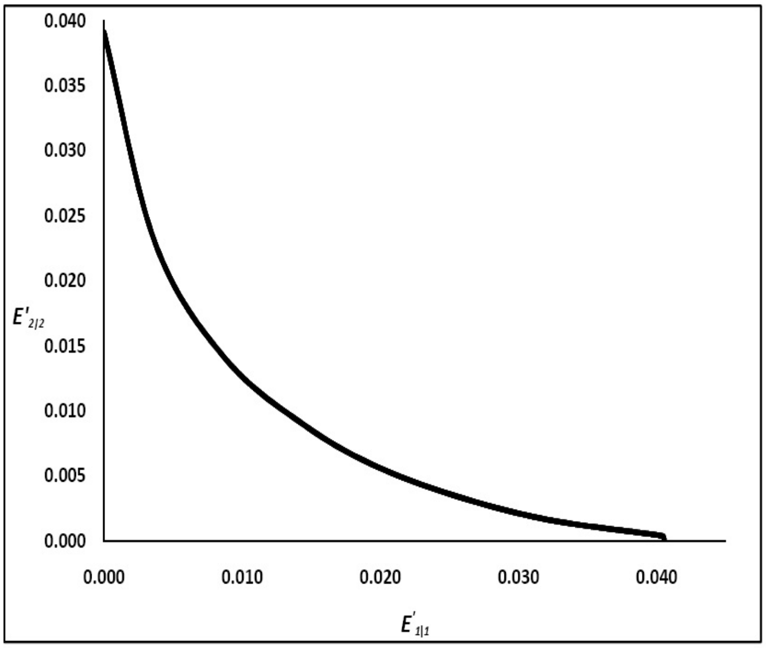


Figure 3: Error probability exponents for the first stage of test with $R = 3$ and $S = 8$

$$\mathbf{E}(\phi^*) = \begin{bmatrix} 0.0050 & 0.0112 & 0.0890 & 0.0762 & 0.2368 & 0.2341 & 0.4365 & 0.0050 \\ 0.0183 & 0.0100 & 0.0100 & 0.0215 & 0.0851 & 0.1094 & 0.2353 & 0.0100 \\ 0.1643 & 0.0223 & 0.0223 & 0.0556 & 0.0245 & 0.0947 & 0.1482 & 0.0223 \\ 0.0674 & 0.0167 & 0.0280 & 0.0070 & 0.0354 & 0.0111 & 0.0769 & 0.0070 \\ 0.2133 & 0.0560 & 0.0027 & 0.0273 & 0.0020 & 0.0189 & 0.0349 & 0.0020 \\ 0.1914 & 0.0926 & 0.0644 & 0.0111 & 0.0322 & 0.0070 & 0.0124 & 0.0070 \\ 0.3189 & 0.1557 & 0.0729 & 0.0517 & 0.0292 & 0.0026 & 0.0004 & 0.0004 \\ 0.4294 & 0.2710 & 0.1818 & 0.1043 & 0.1043 & 0.0157 & 0.0141 & 0.0141 \end{bmatrix}.$$

$$\mathbf{E}'''(\Phi_1^*) = \begin{bmatrix} 0.0050 & 0.0112 & \mathbf{0.0050} & 0.0762 & 0.2368 & 0.2341 & 0.4365 & \mathbf{0.0100} \\ 0.0183 & 0.0100 & 0.0100 & 0.0215 & 0.0851 & 0.1094 & 0.2353 & 0.0100 \\ 0.1643 & 0.0223 & 0.0223 & 0.0556 & 0.0245 & 0.0947 & 0.1482 & \mathbf{0.0100} \\ 0.0674 & 0.0167 & \mathbf{0.0460} & 0.0070 & 0.0354 & 0.0111 & 0.0769 & 0.0070 \\ 0.2133 & 0.0560 & \mathbf{0.0101} & 0.0273 & 0.0020 & 0.0189 & 0.0349 & 0.0020 \\ 0.1914 & 0.0926 & \mathbf{0.0933} & 0.0111 & 0.0322 & 0.0070 & 0.0124 & 0.0070 \\ 0.3189 & 0.1557 & \mathbf{0.1025} & 0.0517 & 0.0292 & 0.0026 & 0.0004 & 0.0004 \\ 0.4294 & 0.2710 & \mathbf{0.2293} & 0.1043 & 0.1043 & 0.0157 & 0.0141 & 0.0141 \end{bmatrix}.$$

The unequal reliabilities of the second matrix are printed in bold-face.

The reliabilities matrices of the one-stage test $\mathbf{E}(\phi^*)$ and the two-stage test with two samples $\mathbf{E}'''(\Phi_2^*)$ with $\psi = 0.1$ and the same diagonal elements, are as follows

$$\mathbf{E}(\phi^*) = \begin{bmatrix} 0.0050 & 0.0112 & 0.0958 & 0.0762 & 0.2368 & 0.2341 & 0.4365 & 0.0050 \\ 0.0183 & 0.0100 & 0.0125 & 0.0215 & 0.0851 & 0.1094 & 0.2353 & 0.0100 \\ 0.1643 & 0.0223 & 0.0185 & 0.0556 & 0.0245 & 0.0947 & 0.1482 & 0.0185 \\ 0.0674 & 0.0167 & 0.0324 & 0.0070 & 0.0354 & 0.0111 & 0.0769 & 0.0070 \\ 0.2133 & 0.0560 & 0.0042 & 0.0273 & 0.0020 & 0.0189 & 0.0349 & 0.0020 \\ 0.1914 & 0.0926 & 0.0721 & 0.0111 & 0.0322 & 0.0070 & 0.0124 & 0.0070 \\ 0.3189 & 0.1557 & 0.0805 & 0.0517 & 0.0292 & 0.0026 & 0.0004 & 0.0004 \\ 0.4294 & 0.2710 & 0.1941 & 0.1043 & 0.1043 & 0.0157 & 0.0141 & 0.0141 \end{bmatrix},$$

$$\mathbf{E}'''(\Phi_2^*) = \begin{bmatrix} 0.0050 & 0.0091 & 0.0050 & 0.0672 & 0.2125 & 0.2075 & 0.3926 & \mathbf{0.0111} \\ 0.0155 & 0.0100 & 0.0100 & 0.0193 & 0.0764 & 0.0968 & 0.2121 & \mathbf{0.0111} \\ 0.1447 & 0.0185 & 0.0185 & 0.0491 & 0.0224 & 0.0842 & 0.1338 & 0.0111 \\ 0.0607 & 0.0154 & \mathbf{0.0411} & 0.0070 & 0.0311 & 0.0091 & 0.0686 & 0.0070 \\ 0.1902 & 0.0491 & \mathbf{0.0090} & 0.0229 & 0.0020 & 0.0160 & 0.0311 & 0.0020 \\ 0.1789 & 0.0896 & \mathbf{0.0898} & 0.0091 & 0.0283 & 0.0070 & 0.0110 & 0.0070 \\ 0.2939 & 0.1465 & \mathbf{0.0990} & 0.0446 & 0.0256 & 0.0019 & 0.0004 & 0.0004 \\ 0.4054 & 0.2617 & \mathbf{0.2240} & 0.0911 & 0.0924 & 0.0130 & 0.0124 & 0.0124 \end{bmatrix},$$

the bold-faced numbers are the reliabilities of two-stage test which are greater than the reliabilities of one-stage test.

The reliabilities matrices of the one-stage test $\mathbf{E}(\phi^*)$ and the two-stage test with two samples $\mathbf{E}'''(\Phi_2^*)$ with $\psi = 0.08$ and the same diagonal elements, are as follows

$$\mathbf{E}(\phi^*) = \begin{bmatrix} 0.0050 & 0.0112 & 0.0944 & 0.0762 & 0.2368 & 0.2341 & 0.4365 & 0.0050 \\ 0.0183 & 0.0100 & 0.0120 & 0.0215 & 0.0851 & 0.1094 & 0.2353 & 0.0100 \\ 0.1643 & 0.0223 & 0.0192 & 0.0556 & 0.0245 & 0.0947 & 0.1482 & 0.0185 \\ 0.0674 & 0.0167 & 0.0315 & 0.0070 & 0.0354 & 0.0111 & 0.0769 & 0.0070 \\ 0.2133 & 0.0560 & 0.0039 & 0.0273 & 0.0020 & 0.0189 & 0.0349 & 0.0020 \\ 0.1914 & 0.0926 & 0.0705 & 0.0111 & 0.0322 & 0.0070 & 0.0124 & 0.0070 \\ 0.3189 & 0.1557 & 0.0790 & 0.0517 & 0.0292 & 0.0026 & 0.0004 & 0.0004 \\ 0.4294 & 0.2710 & 0.1916 & 0.1043 & 0.1043 & 0.0157 & 0.0141 & 0.0141 \end{bmatrix},$$

$$\mathbf{E}'''(\Phi_2^*) = \begin{bmatrix} 0.0050 & 0.0095 & 0.0050 & 0.0690 & 0.2170 & 0.2127 & 0.4014 & \mathbf{0.0109} \\ 0.0160 & 0.0100 & 0.0100 & 0.0197 & 0.0781 & 0.0993 & 0.2165 & \mathbf{0.0109} \\ 0.1487 & 0.0192 & 0.0192 & 0.0504 & 0.0229 & 0.0858 & 0.1366 & 0.0109 \\ 0.0621 & 0.0156 & \mathbf{0.0419} & 0.0070 & 0.0317 & 0.0095 & 0.0704 & 0.0070 \\ 0.1948 & 0.0505 & \mathbf{0.0092} & 0.0236 & 0.0020 & 0.0166 & 0.0318 & 0.0020 \\ 0.1814 & 0.0902 & \mathbf{0.0909} & 0.0095 & 0.0291 & 0.0070 & 0.0113 & 0.0070 \\ 0.2989 & 0.1486 & \mathbf{0.0997} & 0.0460 & 0.0263 & 0.0020 & 0.0004 & 0.0004 \\ 0.4104 & 0.2636 & \mathbf{0.2250} & 0.0937 & 0.0948 & 0.0135 & 0.0128 & 0.0128 \end{bmatrix}.$$

The bold-faced numbers are the reliabilities of two-stage test which are greater than the reliabilities of one-stage test.

# 7 Conclusion

We have shown that the number of the preliminarily given elements of the matrix of reliabilities in both one-stage and two-stage tests with one and two samples are the same. Some elements of the matrix of reliabilities of the two-stage test by one sample and by two samples can be even greater than the corresponding elements of the one-stage test. So, the customer has possibility to use the method which is preferable. It is possible to experimentally choose value of $\psi$ in two-stage test with the pair of samples, but it will be interesting to discover the best $\psi$ in function of $R$ and $S$. We can prove that the number of operations of the two-stage test by one sample is less than that of one-stage test and is greater than the number of operations of two-stage test with a pair of samples. This was observed also during experimental calculations of examples.

In [3] the LAO test for case of two hypotheses is constructed based on Neyman-Pearson Lemma, Sanov's Theorem and Lagrange multipliers. Also for the Bayesian case the Chernoff bound is obtained for the best achievable exponent. It is worth to perform analogical investigation for the case of two-stage multiple hypotheses testing.

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