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Mean-Variance Portfolio Optimization Problem with Fixed Salary and Inflation Protection for a Defined Contribution Pension Scheme

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Abstract

This paper examines a mean-variance portfolio selection problem with fixed salary or income and inflation protection strategy in the accumulation phase of a defined contribution (DC) pension plan. It was assumed that the flow of contributions made by the PPM are invested into a market that is characterized by a cash account, an inflation-linked bond and a stock. Due to the increasing risk of inflation rate and diminishing value of pension benefits, the need for hedging such risk has becomes imperative. In this paper, inflation-linked bond is traded and used to hedge inflation risks associated with the investment. The aim of this paper is to maximize the expected final wealth and minimize its variance. Efficient frontier for the three classes of assets that will enable pension plan members (PPMs) to decide their own wealth and risk in their investment profile at retirement was obtained.

Mathematics Subject Classification: 91B28, 91B30, 91B70, 93E20
Keywords: mean-variance, optimal portfolio, fixed salary, defined contribution, inflation protection, pension plans, efficient frontier

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1 Introduction

A mean-variance optimization is a quantitative method that is adopted by fund managers, consultants and investment advisors to construct portfolios for the investors. When the market is less volatile, mean-variance model seems to be a better and more reasonable way of determining portfolio selection problem. One of the aims of mean-variance optimization is to find portfolio that optimally diversify risk without reducing the expected return and to enhance portfolio construction strategy. This method is based on the pioneering work of [18,19], the Nobel pricing-winning economist, widely recognized as the father of modern portfolio theory. The optimal investment allocation strategy can be found by solving a mean and variance optimization problem.

Today, the world is shifting from the international Pay-As-You-Go public pension scheme to DC pension scheme as a result of the evolution of the equity market. For example, in June 25, 2004, Nigeria replaced her government operation of pension scheme (i.e., Pay-As-You-Go pension scheme) with a privately managed system through making compulsory contribution into their retirement savings account (RSA). This scheme was established by the Nigerian Pension Reform Act (NPRA), 2004,[26]. The aims and objectives of the NPRA are: to ensure that every person who worked in either the public service of the Federation, Federal Capital Territory or private sector receives his/her retirement benefits as and when due; to assist improvident individual by ensuring that they save in order to cater for their livelihood during old age; and to establish a uniform set of rules, regulations and standards for the administration and payments of retirement benefits for the public service of the Federation, Federal Capital Territory and the private sectors. The PPMs make a continuous stochastic income stream into the pension scheme. This cash inflows can be affected by inflation risk thereby reducing the value of pension benefits accrued to PPMs. The PPM bears a considerable risk due to inflation. This inflation risk have negative impact on the real value of PPM pension benefits. Hence, the need to hedge such risk has become imperative. In this paper, we introduced financial derivatives which are linked to inflation such as inflation-linked bonds which are traded upon in other to hedge the inflation risk that is associated with the investment.
There are extensive literature that exist on the area of accumulation phase of

a DC pension plan and optimal investment strategies. See for instance, [6], [8], [16], [4], [2], [5], [10], [13], [23], [11], [9], [1], [20] and [21].

In the context of DC pension plans, the problem of finding the optimal investment strategy with fixed salary or income and inflation protection under mean-variance efficient approach has not been reported in published articles. [14] and [24] assumed a constant flow of contributions into the pension scheme which will not be applicable to salary earners in pension scheme. We assume that the contribution of the PPM grows as the salary grows over time.

In the literature, the problem of determining the minimum variance on trading strategy in continuous-time framework has been studied by [23] via the Martingale approach. [1] used the same approach in a more general framework. [17] solved a mean-variance optimization problem in a discrete-time multi-period framework. [25] considered a mean-variance in a continuous-time framework. They shown the possibility of transforming the difficult problem of mean-variance optimization problem into a tractable one, by embedding the original problem into a stochastic linear-quadratic control problem, that can be solved using standard methods. These approaches have been extended and used by many in the financial literature, see for instance, [24], [3], [15], [14] and [7]. [22] considered a mean-variance portfolio selection problem with inflation hedging strategy for a defined contribution pension scheme. They assumed a constant flow of contribution by pension plan member into the scheme.

In this paper, we study a mean-variance approach (MVA) to portfolio selection problem with fixed salary or income of a PPM and inflation protection strategy in accumulation phase of a DC pension scheme. Our result shows that inflation-linked bond can be used to hedge inflation risk that is associated with the PPM's wealth. We found that our optimal portfolio is efficient in the mean-variance approach.

The remainder of this paper is organized as follows. In section 2, we present the problem formulation and financial market models, We also establish in this section, the dynamics of the wealth process of PPM. In section 3, we present the mean-variance approach. In section 4, we present the optimization processes of our problem and expected wealth at time t and at the terminal period for the PPM. In section 5, we present the efficient frontier of the PPM's wealth at terminal period. Finally, section 6 concludes the paper.

2 Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $\mathbf{F}(\mathcal{F}) = \{\mathcal{F}_t : t \in [0,T]\}$, where $\mathcal{F}_t = \sigma(S(s), B(s, Q(s)) : s \leq t)$, where $S(t)$ is stock price process at time $s \leq t$, $B(s, Q(s))$ is the inflation-linked bond, where $Q(s)$ inflation index at time $s \leq t$. The Brownian motions $W(t) = (W^B(t), W^S(t))'$, $0 \leq t \leq T$ is a 2-dimensional process, defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}(\mathcal{F}), \mathbf{P})$, where $\mathbf{P}$ is the real world probability measure and σ_S and σ_B are the volatility vectors of stock and volatility of the inflation-linked bond with respect to changes in $W^S(t)$ and $W^B(t)$, respectively, referred to as the coefficients of the market and are progressively measurable with respect to the filtration $\mathcal{F}$.

In this paper, we assume that the pension fund administrator (PFA) faces a market that is characterized by a risk-free asset (cash account) and two risky assets, all of whom are trade-able. We allow the stock price to be correlated to inflation. Also, we correlate the salary process of the PPM to stock. Therefore, the dynamics of the underlying assets are given in (1) to (3)

$$dC(t) = rC(t)dt, C(0) = 1, \tag{1}$$

$$dS(t) = S(t)(\mu dt + \sigma \cdot dW(t)), S(0) = s_0 > 0, \tag{2}$$

$$dB(t, Q(t)) = B(t, Q(t))\left((r + \sigma_B\theta_B)dt + \sigma_I \cdot dW(t)\right), B(0) = b > 0 \tag{3}$$

where,
μ is the appreciation rate for stock,
$\sigma = (\rho\sigma_S, \sqrt{1-\rho^2}\sigma_S)$,
$\sigma_I = (\sigma_B, 0)$,
$0 < \rho < 1$,
r is the nominal interest rate,
θ_I is the price of inflation risk,
$C(t)$ is the price process of the cash account at time t,
$S(t)$ is stock price process at time t,
$Q(t)$ is the inflation index at time t and has the dynamics:
$dQ(t) = E(q)Q(t)dt + \sigma_B Q(t)dW^B(t)$,
where $E(q)$ is the expected rate of inflation, which is the difference between nominal interest rate, r and real interest rate R (i.e. $E(q) = r - R$).

$B(t, Q(t))$ is the inflation-indexed bond price process at time t.
Then, the volatility matrix

$$\Sigma := \begin{pmatrix} \sigma_B & 0 \\ \rho\sigma_S & \sqrt{1-\rho^2}\sigma_S \end{pmatrix}$$

corresponding to the two risky assets and satisfies $det(\Sigma) = \sigma_S\sigma_B\sqrt{1-\rho^2} \neq 0$. Therefore, the market is complete and there exists a unique market price θ satisfying

$$\theta := \begin{pmatrix} \theta_B \\ \theta_S \end{pmatrix} = \begin{pmatrix} \theta_B \\ \dfrac{\mu - r - \theta_B\rho\sigma_S}{\sigma_S\sqrt{(1-\rho^2)}} \end{pmatrix}$$

where θ_S is the market price of risks. We assume in this paper that the salary process or income process $Y(t) = y$ at time t is constant in time.

Let $c > 0$ be the proportion of the PPM salary that is contributed into the pension plan, then the amount of contributions made by the PPM is $cY(t)$ at time t. Let $X(t)$ be the wealth process of the PPM at time t and $\Delta(t) = (\Delta^B(t), \Delta^S(t))$ be the portfolio process at time t such that $\Delta^B(t)$ is the proportion of wealth invested in the inflation-linked bond at time t and $\Delta^S(t)$, the proportion of wealth invested in stock at time t. Then, $\Delta_0(t) = 1 - \Delta^B(t) - \Delta^S(t)$ is the proportion of wealth invested in cash account at time t. Therefore, the wealth process of the PPM is governs by the stochastic differential equation (SDE):

$$dX(t) = (rX(t) + \Delta(t)\lambda X(t) + cy)dt + X(t)\Sigma\Delta'(t) \cdot dW(t), X(0) = x_0 > 0, \quad (4)$$

where, $\lambda = (\sigma_B\theta_B, \mu - r)'$.

The amount x_0 is the initial fund paid into PPM's account. If no amount is paid into the PPM account at the beginning, then the initial wealth is null (i.e., $x_0 = 0$). But, in this paper, we assume that at the beginning of the planning horizon, x_0 amount of money is paid into the PPM's account.

3 The Mean-Variance Approach (MVA)

In this section, we assume that the PPM invests his/her contributions through the PFA from time 0 to time T. The aim of the PPM is to maximize his/her expected terminal wealth and simultaneously minimize the variance of the terminal wealth. Hence, the PPM aim at minimizing the vector

$$[-E(X(T)), Var(X(T))].$$

Definition 3.1. *Definition 1: The portfolio strategy* $\Delta(.) = (\Delta^S(.), \Delta^B(.))$ *is said to be admissible if* $\Delta(.) \in L^2_F(0, T; R)$ *such that* $\Delta^B(.) \in L^2_{F^I}(0, T; R)$ *and* $\Delta^S(.) \in L^2_{F^S}(0, T; R)$.

Definition 3.2. *Definition 2: The mean-variance optimization problem is defined as*

$$\min_{\Delta \in (\Delta^B, \Delta^S)} J = [-E(X(T, \Delta)), Var(X(T, \Delta))] \tag{5}$$

subject to:

$$\begin{cases} \Delta(.), & \textit{set of admissible portfolio strategy} \\ X(.), \Delta(.), & \textit{satisfy}(4). \end{cases}$$

Remark 3.3. *Remark: A portfolio is said to be efficient if it is not possible to have a higher return without increasing standard deviation.*

Solving (5) is equivalent to solving the following equation

$$\min_{\Delta \in (\Delta^B, \Delta^S)} J = [-E(X(T, \Delta(.))) + \psi Var(X(T, \Delta(.)))], \psi > 0, \tag{6}$$

see [24].

By definition of variance in elementary statistics, we have

$$Var(X(T)) = E(X(T)^2) - (E(X(T)))^2 \tag{7}$$

Substituting (7) into (6), we obtain

$$\min_{\Delta(.)} J = E[\psi X^2(T) - \beta X(T)], \tag{8}$$

where,

$$\beta^* = 1 + 2\psi E(X^*(T)). \tag{9}$$

(8) is known as a linear-quadratic control problem. Hence, instead of solving (6), we now solve the following

$$\min(J(\Delta(.)), \psi, \beta) = E[\psi X(T, \Delta(.))^2 - \beta X(T, \Delta(.))], \tag{10}$$

subject to:

$$\begin{cases} \Delta(.), & \textit{set of admissible portfolio strategy} \\ X(.), \Delta(.), & \textit{satisfy}(4). \end{cases}$$

4 The Optimization problem

In solving (10), we set $\omega = \dfrac{\beta}{2\psi}$ and $F(t) = X(t) - \omega$, see [14], [23] and [24]. It implies that

$$E[\psi X(t, \Delta(t))^2 - \beta X(t, \Delta(t))] = E[\psi(X(t)F(t) - X(t)^2)].$$

Therefore, our problem is equivalent to solving

$$\min_{\Delta(.)} J(\Delta(.), \psi, \beta) = [\frac{\psi F(T)^2}{2}] \tag{11}$$

where the process $F(t)$ follows the SDE in (12):

$$\begin{cases} dF(t) = ((F(t)+\omega)(r + \Delta(t)\lambda) + cy)dt + (F(t)+\omega)\Sigma\Delta'(t) \cdot dW(t), \\ F(0) = x - \omega = f. \end{cases} \tag{12}$$

(11) is a standard optimal stochastic control problem. Let

$$U(t, f) = \inf_{\Delta(.)} E_{t,f}[\frac{\psi F(T)^2}{2}] = \sup_{\Delta(.)} J(\Delta(.), \psi, \beta) \tag{13}$$

Then, the value function U satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{cases} \inf_{\Delta \in R}\{U_t + [(f+\omega)(r + \Delta(t)\lambda) + cy]U_f \\ +\dfrac{1}{2}(f+\omega)^2(\Sigma\Delta'(t))(\Sigma\Delta'(t))'U_{ff}\} = 0, \\ \\ U(T, f) = \frac{1}{2}\psi f^2. \end{cases} \tag{14}$$

Assuming U to be a convex function of f, then first order conditions lead to the optimal fraction of portfolios to be invested in inflation-linked bond and stock at time t:

$$\Delta'^*(t) = \frac{-(\Sigma\Sigma')^{-1}\lambda U_f}{(f+\omega)U_{ff}}. \tag{15}$$

Substituting (15) into (14), we have

$$U_t + (r(f+\omega) + cy)U_f + (\frac{1}{2}(\Sigma M\lambda)'(\Sigma M\lambda) - \lambda' M\lambda)\frac{U_f^2}{U_{ff}} = 0, \tag{16}$$

where, $M = (\Sigma\Sigma')^{-1}$.

In this paper, we assume a quadratic utility function of the form:

$$U(t,f) = P(t)f^2 + Q(t)f + R(t), \tag{17}$$

see [14] and [23]. Finding the partial derivative of U in (17) and then substitute into (16), we have the following system of ordinary differential equations (ODEs):

$$P'(t) = (2g_1 - 2r - \alpha)P(t) \tag{18}$$

$$Q'(t) = (2g_1 - r)Q(t) - (2r\omega + 2cy)P(t) \tag{19}$$

$$R'(t) = \left(\frac{1}{2}g_1 - \frac{1}{4}\alpha\right)\frac{Q(t)^2}{P(t)} - (r\omega + cy)Q(t) \tag{20}$$

where, $g_1 = \lambda' M\lambda$, $\alpha = (\Sigma M\lambda)'\Sigma M\lambda$ with boundary conditions
$P(T) = \frac{1}{2}\psi$, $Q(T) = 0$, $R(T) = 0$.

Solving the systems of ODEs in (18) to (20) using the boundary conditions, we have the following (21)-(23):

$$P(t) = \frac{1}{2}\psi\exp(-(2g_1 - 2r - \alpha)(T-t)) \tag{21}$$

$$Q(t) = \frac{\psi(r\omega + cy)}{r+\alpha}(\exp((r+\alpha)(T-t)) - 1)\exp(-(2g_1 - r)(T-t)) \tag{22}$$

$$R(t) = \int_T^t \left\{\left(\frac{1}{2}g_1 - \frac{1}{4}\alpha\right)\frac{Q(u)^2}{P(u)} - (r\omega + cy)Q(u)\right\}du \tag{23}$$

We observe that our utility function U is indeed convex, since

$$U_{ff} = 2P(t) > 0, \psi > 0.$$

Now, substituting partial derivative of U into (15), we have the following:

$$\begin{aligned}\Delta'^*(t) = \frac{-(\Sigma\Sigma')^{-1}\lambda}{f+\omega}[(f+\omega) + \frac{cy}{r+\alpha} - \\ \left(\frac{\omega r\alpha}{r+\alpha} + \frac{cy}{r+\alpha}\right)\exp(-(r+\alpha)(T-t))].\end{aligned} \tag{24}$$

Hence, substituting $X^*(t)$ for $f+\omega$ into (24), we have the following:

$$\begin{aligned}\Delta'^*(t) = \frac{-(\Sigma\Sigma')^{-1}\lambda}{X^*(t)}[X^*(t) + \frac{cy}{r+\alpha} - \\ \left(\frac{\omega r\alpha}{r+\alpha} + \frac{cy}{r+\alpha}\right)\exp(-(r+\alpha)(T-t))].\end{aligned} \tag{25}$$

At $t=0$, we have

$$\Delta'^*(0) = \frac{-(\Sigma\Sigma')^{-1}\lambda}{x_0}\left[x_0 + \frac{cy}{r+\alpha} - \left(\frac{\omega r\alpha}{r+\alpha} + \frac{cy}{r+\alpha}\right)\exp(-(r+\alpha)T)\right]. \tag{26}$$

(26) is the optimal portfolio for the investor at time $t=0$.

5 Efficient Frontier

In this section, we derive the efficient frontier for the original mean-variance problem (5). The evolution of the optimal stochastic fund $X^*(t)$ for a PPM under optimal feedback control $\Delta'^*(t)$ can be obtained by substituting (25) into (4) to obtain:

$$\begin{aligned}dX^*(t) = (rX^*(t) - \lambda' M\lambda(X^*(t) + \frac{cy}{r+\alpha} \\ -(\frac{\omega r\alpha}{r+\alpha} + \frac{cy}{r+\alpha})\exp(-(r+\alpha)(T-t))) \\ +cy)dt - \Sigma M\lambda(X^*(t) + \frac{cy}{r+\alpha} \\ -(\frac{\omega r\alpha}{r+\alpha} + \frac{cy}{r+\alpha})\exp(-(r+\alpha)(T-t)))) \cdot dW(t),\end{aligned} \tag{27}$$

By applying *Itô* Lemma on (27), we obtain the SDE that governs the evolution of the square of optimal control $X^*(t)$:

$$\begin{aligned}dX^{*2}(t) = \{(2r - 2\lambda' M\lambda + \Sigma M\lambda\cdot\Sigma M\lambda)X^{*2}(t) + [-2\lambda' M\lambda(\frac{cy}{r+\alpha} \\ -(\frac{\omega r\alpha}{r+\alpha} + \frac{cy}{r+\alpha})\exp(-(r+\alpha)(T-t)) + 2cy \\ +2\Sigma M\lambda\cdot\Sigma M\lambda(\frac{cy}{r+\alpha} - (\frac{\omega r\alpha}{r+\alpha} + \frac{cy}{r+\alpha}) \\ \times\exp(-(r+\alpha)(T-t)))X^*(t) + (\frac{cy}{r+\alpha} - (\frac{\omega r\alpha}{r+\alpha} + \frac{cy}{r+\alpha}) \\ \times\exp(-(r+\alpha)(T-t)))^2\}dt + \{-2\Sigma M\lambda X^{*2}(t) - 2X^*(t)\Sigma M\lambda(\frac{cy}{r+\alpha} \\ -(\frac{\omega r\alpha}{r+\alpha} + \frac{cy}{r+\alpha})\exp(-(r+\alpha)(T-t))\}\cdot dW(t).\end{aligned} \tag{28}$$

Taking the expectation on both sides of (27) and (28), we find that the expected value of the optimal wealth and the expected value of its square follow the following ODEs:

$$\begin{cases} dE(X^*(t)) = ((r-\lambda' M\lambda)E(X^*(t)) + cy \\ \times(1+\dfrac{\lambda' M\lambda}{r+\alpha}(1-(1+\omega r\alpha)\exp(-(r+\alpha)(T-t)))))dt, \\ E(X^*(0)) = x_0. \end{cases} \tag{29}$$

$$\begin{cases} dE(X^{*2}(t)) = \{(2r-2\lambda' M\lambda + \Sigma M\lambda\cdot\Sigma M\lambda)E(X^{*2}(t)) \\ +(2cy+(2\Sigma M\lambda\cdot\Sigma M\lambda - 2\lambda' M\lambda)(\dfrac{cy}{r+\alpha} \\ -(\dfrac{\omega r\alpha}{r+\alpha}+\dfrac{cy}{r+\alpha})\exp(-(r+\alpha)(T-t)))E(X^*(t)) \\ +(\dfrac{cy}{r+\alpha}-(\dfrac{\omega r\alpha}{r+\alpha}+\dfrac{cy}{r+\alpha})\exp(-(r+\alpha)(T-t)))^2\}dt, \\ E(X^{*2}(0)) = x_0^2. \end{cases} \tag{30}$$

Solving the ODEs (29) and (30), we find that the expected value of the wealth under optimal control at time t is

$$\begin{aligned} E(X^*(t)) &= \left(x_0+\frac{cy}{r-g_1}+\frac{cyg_1}{(r-g_1)(r+\alpha)}\right)\exp((r-g_1)t) \\ &+\frac{cyg_1(1+r\omega\alpha)}{(g_1+\alpha)(r+\alpha)}\exp(-r(T-t)-\alpha T-g_1 t) \\ &-\frac{yg_1(1+r\omega\alpha)}{(g_1+\alpha)(r+\alpha)}\exp(-(r+\alpha)(T-t))-\frac{cy(g_1+r+\alpha)}{(r-g_1)(r+\alpha)}, \end{aligned} \tag{31}$$

and

$$\begin{aligned} E(X^{*2}(t)) &= x_0^2\exp((2r-2g_1+\alpha)t) \\ &+2(cy+\alpha-g_1)\exp((2r-2g_1+\alpha)t)\int_0^t \Lambda(u)E(X^*(u))\exp(-(2r-2g_1+\alpha)u)du \\ &+\exp((2r-2g_1+\alpha)t)\int_0^t A(u)\exp((2r-2g_1+\alpha)(t-u))du \\ A(u) &= (\frac{cy}{r+\alpha}-(\frac{\omega r\alpha}{r+\alpha}+\frac{cy}{r+\alpha})\exp(-(r+\alpha)(T-u)))^2, \\ \Lambda(u) &= (\frac{cy}{r+\alpha}-(\frac{\omega r\alpha}{r+\alpha}+\frac{cy}{r+\alpha})\exp(-(r+\alpha)(T-u))). \end{aligned} \tag{32}$$

At $t=T$, we have the expected terminal wealth of the PPM to be

$$\begin{aligned} E(X^*(T)) &= \left(x_0+\frac{cy}{r-g_1}\left(1+\frac{g_1}{(r+\alpha)}\right)\right)\exp((r-g_1)T) \\ &-\frac{yg_1(1+r\omega\alpha)}{(g_1+\alpha)(r+\alpha)}(1-c\exp(-(\alpha+g_1)T)-\frac{cy(g_1+r+\alpha)}{(r-g_1)(r+\alpha)}, \end{aligned} \tag{33}$$

Setting $\tilde{x} = \frac{cy}{r-g_1}\left(1+\frac{g_1}{(r+\alpha)}\right)$, $D = \frac{yg_1(1+r\omega\alpha)}{(g_1+\alpha)(r+\alpha)}$ and $K = \frac{cy(g_1+r+\alpha)}{(r-g_1)(r+\alpha)}$, (33) becomes

$$E(X^*(T)) = (x_0+\tilde{x})\exp((r-g_1)T) - D(1-c\exp(-(\alpha+g_1)T)) - K. \tag{34}$$

Finding the square of both sides of (34), we have

$$\begin{aligned}&[E(X^*(T))]^2 = x_0^2\exp(2(r-g_1)T) + (2x_0\tilde{x}+\tilde{x}^2)\exp(2(r-g_1)T)\\ &+D^2(1-c\exp(-(\alpha+g_1)T))^2 + K^2 - 2\tilde{x}D\exp((r-g_1)T)\\ &\times(1-c\exp(-(\alpha+g_1)T)) - 2DK(1-c\exp(-(\alpha+g_1)T)) + 2\tilde{x}K\exp((r-g_1)T).\end{aligned} \tag{35}$$

The second moment becomes

$$\begin{aligned}&E(X^{*2}(T)) = x_0^2\exp((2r-2g_1+\alpha)T)\\ &+2(cy+\alpha-g_1)\exp((2r-2g_1+\alpha)T)\int_0^T \Lambda(u)E(X^*(u))\exp(-(2r-2g_1+\alpha)u)du\\ &+\exp((2r-2g_1+\alpha)T)\int_0^T A(u)\exp((2r-2g_1+\alpha)(T-u))du\end{aligned} \tag{36}$$

From (9) and (33) and the definition of ω, we have that ω is a decreasing function of ψ:

$$\begin{aligned}\omega^* = &\frac{(g_1+\alpha)(r+\alpha)}{(g_1+\alpha)(r+\alpha)+ryg_1\alpha(1-c\exp(-(g_1+\alpha)T))}\\ &\times(\frac{1}{2\psi}+(x_0+\frac{cy}{r-g_1}(1+\frac{g_1}{(r+\alpha)}))\exp((r-g_1)T)\\ &-\frac{cy(g_1+\alpha)(g_1+r+\alpha)}{(r-g_1)} - yg_1(1-c\exp(-(g_1+\alpha)T))).\end{aligned}$$

Therefore, using (35) and (36), the variance of the investment portfolio can be expressed as a function of the expected final return $E(X(T))$ as:

$$\begin{aligned}&\sigma^2(X^*(T)) = x_0^2\exp(2(r-g_1)T)(\exp(\alpha T)-1) - \tilde{x}(2x_0+\tilde{x})\exp(2(r-g_1)T)\\ &-D^2(1-c\exp(-(\alpha+g_1)T))^2 - K^2 - 2\tilde{x}D\exp((r-g_1)T)\\ &\times(1-c\exp(-(\alpha+g_1)T)) + 2DK(1-c\exp(-(\alpha+g_1)T)) - 2\tilde{x}K\exp((r-g_1)T)\\ &+2(cy+\alpha-g_1)\exp((2r-2g_1+\alpha)T)\int_0^T \Lambda(u)E(X^*(u))\exp(-(2r-2g_1+\alpha)u)du\\ &+\exp((2r-2g_1+\alpha)T)\int_0^T A(u)\exp((2r-2g_1+\alpha)(T-u))du.\end{aligned}$$

6 Numerical Example

Suppose a market has a cash account with nominal annual interest rate 2%, an inflation-linked bond with a nominal annual appreciation rate 3.8% and a standard deviation of 20% and a stock with a nominal annual appreciation rate 9% and a standard deviation 30%. Suppose also that the following parameters (which have been defined earlier) take the values as follows: $\rho = 40\%$, $c = 15\%$, $y = 0.8$ million, $x_0 = 1$ million and $\psi = 1\%$. Using $T = 10$ (years), we have the following results.

A PPM with initial annual flows of income $y = 0.8$ with initial wealth $x_0 = 1$, who contributes 15% of the income into the pension scheme and wishes to obtain an expected wealth between $0 - 1000$ have the portfolio value in inflation-linked bond as obtain in figure 1 and stock as obtain in figure 2. Figure 3 shows the portfolio value of the investor in cash account in the planning horizon. Figure 4 shows the mean-standard deviation of the PPM. The figure shows that when the $E(X^*(T)) = 1.91053$ million, $\sigma^2(X^*(T)) = 1.60239$ million.

In particular, at time $t = 10$, $\Delta^S(t) = -0.78257$ million, $\Delta^B(t) = 0.0198286$ million and $\Delta_0 = 1.76274$ million. These imply that the stock needs to be shorten for an amount 0.78257 million together with the initial endowment 1 million and then invest it in the inflation-linked bond and cash account.

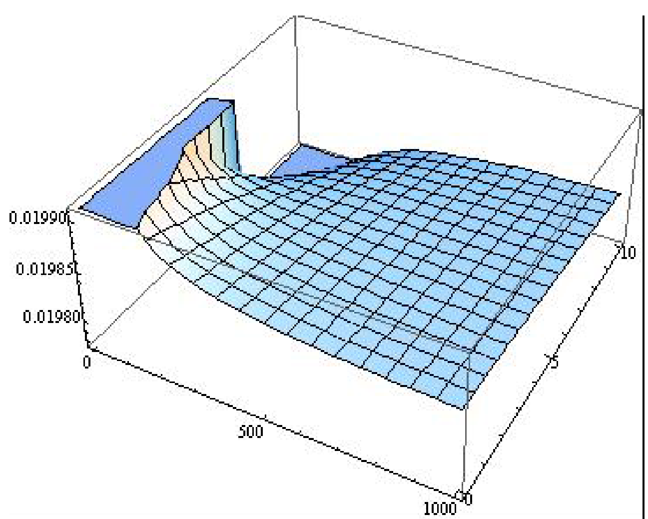

Figure 1: Portfolio Value in Inflation-linked bond

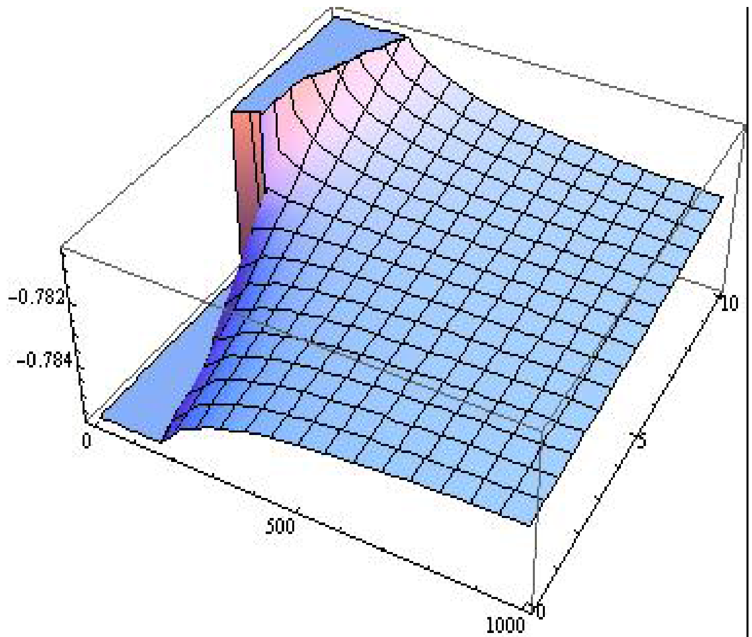

Figure 2: Portfolio Value in Stock

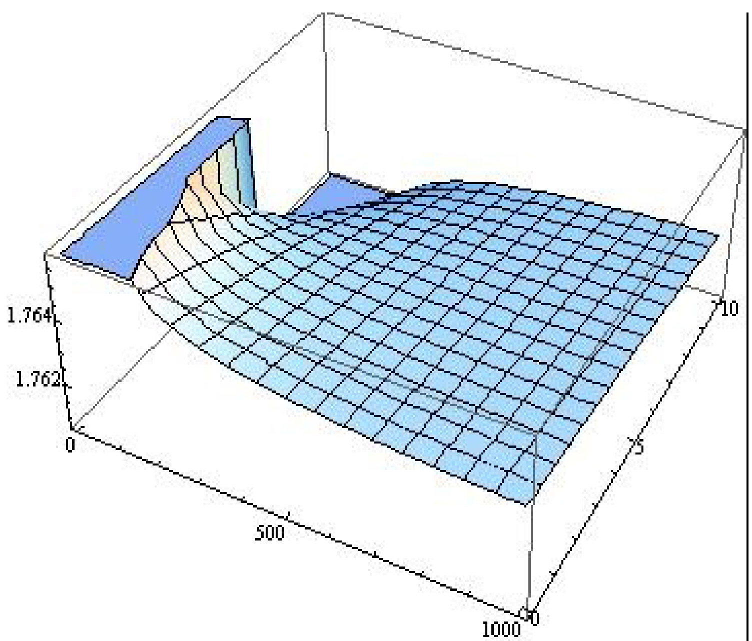

Figure 3: Portfolio Value in Cash Account

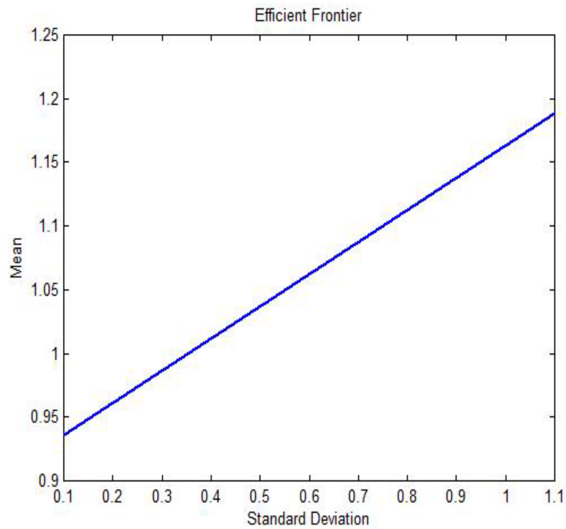

Figure 4: Efficient Frontier (in Million)

7 Conclusion

We considered a mean-variance portfolio selection problem and inflation protection strategy in the accumulation phase of a defined contribution (DC) pension plan. We adopted an hedging strategy against inflation risk. In this paper, inflation-linked bonds are traded and used to hedged inflation risk associated with the investment. We derived stochastic portfolio processes and expected wealth for the PPM. It was found that as time increases the standard deviation continue to decrease. This strategy was found to be suitable for a scheme like pension plan.

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