Journal of Finance and Investment Analysis, vol.6, no.3, 2017, 1-38
ISSN: 2241-0988 (print version), 2241-0996 (online)
Scienpress Ltd, 2017

On Valuing European Option: VAR-COVAR Approach

Ahmad M. Talafha[1] and Emmanuel Thompson[2]

Abstract

Black and Scholes (B-S) in 1973 introduced the famous B-S formula for pricing a European-style stock option. The B-S formula depends on some assumptions that are too restrictive and cannot be entirely met. This paper relaxes some of the assumptions underpinning the B-S model by deriving the equity price process within the framework of a vector autoregressive (VAR) model using stock market indices. The constant risk-free interest rate is replaced by a cointegrated VAR (CO-VAR) model using Treasury securities. Value of a European call option via Monte Carlo simulation is provided. We used antithetic and control variates as variance reduction techniques to improve upon the accuracy of our simulation.

Mathematics Subject Classification: G12; C15; G22
Keywords: Option pricing; Vector Autoregressive; Cointegrated Vector Autoregressive; Monte Carlo; Antithetic Variates; Control Variates

[1] Department of Mathematics, Western Michigan University.
E-mail: a.talafha@wmich.edu
[2] Department of Mathematics, Southeast Missouri State Univeristy.
E-mail: ethompson@semo.edu

Article Info: *Received* : June 14, 2017. *Revised* : July 17, 2017.
Published online : September 1, 2017.

1 Introduction

An option is an agreement that gives the right to sell or buy an underlying risky asset at a given price on a predetermined date. There are two types of options: puts and calls [1]. In terms of style, options are either European or American. While American-style options are exercised at any time before the expiration date, European-style options are exercised only at the expiration date.

The B-S method for modeling European option prices was first introduced in 1973, by [2] and [3], after which the model is named. The pricing model under B-S is premised on several assumptions, such as constant volatility, fixed and known risk-free interest rate. These assumptions are restrictive and cannot be met entirely; therefore; applying the standard B-S formula to real-world situations result in an erroneous price of an option.

There are many econometric models that have been introduced in the literature. Perhaps the most widely applied in modeling the risky asset price, but on no account, the only such model is Autoregressive Conditional Heteroscedasticity (ARCH) models [4]. Then it was followed by [5] who extended the ARCH to a Generalized Autoregressive Conditional Heteroscedasticity (GARCH) model. Both the ARCH and GARCH have improved the pricing performance of the B-S option pricing formula [6, 7].

According to the literature, time-varying interest rate also has attracted considerable attention. It was first introduced by [3], a few empirical studies incorporating stochastic interest rates [8, 9, 10] followed afterward.

Thus, the big question, of course, is: Whether time-varying interest rates incorporating the common stochastic trends among Treasury rates could be indispensable in pricing a European-style option? Inspired by this question, this paper relaxes the constant risk-free interest rate assumption underlying the B-S model by incorporating a time-varying model that follows a COVAR process. As empirical evidence has indicated that the geometric Brownian motion (GBM) process underlying the series of first differences of the log of asset prices are not uncorrelated, this paper introduces correlation in the first differences of the log of asset prices through VAR process.

The purpose of this paper is twofold. The first is to introduce two multivariate time series for modeling the risky asset price and the risk-free interest rate processes. The second is to employ simulation methods to arrive at the value of a European-style option. The remainder of the paper is organized as follows. Section 2 presents VAR and COVAR processes for modeling the risky asset and the risk-free interest rate, respectively. Section 3 demonstrates the empirical results of the VAR process using the Standard & Poor's 500 (S&P 500), the National Association of Securities Dealers Automated Quotations (NASDAQ), and the Dow Jones Industrial Average (DJIA) indices and the COVAR process using the U.S 3-month Treasury bill (TB3MS), 6-month Treasury bill (DTB6), and 1-year Treasury bill (DTB1YR) rates. These rates were from the secondary market. Section 4 introduces risky asset dynamics and pricing formulas of B-S. Section 5 highlights the crude MC method and variance reduction techniques used to arrive at the option price. Extensive results are presented in section 6, and section 7 concludes the paper.

2 Risky Asset Returns and the Risk-free Interest Rate Models

This section presents the VAR model for calculating the risky asset returns and the COVAR model for determining the risk free-interest rates.

2.1 Risky Asset Return model

The risky asset return process follows a VAR model. The log returns of a risky asset is presented as follows:

$$x_t = \log(1 + r_t)$$

where r_t is the actual returns and it can be obtained by the following expression:

$$r_t = \frac{P_t}{P_{t-1}}$$

where P_{t-1} and P_t are the risky asset price at time $t-1$ and t, respectively. A multivariate times series x_t follows a VAR(p) process if it satisfies (1):

$$x_t = b + \Phi_1 \, x_{t-1} + ... + \Phi_k \, x_{t-k} + \epsilon_t \qquad p > 0 \tag{1}$$

where b is k-dimensional vector, Φ is $k \times k$ matrix, and $\{\epsilon_t\}$ is a sequence of serially uncorrelated random vectors with mean zero and a positive definite covariance matrix Σ [1]. The VAR model of order 1 (VAR (1)) can be obtained by letting $p = 1$.

The first step in building a VAR(p) model is to specify an order of p i.e. $p = 0, ..., p = p_{max}$, which by selecting a value of p that minimizes some information criteria. The following criteria were used in this paper: Akaike information criterion (AIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQC) [11, 12, 13]. Once a value of p has been determined, then methods to test the existence of unit-root in time series follows. These tests are the Augmented Dicky Fuller (ADF) test by [14] and Philip and Perrion (PP) test by [15]. Also, several tests are used for diagnostic purpose (residual analyses). The purpose of model checking is to ensure that fitted VAR(p) model are adequate and not over fitted. The portmanteau test in [16] is used to check that there are no cross or auto autocorrelation in vector series x_t.

The estimation of VAR(p) parameters under the assumption of a known value of p can be accomplished by the ordinary least squares (OLS) method or the maximum likelihood (ML) method. Details on OLS and ML methods for a VAR model are provided by [17, 18] and [19] respectively. According to [20] the OLS and ML, under some regularity conditions, the estimates are asymptotically normal. Thus, a test of normality can be performed using Jarque-Bera (JB) test- a statistical test often used for residual analysis [21, 22, 23].

2.2 Risk-free Interest Rate Model

The risk-free interest rate model follows a COVAR process. The VAR model introduced in the previous subsection is appropriate for handling stationary time series ($I(0)$). It is well-known that modeling several unit-root nonstationary ($I(1)$) time series exhibit equilibrium relationship in the long run (cointegration). A systematic process for cointegration in this paper is followed to what is introduced in [1]. To comprehend cointegration in a better way, re-write (1) in such a way that b is replaced by a deterministic function: $b_t = b_0 + b_1 t$, where b_0 and b_1 are k-dimensional constant vectors. The characteristic polynomial is defined as:

$$|\Phi\,(B)| = |I - \Phi_1\,B - ... - \Phi_p\,B^p|$$

if all the zeros of $|\Phi\,(B)|$ are outside the unit circle, then x_t is $I(1)$. Adding the error-correction term Πx_{t-1} to a VAR(p) model in differences produces the vector error-correction (VEC) model:

$$\Delta x_t = b_t + \Pi\,x_{t-1} + \Phi_1^*\,\Delta x_{t-1} + ... + \Phi_{t-p+1}^*\,\Delta x_{t-1} + \epsilon_t \tag{2}$$

Assume $0 < rank(\Pi) = m < k$, then x_t is cointegrated with m linearly independent cointegrating vectors, and $k - m$ unit-roots that gives $k - m$ common stochastic trend of x_t. A COVAR model can be estimated by ML method presented in [19]. The rank of Π can be tested using likelihood ratio (LR) tests in [24]. In this paper, the Johannsen trace and maximal eigenvalue tests were conducted. Critical values of these test statistics are nonstandard but evaluated via simulations, see [25] for details.

3 Empirical Results

This section provides the empirical results of the two processes introduced in section 2. The data used in this paper for the stock returns were the monthly average stock market indices and the risk-free interest rate were the Treasury security bills.

3.1 Descriptive Statistics

Table 1 illustrates minimum and maximum monthly return values, standard deviation, skewness, and kurtosis. The table shows that, the mean returns for the three stock market indices were all positive, ranging from a minimum of 0.006859 (DJIA) to a maximum of 0.010125 (NASDAQ). The table further reveals that, the three series were negatively skewed. It can also be noticed that the sample standard deviation for DJIA stock returns was the highest (0.034493), while that of the NASDAQ was the lowest (0.044224). The three stock returns showed evidence of positive kurtosis, as well as heavy-tailed. The normality test based on the JB statistic is also shown in Table 1. It is indicative from the table that, the probability value of all series is greater than the

Table 1: Descriptive statistics for return series

	S&P 500	**NASDAQ**	**DJIA**
Mean	0.008202	0.010125	0.006859
Median	0.010999	0.011026	0.011602
Min.	-0.085532	-0.086591	-0.099558
Max.	0.102307	0.113711	0.059671
Std. Dev	0.037596	0.044224	0.028459
Variance	0.001413	0.001955	0.000809
Skewness	-0.202624	0.159546	-1.055568
Kurtosis	3.019715	2.689164	4.779707
JB	0.528140	0.636660	24.461000
***p*-value**	0.767900	0.727400	0.000004

Sources: S&P 500, NASDAQ, and DJIA indices for the period December 2009 to August 2016 [27].

5% significant level. Table 2 presents the summary statistics for the 3-month, 6-month, and 1-year Treasury bill rates. The table shows that, the highest mean return is reported for DTB1YR followed by DTB6 and TB3MS. The table further display the three series are positively skewed. Also, they show evidence of positive kurtosis, but also heavy tailed.

3.2 Unit-Root Tests and Lag Length Selection

A crucial issue in practice is distinguishing between $I(0)$ process and one which is $I(1)$. This part of the empirical analysis further attempts to determine whether a time series is consistent with a unit root. In this paper the ADF, and PP tests were used to check the presence of unit roots in both the return in the stock indices and the Treasury securities series.

Table 3 illustrates the test results. The results indicate that the null hypothesis of the presence of unit root in S&P 500, NASDAQ and DJIA return series can be rejected at 1% level of significance, concluding that all the three return series were $I(0)$. According to the AIC and HQC in Table 5, the optimal lag lenght was two with values -23.88, and -23.67, respectively, however,

Table 2: Descriptive Statistics for Treasury Bills

	TB3MS	**DTB6**	**DTB1YR**
Mean	0.000824	0.001431	0.002222
Median	0.000600	0.001150	0.001800
Min.	0.000100	0.000400	0.000900
Max.	0.003100	0.004900	0.006400
Std. Dev	0.000693	0.001016	0.001324
Variance	0.000000	0.000001	0.000002
Skewness	1.444716	1.831815	1.479051
Kurtosis	4.759633	6.109353	4.682721
JB	37.19700	75.04300	37.64100
***p*-value**	8.372×10^{-9}	2.2×10^{-16}	6.704×10^{-9}

Sources: TB3MS, DTB6, and DTB1YR rates. These rates were from secondary market, from December 2009 to June 2016 [28].

the BIC criterion shows an optimal lag length of 1, with value of -23.41. It can be concluded that the smallest value is -23.88, which is associated with $lag\,2$. However, priority is given to $lag\,1$ on the grounds of parsimony.

Table 3: ADF and PP Tests for the Three Stock Markets.

Stock Index	**ADF Test**		**PP Test**	
	Test Statistic	***p*-value**	**Test Statistic**	***p*-value**
S&P 500	-7.2853	0.01	-9.9914	0.01
NASDQ	-7.3896	0.01	-10.2120	0.01
DJIA	-7.1648	0.01	-8.0902	0.01

Table 4: ADF and PP Tests for the Three Treasury Bills.

Treasury bill	**ADF Test**		**PP Test**	
	Test Statistic	***p*-value**	**Test Statistic**	***p*-value**
TB3MS	-1.2330	0.5980	-1.1216	0.9132
DTB6	-0.9893	0.6875	-0.8746	0.9510
DTB1YR	-0.1423	0.9363	-0.4203	0.9830

The ADF and PP unit root tests confirm that the three Treasury bills are $I(1)$. The tests include the case of no trend (constant) as a deterministic term. That is, the $I(1)$ series of Treasury bills can be represented by a COVAR model, and the information criteria were used to determine the order. The order $p = 3$ was chosen by both BIC and HQC.

Table 5: Appropriate lag order for the Three Stock Return

p	**AIC**	**BIC**	**HQC**	***p*-value**
0	-23.1335	-23.1335	-23.1335	0.0000
1	-23.6855	-23.4116	-23.5760	0.0000
2	-23.8893	-23.3414	-23.6701	0.0033
3	-23.7894	-22.9675	-23.4606	0.6203
4	-23.6619	-22.5661	-23.2236	0.8010
5	-23.5299	-22.1601	-22.9820	0.8488

Table 6: Appropriate lag order for the Three Treasury Bills

p	**AIC**	**BIC**	**HQC**	***p*-value**
0	-46.2975	-46.2975	-46.2975	0.0000
1	-50.4459	-50.1759	-50.3377	0.0000
2	-50.6798	-50.1399	-50.4635	0.0014
3	-51.1507	-50.3408	-50.8262	0.0000
4	-51.2349	-50.1552	-50.8023	0.0593
5	-51.2292	-49.8795	-50.6885	0.2760
6	-51.2735	-49.6539	-50.6246	0.1789
7	-51.1755	-49.2860	-50.4185	0.7744
8	-51.1920	-49.0325	-50.3269	0.3589
9	-51.2786	-48.8492	-50.3053	0.2254
10	-51.4114	-48.7120	-50.3299	0.1896
11	-51.4820	-48.5127	-50.2924	0.4009
12	-51.6226	-48.3833	-50.3249	0.3116
13	-51.7315	-48.2223	-50.3256	0.4762

3.3 The VAR (1) Process

The estimated matrix equations from the three stock returns are as follows:

$$\begin{bmatrix} SP500_t \\ NASDQ_t \\ DIJN_t \end{bmatrix} = \begin{bmatrix} 0.009 \\ (0.047) \\ 0.011 \\ (0.034) \\ 0.006 \\ (0.077) \end{bmatrix} + \begin{bmatrix} -0.137 & -0.366 & 0.682 \\ (0.742) & (0.453) & (0.021) \\ -0.073 & 0.078 & 0.244 \\ (0.835) & (0.849) & (0.320) \\ 0.288 & 0.274 & -0.260 \\ (0.0.249) & (0.236) & (0.061) \end{bmatrix} \begin{bmatrix} SP500_{t-1} \\ NASDQ_{t-1} \\ DJIA_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \epsilon_{3t} \end{bmatrix} \quad (3)$$

where the p-values are in parenthesis below each coefficient and the residual covariance matrix is

$$\hat{\Sigma}_\epsilon = \begin{bmatrix} 0.0014 & 0.0016 & 0.0008 \\ 0.0016 & 0.0019 & 0.0009 \\ 0.0008 & 0.0009 & 0.0007 \end{bmatrix}$$

where $\epsilon_t \sim \mathcal{N}(0, \Sigma)$

3.4 Cointegration Test and Vector ECM Representation

In empirical studies of multivariate time series, the number of linearly independent vectors in a COVAR model affects the model set up and the parameter estimation procedures at other stages of the analysis. Therefore, the trace and maximum eigenvalue statistical tests were preformed in this subsection to verify the exact number. Table 7 illustrates the test for cointegration ranks. The three eigenvalues for the null hypothesis were 0.295, 0.168, and 0.032, respectively. It can be noticed that the eigenvalues were all less than 1, indicating that the test is stable. If the the maximum eigenvalue test in [24] is used, then it can achieved that $LR_{max}(2) = 2.47$, $LR_{max}(1) = 14.14$, and $LR_{max}(0) = 26.95$. The maximum eigenvalue tests reject with critical values $r = 0$, and $r = 1$, but fail to reject at critical value $r = 2$. Therefore, there exist 2 linearly independent cointegrating vectors and one common stochastic trend. The trace statistic test in [24] were used and reported the following results: $LR_{trace}(2) = 2.47$, $LR_{trace}(1) = 16.61$, and $LR_{trace}(0) = 43.56$. In the trace test statistic, the null hypothesis of a cointgeration was not rejected.

Table 7: Cointegration tests for interest rates

Hypothesis	**Test Statistic**				**Test Statistic**			
	Max	**10%**	**5%**	**1%**	**Trace**	**10%**	**5%**	**1%**
r $\leq$ 2	2.47	6.50	8.18	11.65	2.47	6.50	8.18	11.65
r $\leq$ 1	14.14	12.91	14.90	19.19	16.61	15.66	17.95	23.52
r = 0	26.95	18.90	21.07	25.75	43.56	28.71	31.52	37.22

The fitted vector ECM is given as:

$$\begin{bmatrix} TB3MS_t \\ DTB6_t \\ DTB1YR_t \end{bmatrix} = \begin{bmatrix} TB3MS_{t-1} \\ DTB6_{t-1} \\ DTB1YR_{t-1} \end{bmatrix} + \begin{bmatrix} -0.691 \\ -0.646 \\ -0.840 \end{bmatrix} \omega_t + \begin{bmatrix} 0.518 & -0.205 & -0.169 \\ 0.640 & 0.000 & -0.365 \\ 0.794 & 0.000 & -0.499 \end{bmatrix} \Delta x_{t-1} + \begin{bmatrix} 0.000 & -0.195 & 0.000 \\ 0.275 & -1.019 & 0.338 \\ 0.623 & -1.333 & 0.284 \end{bmatrix} \Delta x_{t-2} + \epsilon_t, \quad (4)$$

where

Lag 1 and *lag* 2 are obtained by replacing t with $t-1$ and $t-2$ respectively in x_t.

$\omega_t = [1.000, -1.027, 0.254]' \; x_t$.

$$\hat{\Sigma}_\epsilon = \frac{1}{10^7} \begin{bmatrix} 0.50 & 0.57 & 0.39 \\ 0.57 & 1.01 & 0.88 \\ 0.39 & 0.88 & 1.16 \end{bmatrix}$$

The ADF test for ω_t indicates that the series has no unit root. The test statistic is -2.156 with p-value 0.033.

4 B-S Model

This section introduces the B-S model for pricing a European-style option. The pricing model is presented below.

$$V_c\,(T) = P_t \Phi(d_1) - K e^{-r\,(T\,-\,t)} \Phi(d_2) \quad (5)$$

Using (5), then,

$$V_p\,(T) = Ke^{-r\,(T\,-\,t)}\Phi(-d_2) - P_t\Phi(-d_1) \tag{6}$$

$$d_1 = \frac{\ln\left(\frac{P_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{(T - t)}}$$

$$d_2 = \frac{\ln\left(\frac{P_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T}} = d_1 - \sigma\ \sqrt{(T - t)}$$

where $V_c(T)$ and $V_p(T)$ are the value of the call and put option, respectively. Also, P_t is the current risky-asset price, K is the exercise price of the call or the put option, r is the annualized risk-free interest rate, T is the future time to a call or a put option expiration and t is the current time of the risky-asset, σ is the standard deviation of the logarithmic risky-asset return, and Φ is the cumulative normal distribution function; that is,

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\dfrac{-t^2}{2}}\, dt$$

The B-S model assumes that the underlying risky-asset moves randomly, following a GBM process, that is:

$$dP_t = \mu\, P_t\, dt + \sigma\, P_t\, dw_t \tag{7}$$

where w_t is a Wiener process [19]. Estimation of the parameters μ and σ is presented in [1].

Therefore, in the risk-neutral world the value at time $t < T$ of the call option with payoff at time T is the expected value of the payoff, discounted to time t, that is

$$e^{-r(T-t)}\,[max\,(P - K, 0)] \tag{8}$$

Similarly, for the put option

$$e^{-r(T-t)}\,[max\,(K - P, 0)] \tag{9}$$

5 MC Simulation

This section illustrates the concepts of crude Monte Carlo simulation. It also discusses the antithetic variates, and control variates as ways of reducing the variance of MC estimators. Again, it highlights a quite general technique for pricing a European-style option using crude Monte Carlo methods.

5.1 Crude Monte Carlo (MC)

Let h be a real-valued function and $U = (U_1, U_2, ..., U_k)$ is a random vector of independent and identically distributed (i.i.d) $U(0,1)$ random variables. Run a simulation of the form $X = h(U)$. Then, the crude MC estimator q is the sample mean of $\{X_i\}$ as it is given by [29]

$$\hat{q}^{(cmc)} = \frac{1}{k}\sum_{i=1}^{k} h(U_i) \tag{10}$$

$$= \frac{1}{k}\sum_{i=1}^{k} X_i \tag{11}$$

where $q = \mathbb{E}h(U)$, with U a random variable uniformly distributed on $[0,1]$. It can be noted that the X_i's are the results of k independent experiments that have the same probability distributions as X. In other words, each X_i is distributed as X. Given that $Var(X) = \sigma^2$, then

$$Var(\hat{q}) = \frac{1}{k^2}\sum_{i=1}^{k} Var(X_i) = \frac{\sigma^2}{k}.$$

Assuming that σ^2 is unknown, it can be estimated through the sample variance of $\{X_i\}$; that is,

$$S^2 = \frac{1}{k-1}\sum_{i=1}^{k}\left(h(V_i) - \hat{q}\right)^2.$$

For large k, the central limit theorem (CTL) [29, 30] may be used to form approximate confidence interval for q as follows:

$$\left(\frac{1}{k}\sum_{i=1}^{k} X_i - z_{1-\alpha/2}\frac{\sigma}{\sqrt{k}}, \frac{1}{k}\sum_{i=1}^{k} X_i + z_{1-\alpha/2}\frac{\sigma}{\sqrt{k}}\right),$$

where z_γ denotes the γ-quantile of the $\mathcal{N}(0,1)$ distribution.

5.2 Variance Reduction

In this section methods to reduce the variance of $\hat{q}$ are presented, specifically antithetic, and control variates will be highlighted.

5.2.1 Antithetic Variates (AV)

Let $U^* = (U_1^*, U_2^*, ...)$ be a random vector of i.i.d $U(0,1)$ random variable which is independent of U. A pair of real-valued random variables (X, X^*) are said to be an antithetic pair if X and X^* for which X and $X^* = h(U^*)$ are negatively correlated and have the same distribution. Let $k = 2m$ or $k/2 = m$, for some $m \geq 1$, that is k is even and $(X_1, X_1^*), ..., (X_m, X_m^*)$ are independent antithetic pairs of random variables, where X_i and X_i^* share the same distribution, say X, then the antithetic estimator

$$\hat{q}^{(a)} = \frac{1}{2m}\sum_{i=1}^{m}(X_i + X_i^*) \tag{12}$$

is an unbiased estimator of $q = \mathbb{E}(X)$ with variance

$$\begin{aligned} Var(\hat{q}) &= \frac{Var(X) + Var(X^*) + 2\ Cov(X, X^*)}{4m^2} \\ &= \frac{Var(X) + Cov(X, X^*)}{2m} \\ &= \frac{Var(X)}{2m}(1+\rho), \end{aligned} \tag{13}$$

where $\rho = \mathbb{C}orr(X, X^*)$. Recall the well known fact that the correlation estimate between any two random variables should be less than the absolute value of 1, in other words

$$-1 \leq \rho \leq 1$$

Since the pair of (X, X^*) are antithetic variables, then we have

$$-1 \leq \rho \leq 0. \tag{14}$$

Which implies

$$0 \leq (1+\rho) \leq 1. \tag{15}$$

The variance obtained in (13) leads to a similar result for the crude MC solution in a manner that as k becomes arbitrarily large, $Var(\hat{q})$ shrinks towards 0, therefore, (13) shrinks as the absolute correlation between X and X^* increases as a result of (15). Furthermore, results in (13) and (15) is enhancing the antithetic estimation, which makes it more accurate than the standard crude MC [31].

The antithetic estimation process is as follows:

1. Generate $X_1 = h(U_1), ..., X_m = h(U_m)$.

2. Let $X_1^* = h(1-U_1), ..., X_m^* = h(1-U_m)$ via independent simulations.

3. Calculate the sample covariance matrix for each pair $\{(X_i, X_i^*)\}$:

$$Q = \begin{pmatrix} \frac{1}{m-1}\sum_{i=1}^{m}(X_i-\bar{X})^2 & \frac{1}{m-1}\sum_{i=1}^{m}\left[(X_i-\bar{X})(X_i^*-\bar{X}^*)\right] \\ \frac{1}{m-1}\sum_{i=1}^{m}\left[(X_i-\bar{X})(X_i^*-\bar{X}^*)\right] & \frac{1}{m-1}\sum_{i=1}^{m}(X_i^*-\bar{X}^*)^2 \end{pmatrix}.$$

4. Estimate the mean, q using the antithetic estimator $\hat{q}^{(a)}$ given in (12) and determine an approximate $(1-\alpha)$ confidence interval as

$$\left(\hat{q}^{(a)} - z_\gamma \cdot SE, \hat{q}^{(a)} + z_\gamma \cdot SE\right). \tag{16}$$

Where SE denotes the standard error, which is given by

$$SE = \sqrt{\frac{Q_{1,1} + Q_{2,2} + 2Q_{1,2}}{4m}}, \tag{17}$$

and z_γ is the γ-quantile of the standard normal distribution, i.e., $\mathcal{N}(0,1)$ [31].

5.2.2 Control Variates (CV)

Let X be the output of a simulation run. A random variable $\tilde{X}$ achieved from the same simulation run, is called a control variable for X if X and $\tilde{X}$ are negatively or positively correlated and the exception of $\tilde{X}$ is known i.e $\mathbb{E}\tilde{X}_i = \hat{q}^{(c)}$. Thus, the following illustrates the use of control variables as a variance reduction method:

Let $X_1, ..., X_k$ and $\tilde{X}_1, ..., \tilde{X}_k$ be the output of k independent simulation, and the corresponding control variables, respectively.

Let $\alpha \in \mathbb{R}$ then the estimator:

$$\hat{q}^{(c)} = \frac{1}{k}\sum_{i=1}^{k}\left[X_i - \alpha\left(\tilde{X}_i - \tilde{q}\right)\right] \tag{18}$$

is an unbiased estimator of $q = \mathbb{E}(X)$, where $\tilde{q} = \mathbb{E}\tilde{X}_i$ and $\alpha = \frac{Cov\left(X,\tilde{X}\right)}{Var\left(\tilde{X}\right)}$.

with the minimal variance

$$Var(\hat{q}^{(c)}) = \frac{1}{k}\left(1-\rho^2\right)Var(X), \tag{19}$$

where $\rho = \mathbb{C}orr\left(X_i, \tilde{X}_i\right)$. The control estimation process is as follows:

1. From k independent simulation runs generate $X_1, ..., X_k$ and the control variables $\tilde{X}_1, ..., \tilde{X}_k$
2. Compute the sample covariance matrix for each pair $\{(X_i, \tilde{X}_i)\}$:

$$C = \begin{pmatrix} \frac{1}{k-1}\sum_{i=1}^{k}(X_i - \bar{X})^2 & \frac{1}{k-1}\sum_{i=1}^{k}\left[(X_i - \bar{X})(\tilde{X}_i - \bar{\tilde{X}})\right] \\ \frac{1}{k-1}\sum_{i=1}^{k}\left[(X_i - \bar{X})(\tilde{X}_i - \bar{\tilde{X}})\right] & \frac{1}{k-1}\sum_{i=1}^{k}(\tilde{X}_i - \bar{\tilde{X}})^2 \end{pmatrix}.$$

3. Estimate the mean, $\hat{q}^{(c)}$ given in (18) with $\alpha = \frac{C_{1,2}}{C_{2,2}}$ and determine an approximate $(1-\alpha)$ confidence interval as

$$\left(\hat{q}^{(c)} - z_\gamma \cdot SE, \hat{q}^{(c)} + z_\gamma \cdot SE\right). \tag{20}$$

Where SE is given by

$$SE = \sqrt{\frac{1}{k}\left(1 - \frac{C_{1,2}^2}{C_{1,1}C_{2,2}}C_{1,1}\right)}, \tag{21}$$

and z_γ is the γ-quantile of the standard normal distribution, i.e., $\mathcal{N}(0,1)$.

Recall (8), the discounted payoff for a European call. To obtain the crude MC European call option price, let

$$X_i = e^{-r(T-t)} \cdot max\{P_T^{(i)} - K, 0\}$$

for $i = 1, ..., k$.

To arrive at the AV European call option option price , let

$$X_i = e^{-r(T-t)} \cdot max\{P_T^{(i)} - K, 0\}$$

for $i = 1, ..., m$.

Similarly,

$$X_i^* = e^{-r(T-t)} \cdot max\{P_T^{*(i)} - K, 0\}$$

for $i = 1, ..., m$.

Also, to arrive at the CV European call option price , let

$$X_i = e^{-r(T-t)} \cdot max\{P_T^{(i)} - K, 0\}$$

for $i = 1, ..., k$. while,

$$\tilde{X}_i = e^{r(T-t)}\, P_T - P_t$$

6 Results

This section provides the paper results and starts with the specification of the simulation parameters used in computing the crude MC $\left(V_c^{(cmc)}\right)$, the AV $\left(V_c^{(a)}\right)$, and the CV $\left(V_c^{(c)}\right)$ based European call option price. Results of the simulated European call option prices in the case of VAR-COVAR was compared to the BS-COVAR using crude MC method, AV, and CV techniques.

6.1 Simulation Parameters

To compare the European call option price from VAR-COVAR and BS-COVAR models, we used the DJIA for the asset return and the 3-month risk-free interest rate models respectively. For ease of reference, refer to (3) and (4).

The $lag\,1$ log-returns in the simulation of the European call option price for the stock part were -0.00127, 0.00426, and -0.00150 for the S&P 500, NASDAQ, and DJIA respectively. Also, the DJIA mean and standard deviation of the log-returns were 0.00600 and 0.02645 respectively.

A similar approach was followed in the case of the risk-free interest rate model. The $lag\,1$ rates for the model were fixed at 0.00270, 0.0040, 0.00540 for the 3-month, 6-month, and the 1-year, respectively. The $lag\,2$ rates were respectively 0.00270, 0.00410, and 0.00570 for the 3-month, 6-month, and the 1-year. Also, the 3-month mean and standard deviation of the rates were -0.691 and $\sqrt{(0.50)(10^{-7})}$, respectively.

In the case of the BS-COVAR model, we allowed the asset return process to follow a normal distribution. That is, the stock part of the option pricing model was driven by a GBM in (7) with the parameters estimated using the sample mean and standard deviation formulas in [1]. The risk-free interest rate model follows the COVAR model for pricing the option under the BS-COVAR model.

6.2 European Call Option Value

Using the parameters provided under 6.1. This section provides the European call option for the 3-month, 6-month, 9-month, and 1-year expiration date. The tables comprise of the call option prices, their standard errors ($SE(V_c)$), and corresponding 95% confidence intervals ($CI(V_c)$). In these tables, the European call option prices were obtained by using BS-COVAR and VAR-COVAR models.

6.2.1 3-month Call Option

Tables 8, 9, and 10 give results for European call option price for 3-month expiration date. For Table 8, it is obvious that the BS-COVAR price was lower compared to the VAR-COVAR price for all k. While BS-COVAR call option price was in the range of \$71.49 to \$72.34, the VAR-COVAR was in the range of \$161.62 to \$181.58. For BS-COVAR and VAR-COVAR option price as k increases the $SE(V_c^{(cmc)})$ decreases as well. It is clear the BS-COVAR model was estimated at \$71.99 ($N = 100000$, $SE(V_c^{(cmc)}) = 0.0004157$, and 95% $CI(V_c^{(cmc)}) = (\$71.989, \$71.991)$). While it is notable that VAR-COVAR model for a European call option price for the same period was reported at \$165.83 ($k = 100000$, $SE(V_c^{(cmc)}) = 0.0009575$, and 95% $CI(V_c^{(cmc)}) = (\$165.828, \$165.832)$).

Table 9 shows that the BS-COVR price was relatively closer to the VAR-COVAR price for all k. While BS-COVAR call option price was in the range of \$35.75 to \$35.82, the VAR-COVAR was in the range of \$56.41 to \$56.66. Furthermore, for the BS-COVAR and the VAR-COVAR option prices as k goes up the $SE(V_c^{(a)})$ goes down. It is clear the BS-COVAR model was estimated at \$35.75 ($k = 100000$, $SE(V_c^{(a)}) = 0.000007$, and 95% $CI(V_c^{(a)}) = (\$35.75, \$35.75)$). While it is notable that VAR-COVAR model for the same period was reported at \$56.44 ($k = 100000$, $SE(V_c^{(a)}) = 0.0001400$, and 95% $CI(V_c^{(a)}) = (\$56.44, \$56.44)$).

Table 10 indicates that the BS-COVAR was slightly higher than the value of VAR-COVAR for all k. BS-VAR reported values from \$151.34 to \$151.46. While the VAR-COVAR value converged to \$150.60. Besides, for BS-COVAR and VAR-COVAR option price as k increases the $SE(V_c^{(c)})$ decreases as well. The BS-COVAR model was estimated at \$151.42 ($N = 100000$, $SE(V_c^{(c)}) = 0.022950$, and 95% $CI(V_c^{(c)}) = (\$151.37336, \$151.46330)$). While it is clear that VAR-COVAR model for a European call option price for the same period was reported at \$150.60 ($k = 100000$, $SE(V_c^{(c)}) = 0.000005$, and 95% $CI(V_c^{(c)}) = (\$150.60294, \$150.60296)$).

Table 8: Call option: Crude MC with 3-month expiration.

	BS-COVAR			VAR-COVAR		
k	$V_c^{(cmc)}$	$SE\left(V_c^{(cmc)}\right)$	$95\%\ CI\left(V_c^{(cmc)}\right)$	$V_c^{(cmc)}$	$SE\left(V_c^{(cmc)}\right)$	$95\%\ CI\left(V_c^{(cmc)}\right)$
1000	71.86	0.041179	(71.779, 71.941)	181.58	0.100398	(181.383, 181.777)
5000	71.49	0.008268	(71.474, 71.506)	160.62	0.018693	(160.583, 160.657)
10000	72.01	0.00413	(72.002, 72.018)	165.78	0.009549	(165.761, 165.799)
15000	72.34	0.002777	(72.335, 72.345)	165.35	0.006401	(165.338, 165.363)
20000	72.3	0.002087	(72.296, 72.304)	166.3	0.004851	(166.291, 166.31)
50000	71.91	0.000831	(71.908, 71.912)	164.71	0.001913	(164.706, 164.714)
100000	71.99	0.000416	(71.989, 71.991)	165.83	0.000958	(165.828, 165.832)

Table 9: Call option: AV with 3-month expiration.

	BS-COVAR			VAR-COVAR		
k	$V_c^{(a)}$	$SE\left(V_c^{(a)}\right)$	$95\%\ CI\left(V_c^{(a)}\right)$	$V_c^{(a)}$	$SE\left(V_c^{(a)}\right)$	$95\%\ CI\left(V_c^{(a)}\right)$
1000	35.82	0.001271	(35.81792, 35.82208)	56.66	0.019993	(56.57996, 56.62004)
5000	35.76	0.000149	(35.76957, 35.77043)	56.42	0.001152	(56.44045, 56.45955)
10000	35.76	0.000043	(35.75986, 35.76014)	56.44	0.002177	(56.42585, 56.43415)
15000	35.75	0.000099	(35.74994, 35.75006)	56.41	0.000788	(56.46752, 56.47248)
20000	35.75	0.000039	(35.74997, 35.75003)	56.46	0.000837	(56.47848, 56.48152)
50000	35.75	0.000018	(35.74997, 35.75003)	56.45	0.000317	(56.42950, 56.43050)
100000	35.75	0.000007	(35.74999, 35.75001)	56.44	0.00014	(56.43975, 56.44025)

Table 10: Call option: CV with 3-month Expiration.

	BS-COVAR			VAR-COVAR		
k	$V_c^{(c)}$	$SE\left(V_c^{(c)}\right)$	$95\%\ CI\left(V_c^{(c)}\right)$	$V_c^{(c)}$	$SE\left(V_c^{(c)}\right)$	$95\%\ CI\left(V_c^{(c)}\right)$
1000	151.40	0.23460	(150.94040, 151.85980)	150.6	0.000048	(150.60286, 150.60305)
5000	151.39	0.10120	(151.19240, 151.58890)	150.6	0.000021	(150.60291, 150.60299)
10000	151.35	0.06749	(151.21900, 151.48358)	150.6	0.000015	(150.60292, 150.60298)
15000	151.46	0.06245	(151.33888, 151.58369)	150.6	0.000012	(150.60293, 150.60298)
20000	151.45	0.04968	(151.35078, 151.54554)	150.6	0.000011	(150.60293, 150.60297)
50000	151.37	0.03032	(151.31204, 151.43088)	150.6	0.000007	(150.60294, 150.60297)
100000	151.42	0.02295	(151.37336, 151.46330)	150.6	0.000005	(150.60294, 150.60296)

6.2.2 6-month Call Option

Tables 11, 12, and 13 show results for European call option price for 6-month expiration date. Table 11 indicates that the VAR-COVAR model always produced larger values of a European call option price than the BS-COVAR model. However, the BS-VAR and the VAR-COVAR model obtain value in the range \$76.46 to \$79.19 and \$163.02 to \$184.30 respectively. It should be noted that BS-COVAR and VAR-COVAR option prices as k increases the $SE(V_c^{(cmc)})$ decrease as well. Also, it can be seen that the BS-COVAR was estimated at \$76.63 ($k = 100000$, $SE(V_c^{(cmc)}) = 0.0.0004434$, and 95% $CI(V_c^{(cmc)}) = (\$76.629, \$76.631)$). Meanwhile, the VAR-COVAR option price for the same maturity period was estimated at \$168.31 ($k = 100000$, $SE(V_c^{(cmc)}) = 0.0.0009718$, and 95% $CI(V_c^{(cmc)}) = (\$168.308, \$168.132)$).

Table 12 indicates that the value of the VAR-COVAR model is quit similar to the value from the BS-COVAR model. However, with BS-COVAR and VAR-COVAR model, the European call option achieved value in the range \$37.18 to \$37.26 and \$57.52 to \$57.76, respectively. The BS-COVAR and VAR-COVAR option prices as the simuation path increases, the $SE(V_c^{(a)})$ decreases. It is observed that the BS-COVAR was estimated at \$37.18 ($k = 100000$, $SE(V_c^{(a)}) = 0.000005$, and 95% $CI(V_c^{(a)}) = (\$37.18, \$37.18)$). Meanwhile, the VAR-COVAR option price for the same expriation was estimated at \$57.55 ($k = 100000$, $SE(V_c^{(a)}) = 0.000143$, and 95% $CI(V_c^{(a)}) = (\$57.55, \$57.55)$).

In Table 13, it is obvious that the BS-COVAR price was higher compared to the VAR-COVAR price for all k. While BS-COVAR call option price was in the range \$166.70 to \$167.13, the VAR-COVAR converged to \$163.30. For BS-COVAR and VAR-COVAR option price as k increases the $SE(V_c^{(c)})$ decreases as well. The BS-COVAR model was estimated at \$167.01 ($N = 100000$, $SE(V_c^{(c)}) = 0.057930$, and 95% $CI(V_c^{(c)}) = (\$166.89219, \$167.11928)$). While it is notable that VAR-COVAR model for a European call option prices for the same period was reported at \$163.30 ($k = 100000$, $SE(V_c^{(c)}) = 0.000008$, and 95% $CI(V_c^{(c)}) = (\$163.29776, \$163.29779)$).

Table 11: Call option: Crude MC with 6-month Expiration.

	BS-COVAR			VAR-COVAR		
k	$V_c^{(cmc)}$	$SE\left(V_c^{(cmc)}\right)$	$95\%\ CI\left(V_c^{(cmc)}\right)$	$V_c^{(cmc)}$	$SE\left(V_c^{(cmc)}\right)$	$95\%\ CI\left(V_c^{(cmc)}\right)$
1000	79.19	0.045396	(79.101, 79.279)	184.30	0.101899	(184.100, 184.500)
5000	77.11	0.008918	(77.093, 77.127)	163.02	0.018972	(162.983, 163.057)
10000	76.46	0.004420	(76.451, 76.469)	168.26	0.009692	(168.241, 168.279)
15000	76.92	0.002958	(76.914, 76.926)	167.82	0.006497	(167.807, 167.833)
20000	76.74	0.002222	(76.736, 76.744)	168.78	0.004923	(168.770, 168.790)
50000	77.16	0.000891	(77.158, 77.162)	167.17	0.001942	(167.166, 167.174)
100000	76.63	0.000443	(76.629, 76.631)	168.31	0.000972	(168.308, 168.312)

Table 12: Call option: AV with 6-month Expiration.

	BS-COVAR			VAR-COVAR		
k	$V_c^{(a)}$	$SE\left(V_c^{(a)}\right)$	$95\%\ CI\left(V_c^{(a)}\right)$	$V_c^{(a)}$	$SE\left(V_c^{(a)}\right)$	$95\%\ CI\left(V_c^{(a)}\right)$
1000	37.26	0.003526	(37.25637, 37.26363)	57.76	0.020430	(57.67952, 57.72048)
5000	37.20	0.000595	(37.19924, 37.20076)	57.52	0.001177	(57.54024, 57.55976)
10000	37.19	0.000084	(37.18975, 37.19025)	57.54	0.002225	(57.52576, 57.53424)
15000	37.19	0.000093	(37.18988, 37.19012)	57.52	0.000806	(57.57746, 57.58254)
20000	37.19	0.000023	(37.18994, 37.19006)	57.57	0.000855	(57.57844, 57.58156)
50000	37.18	0.000035	(37.18995, 37.19005)	57.55	0.000324	(57.52949, 57.53051)
100000	37.18	0.000005	(37.18999, 37.19001)	57.55	0.000143	(57.53974, 57.54026)

Table 13: Call option: CV with 6-month Expiration.

	BS-COVAR			VAR-COVAR		
k	$V_c^{(c)}$	$SE\left(V_c^{(c)}\right)$	$95\%\ CI\left(V_c^{(c)}\right)$	$V_c^{(c)}$	$SE\left(V_c^{(c)}\right)$	$95\%\ CI\left(V_c^{(cmc)}\right)$
1000	167.04	0.58080	(165.89990, 168.17670)	163.30	0.000081	(163.29762, 163.29793)
5000	166.90	0.25590	(166.40060, 167.40390)	163.30	0.000036	(163.29770, 163.29785)
10000	166.70	0.17430	(166.36250, 167.04590)	163.30	0.000026	(163.29772, 163.29783)
15000	167.03	0.15340	(166.73240, 167.33370)	163.30	0.000021	(163.29773, 163.29782)
20000	167.13	0.12790	(166.88220, 167.38340)	163.30	0.000018	(163.29774, 163.29781)
50000	166.90	0.07904	(166.74406, 167.05390)	163.30	0.000011	(163.29775, 163.2978)
100000	167.01	0.05793	(166.89219, 167.11928)	163.30	0.000008	(163.29776, 163.29779)

6.2.3 9-month Call Option

Tables 14, 15, and 16 provide results for European call option price for 9-month expiration date. Table 14 shows that regardless of the number of simulations of the European call option value, the BS-COVAR model produce values lower than using the VAR-COVAR model. Using BS-COVAR model gave values in the range of \$82.01 to \$85.27. While VAR-COVAR reports values in the range of \$165.59 to \$187.20, for the same expiration date. Noted that for the BS-COVAR and the VAR-COVAR option prices as k increases the $SE(V_c^{(cmc)})$ reduces as well. There is a very conspicuous difference in the value of the European call option of the BS-COVAR model and the VAR-COVAR model, which were reported as \$82.19 ($k = 100000$, $SE(V_c^{(cmc)}) = 0.0004756$, and 95% $CI(V_c^{(cmc)}) = (\$82.189, \$82.191)$), and \$170.96 ($k = 100000$, $SE(V_c^{(cmc)}) = 0.000927$, and 95% $CI(V_c^{(cmc)}) = (\$170.958, \$170.963)$), respectively.

Table 15 indicates that the difference between the value of the call option of the BS-COVAR and the VAR-COVAR stays about the same regardless to the increases of the value of k. However, the value of BS-COVAR model was quite lower compared to the value of the VAR-COVAR model. It has to be noted that for BS-COVAR and VAR-COVAR option prices as k increases the $SE(V_c^{(a)})$ significantly decrease as well. Using BS-COVAR model produced values in the range of \$38.79 to \$38.86. While VAR-COVAR reported values in the range \$58.78 to \$58.95, for the same expiry date. The value of the European call option based on the the BS-COVAR and the VAR-COVAR models, were reported as \$38.79 ($k = 100000$, $SE(V_c^{(a)}) = 0.0001$, and 95% $CI(V_c^{(a)}) = (\$38.79, \$38.79)$), and \$58.79 ($k = 100000$, $SE(V_c^{(a)}) = 0.000134$, and 95% $CI(V_c^{(a)}) = (\$58.79, \$58.79)$), respectively.

Table 16 illustrates that the BS-COVAR model always produced larger values of a European call option price than the VAR-COVAR model. However, the BS-VAR and VAR-COVAR model obtain value in the range \$183.79 to \$184.63 and \$165.59 to \$187.20 respectively. It should be noted that BS-COVAR and VAR-COVAR option prices as k increases the $SE(V_c^{(c)})$ decrease as well. Bessides, the BS-COVAR was estimated at \$170.96 ($k = 100000$, $SE(V_c^{(c)}) = 0.087690$, and 95% $CI(V_c^{(c)}) = (\$184.04738, \$184.39113)$). Meanwhile, the VAR-COVAR option price for the same maturity period was estimated at \$170.96 ($k = 100000$, $SE(V_c^{(c)}) = 0.000987$, and 95% $CI(V_c^{(c)}) =$

$(\$170.95807, \$170.96193))$.

Table 14: Call option: Crude MC with 9-month Expiration.

	BS-COVAR			VAR-COVAR		
k	$V_c^{(cmc)}$	$SE\left(V_c^{(cmc)}\right)$	$95\%\ CI\left(V_c^{(cmc)}\right)$	$V_c^{(cmc)}$	$SE\left(V_c^{(cmc)}\right)$	$95\%\ CI\left(V_c^{(cmc)}\right)$
1000	85.27	0.048814	$(85.174, 85.366)$	187.20	0.103503	$(186.997, 187.403)$
5000	82.73	0.009574	$(82.711, 82.749)$	165.59	0.019270	$(165.552, 165.628)$
10000	82.01	0.004739	$(82.001, 82.019)$	170.91	0.009845	$(170.891, 170.929)$
15000	82.60	0.003174	$(82.594, 82.606)$	170.47	0.006599	$(170.457, 170.483)$
20000	82.31	0.002385	$(82.305, 82.315)$	171.44	0.005001	$(171.430, 171.450)$
50000	82.82	0.000957	$(82.818, 82.822)$	169.80	0.001973	$(169.796, 169.804)$
100000	82.19	0.000476	$(82.189, 82.191)$	170.96	0.000987	$(170.958, 170.962)$

Table 15: Call option: AV with 9-month Expiration.

	BS-COVAR			VAR-COVAR		
k	$V_c^{(a)}$	$SE\left(V_c^{(a)}\right)$	$95\%\ CI\left(V_c^{(a)}\right)$	$V_c^{(a)}$	$SE\left(V_c^{(a)}\right)$	$95\%\ CI\left(V_c^{(a)}\right)$
1000	38.86	0.002599	(38.85491, 38.86509)	58.95	0.010677	(58.92907, 58.97093)
5000	38.81	0.000543	(38.80894, 38.81106)	58.80	0.005089	(58.79003, 58.80997)
10000	38.79	0.000180	(38.78965, 38.79035)	58.78	0.002213	(58.77566, 58.78434)
15000	38.79	0.000086	(38.78983, 38.79017)	58.82	0.001324	(58.81741, 58.82259)
20000	38.79	0.000049	(38.7899, 38.79010)	58.83	0.000812	(58.82841, 58.83159)
50000	38.79	0.000037	(38.78993, 38.79007)	58.78	0.000266	(58.77948, 58.78052)
100000	38.79	0.000010	(38.78998, 38.79002)	58.79	0.000134	(58.78974, 58.79026)

Table 16: Call option: CV with 9-month Expiration.

	BS-COVAR			VAR-COVAR		
k	$V_c^{(c)}$	$SE\left(V_c^{(c)}\right)$	$95\%\ CI\left(V_c^{(c)}\right)$	$V_c^{(c)}$	$SE\left(V_c^{(c)}\right)$	$95\%\ CI\left(V_c^{(c)}\right)$
1000	184.63	0.8352	(182.98820, 186.26200)	187.20	0.103500	(186.99710, 187.40290)
5000	184.05	0.3852	(183.29460, 184.80440)	165.59	0.019270	(165.55223, 165.62777)
10000	183.79	0.2686	(183.26740, 184.32030)	170.91	0.009844	(170.89071, 170.92930)
15000	184.12	0.2196	(183.68500, 184.54600)	170.47	0.006599	(170.45707, 170.48293)
20000	184.05	0.1939	(183.67450, 184.43470)	171.44	0.005001	(171.43020, 171.44980)
50000	184.14	0.1239	(183.89920, 184.38500)	169.80	0.001973	(169.79613, 169.80387)
100000	184.22	0.0876	(184.04738, 184.39113)	170.96	0.000987	(170.95807, 170.96193)

6.2.4 1-year Call Option

Tables 17, 18, and 19 display results for European call option price for 1-year expiration date. Table 17 shows that the BS-COVAR model offers a lower value of the European call option than the VAR-COVAR model, regardless of the value of k. Notice that for the BS-COVAR and the VAR-COVAR option prices as the number of simulations rises the $SE(V_c^{(cmc)})$ falls. The results of the simulation presented in the table report the value of the BS-COVAR European call option to be in the range \$87.73 to \$91.46, while the VAR-COVAR model provide a value in the range \$172.40 to \$190.06. Observe that the BS-COVAR call option price was reported as \$87.90 ($k = 100000$, $SE(V_c^{(cmc)}) = 0.0005087$, and 95% $CI(V_c^{(cmc)}) = (\$87.899, \$87.901)$), while the VAR-COVAR was reported at \$173.58 ($k = 100000$, $SE(V_c^{(cmc)}) = 0.0010022$, and 95% $CI(V_c^{(cmc)}) = (\$173.578, \$173.582)$).

Table 18 displays that the BS-COVAR model offer a closer value of the European call option with 1-year expiry date to the VAR-COVAR model as the value of k increases. The results of the simulation presented in the table reports the value of the BS-COVAR model of a European call option price was in the range \$40.42 to \$40.49. For BS-COVAR and VAR-COVAR option prices as the number of simulations paths increases the $SE(V_c^{(a)})$ decreases. The VAR-COVAR model provide a value in the range \$60.03 to \$60.21. Observe that the BS-COVAR call option was reported as \$40.42 ($k = 100000$, $SE(V_c^{(a)}) = 0.000013$, and 95% $CI(V_c^{(a)}) = (\$40.42, \$40.42)$), while the VAR-COVAR was esitmated at \$60.04 ($k = 100000$, $SE(V_c^{(a)}) = 0.000137$, and 95% $CI(V_c^{(a)}) = (\$60.04, \$60.04)$).

Table 19 indicate that regardless of the number of simulations of the European call option value, the BS-COVAR model produce values higher than using the VAR-COVAR model. Using BS-COVAR model reported values in the range of \$201.65 to \$202.97. While VAR-COVAR reports value in that converges to \$192.10, for the same expiration date. The BS-COVAR and the VAR-COVAR option prices as k increases the $SE(V_c^{(c)})$ reduces as well. Meanwhile, the BS-COVAR model and the VAR-COVAR model, which were reported as \$201.97 ($k = 100000$, $SE(V_c^{(c)}) = 0.113500$, and 95% $CI(V_c^{(c)}) = (\$201.75210, \$202.19690)$), and \$192.10 ($k = 100000$, $SE(V_c^{(c)}) = 0.000014$, and 95% $CI(V_c^{(c)}) = (\$192.09983, \$192.09988)$), respectively.

Table 17: Call option: Crude MC with 1-year Expiration.

	BS-COVAR			VAR-COVAR		
k	$V_c^{(cmc)}$	$SE\left(V_c^{(cmc)}\right)$	$95\%\ CI\left(V_c^{(cmc)}\right)$	$V_c^{(cmc)}$	$SE\left(V_c^{(cmc)}\right)$	$95\%\ CI\left(V_c^{(cmc)}\right)$
1000	91.46	0.052308	(91.357, 91.563)	190.06	0.105087	(189.854, 190.266)
5000	88.50	0.010246	(88.480, 88.520)	168.12	0.019565	(168.082, 168.158)
10000	87.73	0.005068	(87.720, 87.740)	173.53	0.009995	(173.510, 173.550)
15000	88.42	0.003397	(88.413, 88.427)	173.08	0.006700	(173.067, 173.093)
20000	88.02	0.002551	(88.015, 88.025)	174.06	0.005077	(174.050, 174.070)
50000	88.63	0.001024	(88.628, 88.632)	172.4	0.002003	(172.396, 172.404)
100000	87.90	0.000509	(87.899, 87.901)	173.58	0.001002	(173.578, 173.582)

Table 18: Call option: AV with 1-year Expiration.

	BS-COVAR			VAR-COVAR		
k	$V_c^{(a)}$	$SE\left(V_c^{(a)}\right)$	$95\%\ CI\left(V_c^{(a)}\right)$	$V_c^{(a)}$	$SE\left(V_c^{(a)}\right)$	$95\%\ CI\left(V_c^{(a)}\right)$
1000	40.49	0.003328	(40.48114, 40.49886)	60.21	0.010911	(60.18862, 60.23139)
5000	40.44	0.000697	(40.42978, 40.43022)	60.05	0.005200	(60.03981, 60.06019)
10000	40.42	0.000233	(40.41932, 40.42068)	60.03	0.002261	(60.02557, 60.03443)
15000	40.42	0.000114	(40.41995, 40.42005)	60.07	0.001353	(60.06735, 60.07265)
20000	40.42	0.000066	(40.41977, 40.42023)	60.08	0.000830	(60.07837, 60.08163)
50000	40.42	0.000048	(40.41993, 40.42007)	60.03	0.000272	(60.02947, 60.03053)
100000	40.42	0.000013	(40.41996, 40.42004)	60.04	0.000137	(60.03973, 60.04027)

Table 19: Call option: CV with 1-year Expiration.MMMM

	BS-COVAR			VAR-COVAR		
k	$V_c^{(c)}$	$SE\left(V_c^{(c)}\right)$	$95\%\ CI\left(V_c^{(c)}\right)$	$V_c^{(c)}$	$SE\left(V_c^{(c)}\right)$	$95\%\ CI\left(V_c^{(c)}\right)$
1000	202.06	1.1400	(199.8200, 204.2900)	192.10	0.000136	(192.09959, 192.10012)
5000	201.65	0.5017	(200.6617, 202.6285)	192.10	0.000061	(192.09974, 192.09998)
10000	201.35	0.3451	(200.6778, 202.0307)	192.10	0.000043	(192.09977, 192.09994)
15000	202.02	0.2971	(201.4346, 202.5994)	192.10	0.000035	(192.09979, 192.09992)
20000	202.21	0.2521	(201.7184, 202.7067)	192.10	0.000030	(192.0998, 192.09991)
50000	201.82	0.1566	(201.5093, 202.1232)	192.10	0.000019	(192.09982, 192.09989)
100000	201.97	0.1135	(201.7521, 202.1969)	192.10	0.000014	(192.09983, 192.09988)

7 Conclusions

In this paper, the European call option price based on the BS-COVAR model retained all the B-S properties with the exception of the constant risk-free interest rate assumption. Rather, the 3-month Treasury bill rate model replaced the constant risk-free interest rate. This modeling approach for the BS-COVAR model was adopted to provide a common ground for comparison between the BS-COVAR and the VAR-COVAR based European call option prices at varying maturity dates.

The crude MC simulation results of the comparison between the BS-COVAR and VAR-COVAR European call option prices showed that the price of the BS-COVAR model was consistently lower than that of the VAR-COVAR model for all expiration dates (3-month, 6-month, 9-month, and 1-year). Though, the results of the simulation incorporating AV indicated that the European call option price from the VAR-COVAR and BS-COVAR models for all expiration dates were closer compared to the results from the crude MC method. Besides, the effects of the simulation including CV showed that the European call option price from the VAR-COVAR and BS-COVAR models for all expiration dates were closer compared to the results from the crude MC method.

In conclusion, this research makes two contributions. First, the constant risk-free rate assumption under the B-S formula was relaxed via the COVAR model. Second, the B-S formula relies on a single stock price. However, this paper has considered stock market indices within the framework of a VAR model where the selected model in the valuation process was permitted to be dependent on other stock indices. With these modifications to the standard B-S formula, it can be inferred that the European call option price emanating from the VAR-COVAR model provides a more realistic market value of the call option in comparison with the value based on the BS-COVAR model.

References

[1] Ruey S. Tsay, *Analysis of Financial Time Series*, Wiley, New Jersey, 2010.

[2] Fischer Black, Myron Scholes, The Pricing of Options and Corporate Liabilities, *The Journal of Political Economy*, **81**(3), (1973), 637-654.

[3] Robert C. Merton, Theory of Rational Option Pricing, *The Bell Journal of Economics and Management Science*, **4**(1), (1973), 141-183.

[4] Robert F. Engle, Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation, *Econometrica*, **50**(4), (1982), 987-1008.

[5] Tim Bollerslev, Generalized Autoregressive Conditional Heteroskedasticity, *Journal of Econometrics*, **31**, (1986), 307-327.

[6] Jan Kallsen, Murad S. Taqqu, Option Pricing in ARCH-type Models, *Mathematical Finance*, **8**(1), (1998), 13-26.

[7] Jaesun Noh, Robert F. Engle, Alex Kane, A Test of Efficiency for the S&P Index Option Market Using Variance Forecasts, *National Bureau of Economic Research*, **2**, (1993), 17-30.

[8] K. I. Amin, Pricing Options On Risky Assets In A Stochastic Interest Rate Economy, *Math. Finance*, **2**(4), (1992), 217-237.

[9] Gurdip Bakshia, Charles Caob, Zhiwu Chenc, Pricing and hedging long-term options, *Econometric*, **94**(1-2), (2000), 277-318.

[10] R. Rabinovitch, Pricing Stock and Bond Options when the Default-Free Rate is Stochastic, *J. Financ.Quant. Anal*, **24**(4), (1989), 447-457.

[11] H. Akaike, A new look at the statistical model identification, *IEEE Transactions on Automatic Control*, **19**(6), (1974), 716-723.

[12] Gideon Schwarz, Estimating the Dimension of a Model, *The Annals of Statistics*, **6**(2), (1978), 461-464.

[13] E. J. Hannan, B. G. Quinn, The Determination of the Order of an Autoregression, *Journal of the Royal Statistical Society Series B (Methodological)*, **41**(2), (1979), 190-195.

[14] Said E. Said, David A. Dickey, Testing for Unit Roots in Autoregressive-Moving Average Models of Unknown Order, *Biometrika*, **71**(3), (1984), 599-607.

[15] Peter C. B. Phillips, Pierre Perron, Testing for Unit Root in Time Series Regression, *Biometrika*, **75**(2), (1988), 335-346.

[16] G. C. Tiao and G. E. P. Box, Modeling Multiple Times Series with Applications, *Journal of the American Statistical Association*, **76**(376), (1981), 802-816.

[17] James D. Hamilton, *Time Series Analysis*, Princeton University Press, New Jersey, 1994.

[18] Helmut Lutkepohl, *Handbook of Matrices*, Wiley, New York, 1997.

[19] R. Tsay, *Analysis of Financial Time Series*, Wiley, New York, 2005.

[20] Gregory C. Reinsel, Sung K. Ahn, Vector autoregressive models with unit roots and redused rank structure: estimation. Likelihood ratio test, and forecasting, *Journal of Time Series Analysis*, **13**(4), (1992), 353-375.

[21] Carlos M. Jarque, Anil K. Bera, A Test for Normality of Observations and Regression Residuals, *International Statistical Institute (ISI)*, **55**(2), (1987), 163-172.

[22] Carlos M. Jarque, Anil K. Bera, Efficient tests for normality, homoscedasticity and serial independence of regression residuals, *Economics Letters*, **6**(3), (1980), 255-259.

[23] Anil K. Bera, Carlos M. Jarque, Efficient tests for normality, homoscedasticity and serial independence of regression residuals: Monte Carlo Evidence, *Economics Letters*, **7**(4), (1981), 313-318.

[24] Sren Johansen, Katarina Juselius, Maximum Likelihood Estimation and Inference on Cointegration With Application to Demand for Money, *Oxford Bulletin of Economics and Statistics*, **52**(2), (1990), 169-210.

[25] James G. MacKinnon, Critical Values for Cointegration Tests, *Queens Economics Department Working Paper*, **1227**, Department of Economics, (2010).

[26] Richard A. Johnson, Dean W. Wichern, *Applied Multivariate Statistical Analysis*, Pearson, New Jersey, 2007.

[27] Yahoo https://finance.yahoo.com.

[28] Federal Reserve Economic Data-St. Louis https://fred.stlouisfed.org/.

[29] Elke Korn, Ralf Korn, *Monte Carlo Methods and Models in Finance and Insurance*, CRC Press, Florida, 2010.

[30] Sheldon M. Ross, *Introduction to probability models*, Elsevier Inc., Amsterdam, 2010.

[31] Dirk P. Kroese, Thomas Taimre, Zdravko I. Botev, *Handbook of monte carlo methods*, Wiley, New York, 2013.