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Using Intraday Statistics for the Estimation of the Return Variance

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Abstract

This paper proposes new estimators for the daily return variance which are based on common intraday statistics (opening, high, low, and closing prices). These estimators utilize information contained in products of absolute values of uncorrelated intraday statistics. An empirical study of nine components of the Dow Jones Industrial Average from 1962-01-02 to 2013-03-13 shows that they outperform existing estimators over all stocks and time periods.

JEL classification numbers: C13, G1
Keywords: Range-based estimation, Brownian motion, Folded normal distribution

1 Introduction

Lopez [7] showed that the daily squared return is an unbiased but extremely imprecise estimator for daily volatility. Estimators based on intraday statistics are, of course, much more precise. Parkinson [8] proposed a range-based estimator and Garman and Klass [6] optimized this estimator by introducing weights and taking joint effects into account. Unlike estimators obtained from squared high-frequency intraday returns (realized volatility; see, e.g., [2]), these range-based estimators are relatively robust to various types of intraday patterns (for a more detailed discussion, see [9]).
This paper proposes to improve the optimized estimator of Garman and Klass [6] by including products of absolute values of uncorrelated intraday statistics in addition to the already included squares of intraday statistics and products of correlated intraday statistics. The new estimators are introduced in the third subsection of the next section. The first

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two subsections discuss several existing estimators. The estimators considered in Subsection 2.1 are fully specified whereas those considered in Subsection 2.2 depend on an unknown parameter. The latter subsection also investigates the consequences of misspecification of the unknown parameter. Section 3 presents the results of an empirical study which compares the performance of the different estimators. Section 4 concludes.

2 Methods

2.1 Fully Specified Estimators for the Return Variance

Feller [5] derived the asymptotic density function:

$$\delta(r|t) = 8\sum_{k=1}^{\infty}(-1)^{k-1}k^2\phi\left(\tfrac{kr}{\sqrt{t}}\right)$$

of the range:

$$R(t) = \max_{0\le s\le t}(W(s)) - \min_{0\le s\le t}(W(s))$$

of a Wiener process $W(s)$, $s\geq 0$, in the interval $[0,t]$, where ϕ denotes the normal density function with zero mean and unit variance. It follows that:

$$E(R^k(t)) = \tfrac{4}{\sqrt{\pi}}\Gamma\left(\tfrac{k+1}{2}\right)(1-\tfrac{4}{2^k})\zeta(k-1)\sqrt{(2t)^k}$$

for $k\in\{1,3,4,5,...\}$ and

$$E(R^k(t)) = \tfrac{4}{\sqrt{\pi}}\Gamma\left(\tfrac{k+1}{2}\right)\eta(k-1)\sqrt{(2t)^k}$$

for k=2 [8], where:

$$\zeta(u) = \sum_{j=1}^{\infty} j^{-u}$$

is the Riemann zeta function and:

$$\eta(u) = \sum_{j=1}^{\infty}(-1)^{k-1} j^{-u}$$

(for tables of values of ζ see [1]).
In particular, we have:

$$\zeta(0) = -\tfrac{1}{2},\ \ \eta(1) = \log(2),\ \ \zeta(2) = \tfrac{\pi^2}{6},\ \ \zeta(3) \approx 1.2020569.$$

Thus,

$$E(R(t)) = 2\sqrt{\tfrac{2}{\pi}}\sqrt{t}\ ,$$

$$E(R^2(t)) = 4\log(2)\,t\ ,$$

$E(R^3(t)) = \frac{2}{3}\sqrt{2\pi^3}\sqrt{t^3}$,

and $E(R^4(t)) = 9\zeta(3)\,t^2$.

Because log stock prices are commonly modeled as a Brownian motion with unknown volatility σ (and drift μ), the moments

$E(R) = 2\sigma\sqrt{\frac{2}{\pi}}$, $E(R^2) = 4\sigma^2 \log(2)$, $E(R^3(t)) = \frac{2}{3}\sqrt{2\pi^3}\sigma^3$, $E(R^4) = 9\zeta(3)\sigma^4$,

of the range R of the Brownian motion $B(s)=\sigma W(s)$, $s\geq 0$, on the interval [0,1] can be used for the estimation of the variance σ^2 of the return over a unit time period as well as for the evaluation of the properties of the respective estimators. In the case of daily prices, μ is negligible compared to σ and can therefore be disregarded. Suppose that the opening, high, low, and closing prices of N trading days are available. Because volatility changes over time, only the last $n<N$ trading days are used for the estimation of the present volatility. Let (O_i,H_i,L_i,C_i), $i=1,...,n$, be the corresponding sample of the log intraday statistics. The most obvious improvement over the standard estimator

$$\hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^{n}\left(C_i - C_{i-1}\right)^2 \tag{1}$$

is obtained simply by dividing each trading day in a period when the market is closed and a period when it is open. Denoting the corresponding return variances by σ_1^2 and σ_2^2, respectively, and assuming that $\sigma^2 = \sigma_1^2 + \sigma_2^2$, leads straightforwardly to the estimator

$$\hat{\sigma}^2 = \hat{\sigma}_1^2 + \hat{\sigma}_2^2 = \frac{1}{n}\sum_{i=1}^{n}(O_i - C_{i-1})^2 + \frac{1}{n}\sum_{i=1}^{n}(C_i - O_i)^2 . \tag{2}$$

A further improvement is obtained by utilizing additional information which is available for the second period. Assume for the moment that the time interval [0,1] represents only the time of the trading session, not the whole 24 hours trading day. Then the application of the formula for the second noncentral moment of the range of a Brownian motion yields estimators for σ_2^2 and σ^2, respectively, which are based on the intraday ranges $R_i=H_i-L_i$, $i=1,...,n$, i.e.,

$$\hat{\sigma}_P^2 = \frac{1}{4\log(2)}\frac{1}{n}\sum_{i=1}^{n}R_i^2 \tag{3}$$

[8] and

$$\hat{\sigma}_R^2 = \hat{\sigma}_1^2 + \hat{\sigma}_P^2 . \tag{4}$$

A less obvious but possibly more efficient version of (4) was proposed by Garman and Klass [6]. Their estimator will be discussed in more detail in the next subsection.

As pointed out by Yang and Zhang [12], the no drift assumption may not be appropriate in certain situations, e.g., in strong bull markets. An estimator for σ_2^2 which is independent of μ is given by

$$\hat{\sigma}_{RS}^2 = \frac{1}{n}\sum_{i=1}^{n}\Big(H_i^*(H_i^* - C_i^*) + L_i^*(L_i^* - C_i^*)\Big), \tag{5}$$

where $H_i^* = H_i - O_i$, $L_i^* = L_i - O_i$, and $C_i^* = C_i - O_i$ [10, 11]. However, the composite estimator

$$\hat{\sigma}_{cRS}^2 = \hat{\sigma}_1^2 + \hat{\sigma}_{RS}^2 \tag{6}$$

again depends on μ, albeit to a lesser degree than the other estimators. Yang and Zhang [12] therefore proposed the further development

$$\hat{\sigma}_{YZ}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(O_i^* - \overline{O^*})^2 + k\frac{1}{n-1}\sum_{i=1}^{n}(C_i^* - \overline{C^*})^2 + (1-k)\hat{\sigma}_{RS}^2, \tag{7}$$

Where: $\overline{O^*}$ denotes the sample mean of $O_i^* = O_i - C_{i-1}$, i=1,...,n, which is truly independent of μ.. Not surprisingly, this independence comes with a price. The estimator (7) cannot be based on a single day, i.e., n must be greater than one. Yang and Zhang [12] recommended to set the constant k to

$$k_0 = \frac{0.34}{1.34 + \frac{n+1}{n-1}}$$

in order to minimize the variance. In the case of n=10 (two weeks), they expected that the variance of the resulting estimator is typically more than seven times smaller than that of the benchmark estimator (1).

Volatility estimators based on intraday statistics are often used for the evaluation of volatility models such as ARCH [4] and GARCH [3] models. These models differ in the way they describe the dependence of today's variance on the volatility in the past. If n is large, the variance estimates produced by the estimators discussed above might closely resemble the conditional variance estimates produced by some of the volatility models. Only in the case n=1, a fair comparison of volatility models is therefore possible. Estimators such as (7) are therefore not universally usable and will therefore not be considered in the rest of this paper.

2.2 Estimators Depending on an Unknown Parameter

The estimators of the previous subsection were derived under idealizing assumptions, e.g., the normality and serial uncorrelatedness of returns. More efficient estimators can be derived under additional assumptions. The key assumption in this subsection is that the parameter $f = \sigma_1^2 / \sigma^2$ is known. Under this assumption, Garman and Klass [6] showed that even the simple estimator

$$\hat{\sigma}_2^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2f}(O_i - C_{i-1})^2 + \frac{1}{2(1-f)}(C_i - O_i)^2\right) \tag{8}$$

is already "two times better" than the benchmark estimator. Indeed, the expected value of each term is given by

$$\frac{1}{2f}E(O_i - C_{i-1})^2 + \frac{1}{2(1-f)}E(C_i - O_i)^2 = \sigma^2 = E(C_i - C_{i-1})^2$$

and the variance by

$$\frac{\sigma^4}{4}Var\frac{(O_i - C_{i-1})^2}{f\sigma^2} + \frac{\sigma^4}{4}Var\frac{(C_i - O_i)^2}{(1-f)\sigma^2} = \frac{\sigma^4}{4}2 + \frac{\sigma^4}{4}2$$

$$< \sigma^4 2 = \sigma^4 Var\frac{(C_i - C_{i-1})^2}{\sigma^2} = Var(C_i - C_{i-1})^2 .$$

To further improve efficiency, Garman and Klass [6] proposed their composite estimator:

$$\hat{\sigma}^2_{cGK} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{a}{f}O_i^{*2} + \frac{1-a}{1-f}\frac{R_i^2}{4\log(2)}\right), \tag{9}$$

and their "best" estimator

$$\hat{\sigma}^2_{GK} = \frac{a}{f}\frac{1}{n}\sum_{i=1}^{n} O_i^{*2} + \frac{1-a}{1-f}\frac{1}{n}\sum_{i=1}^{n}\left(0.511R_i^2 - 0.019\left(C_i^*(H_i^* + L_i^*) - 2H_i^*L_i^*\right) - 0.383C_i^{*2}\right) \tag{10}$$

The first is closely related to estimator (4) and the second takes also into account the joint effects between the different intraday statistics. The variances of the estimators (9) and (10) are minimized when a=0.17 and a=0.12, respectively, and in these cases they are more than six times and more than eight times, respectively, "better" than the benchmark estimator (1).

The application of the estimators discussed in this subsection requires the specification of the unknown parameter f. Clearly, it cannot simply be obtained from the physical time interval during which markets are closed. It must rather be estimated from historical data sets. Yang and Zhang [12] did just that and found values between 0.18 and 0.30. The value chosen in a real applications might therefore differ significantly from the true value. To illustrate the possible consequences of such a misspecification, the simple estimator (8) is used. If a different value g is used instead of f, the bias of this estimator is for each term given by

$$E\left(\frac{1}{2g}(O_i - C_{i-1})^2 + \frac{1}{2(1-g)}(C_i - O_i)^2\right) - \sigma^2 = \left(\frac{f}{2g} + \frac{1-f}{2(1-g)} - 1\right)\sigma^2,$$

its variance by

$$\frac{\sigma^4}{4}\frac{f^2}{g^2}Var\frac{(O_i - C_{i-1})^2}{f\sigma^2} + \frac{\sigma^4}{4}\frac{(1-f)^2}{(1-g)^2}Var\frac{(C_i - O_i)^2}{(1-f)\sigma^2} = \frac{\sigma^4}{4}\frac{f^2}{g^2}2 + \frac{\sigma^4}{4}\frac{(1-f)^2}{(1-g)^2}2,$$

and its mean squared error (MSE) by

$$\frac{\sigma^4}{4}\left(2\frac{f^2}{g^2} + 2\frac{(1-f)^2}{(1-g)^2} + \left(\frac{f}{g} + \frac{1-f}{(1-g)} - 2\right)^2\right).$$

Figure 1.e shows that g=0.5 is a safe choice. Indeed, the MSE is in this case always smaller than that of the benchmark estimator (1), because

$$\frac{\sigma^4}{4}\left(2\frac{f^2}{\frac{1}{4}} + 2\frac{(1-f)^2}{\frac{1}{4}} + \left(\frac{f}{\frac{1}{2}} + \frac{1-f}{\frac{1}{2}} - 2\right)^2\right) = 2\sigma^4\left(f^2 + (1-f)^2\right) < 2\sigma^4$$

if $0<f<1$. Figures 1.a-c show the MSE as a function of g for f=0.3, f=0.5, and f=0.7, respectively. Not surprisingly, the MSE is small whenever g is close to f, but it increases quickly as g moves away from the true value. Note that the choice $g=f$ is optimal only if f=0.5. Under the plausible assumption that $f\leq 0.5$, a choice of $g<0.5$ would also be safe. The solutions of

$$\frac{\sigma^4}{4}\left(2\frac{f^2}{g^2} + 2\frac{(1-f)^2}{(1-g)^2} + \left(\frac{f}{g} + \frac{1-f}{(1-g)} - 2\right)^2\right) = 2\sigma^4$$

are just the roots of the polynomial

$$P(g) = 4g^4 + (8f-12)g^3 - (8f^2+4f-5)g^2 + (8f^2+2f)g - 3f^2 = 0.$$

In the worst case, i.e., f=0.5, there are four real roots, two of which are in the interval (0,1), namely g_1=0.2715749 and g_2=1-g1=0.7284251. Figures 2.d-f show the MSE as a function of f for g=g_1, g=0.5, and g=g_2, respectively. The MSE is for g=g_1 always smaller than that of the benchmark estimator if f<0.5, equality holds if f=0.5.

Figure 2 shows for each calendar year from 1962 to 2012 and for each of nine stocks the estimate of f obtained by dividing the sample variance of $O_i^* = O_i - C_{i-1}$ by the sample variance of $C_i - C_{i-1}$. Similar estimates were obtained when non-central second moments were used or when the sum of the second moments of $O_i^* = O_i - C_{i-1}$ and $C_i^* = C_i - O_i$ was used as divisor. For each stock, the estimates were well below 0.5 for most of the time, hence choosing f=g_1 seems reasonable. In practice, it will hardly make any difference whether g_1 is used or simply 0.25 or 0.3.

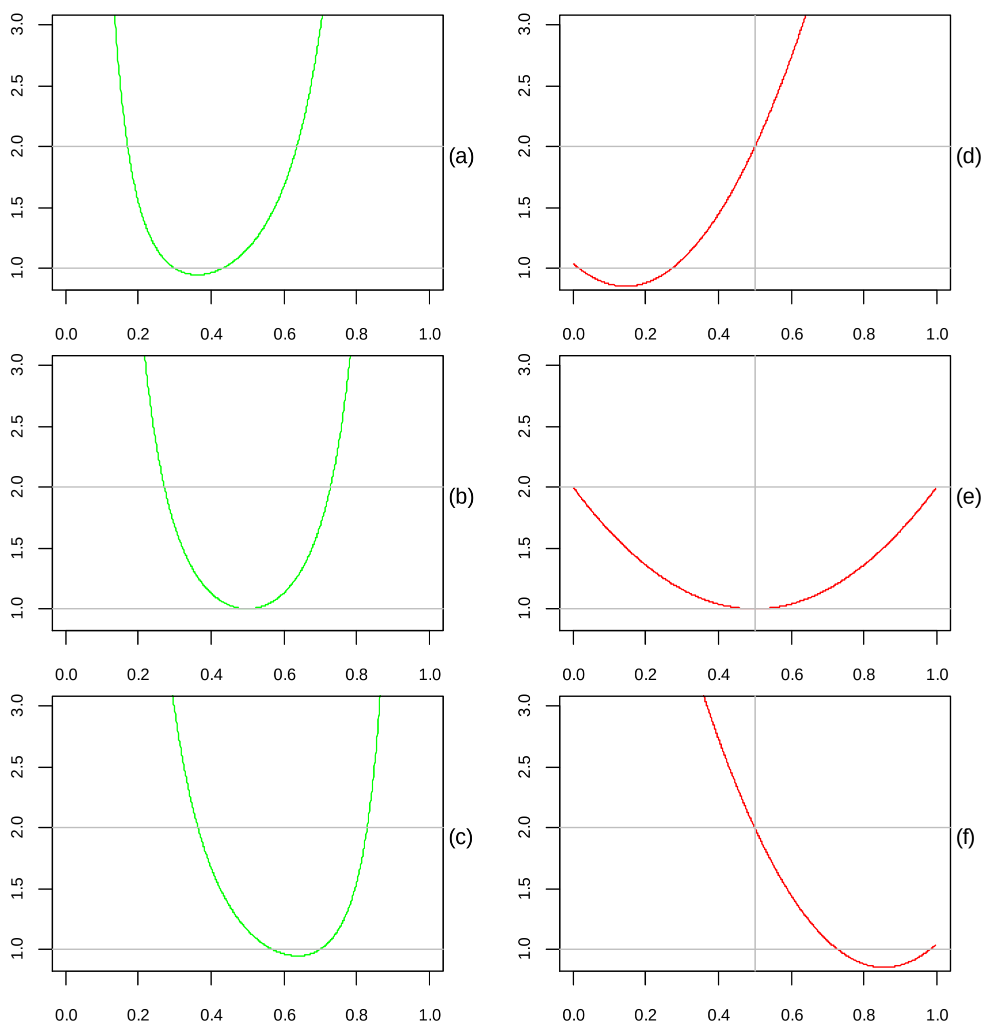

Figure 1: MSE of the estimator (8) as a function of g for f=0.3 (a), f=0.5 (b), f=0.7 (c) and as a function of f for g=g_1 (d), g=0.5 (e), g=g_2 (f), respectively

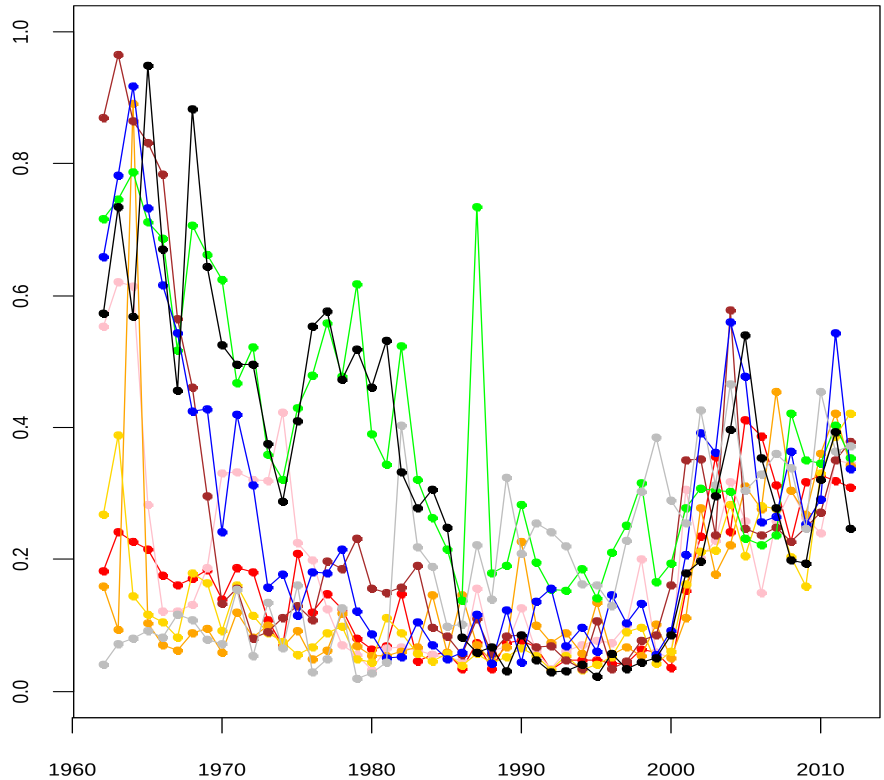

Figure 2: Estimates of the parameter *f* obtained by dividing the sample variance of $O_i - C_{i-1}$ by the sample variance of $C_i - C_{i-1}$ for the calendar years from 1962 to 2012. Data: AA (red), BA (pink), CAT (orange), DD (gold), DIS (brown), GE (green), HPQ (blue), IBM (gray), KO (black)

2.3 Estimators using Additional Information

In the case of known *f*, the estimators (9) and (10) seem to be the methods of choice. The main advantage of (9) is its simplicity, whereas (10) uses more of the available information by taking also the joint effects between H_i^*, L_i^*, and C_i^* into account. However, there are other product terms which are possibly more important. Because of the approximate uncorrelatedness of the factors, it may be justified to ignore products such as $O_i^* C_i^*$ or $O_i^* R_i$. But this is certainly not true for products such as $|O_i^*||C_i^*|$ or $|O_i^*|R_i$. The construction of improved estimators which utilize the latter quantities is straightforward. Under the usual idealizing assumptions, the identities:

$$E|O_i^*|\,E|C_i^*| = \sqrt{f\sigma^2}\sqrt{\tfrac{2}{\pi}}\sqrt{(1-f)\sigma^2}\sqrt{\tfrac{2}{\pi}} = \tfrac{2}{\pi}\sqrt{f(1-f)}\,\sigma^2$$

and $E\left|O_i^*\right| ER_i = \sqrt{f\sigma^2}\sqrt{\frac{2}{\pi}}\sqrt{(1-f)\sigma^2}\,2\sqrt{\frac{2}{\pi}} = \frac{4}{\pi}\sqrt{f(1-f)}\,\sigma^2$

can immediately be derived just by noting that the expected value of the range of a Brownian motion is given by $2\sigma\sqrt{2/\pi}$ and the expected value of the folded normal distribution by $\sigma\sqrt{2/\pi}$. This motivates the introduction of the simple estimators

$$\hat{\sigma}^2_{OC} = \frac{\pi}{2\sqrt{f(1-f)}}\overline{\left|O^*\right|}\,\overline{\left|C^*\right|} \tag{11}$$

and $$\hat{\sigma}^2_{OR} = \frac{\pi}{4\sqrt{f(1-f)}}\overline{\left|O^*\right|}\,\overline{R}\,, \tag{12}$$

as well as of composite estimators such as:

$$\hat{\sigma}^2_{cOC} = a\hat{\sigma}^2_{GK} + (1-a)\hat{\sigma}^2_{OC} \tag{13}$$

or $$\hat{\sigma}^2_{cOR} = a\hat{\sigma}^2_{GK} + (1-a)\hat{\sigma}^2_{OR}. \tag{14}$$

It is not clear whether trying to choose a in order to minimize the variances of (13) and (14) is a worthwhile exercise. Not only is f unknown, which is aggravated by the fact that this parameter is neither constant across all stocks nor across all time periods (see Figure 2), but the other multiplicative constants are also uncertain, because they have been derived under the unrealistic assumption of normality. Finally, the restriction that the sum of the "weights" equals one is also questionable. In view of the many uncertainties, this restriction cannot reliably ensure unbiasedness. Moreover, accepting a small bias is usually an effective way to reduce the MSE. Basically, there are two alternatives. The first is to use equal weights (i.e., a=0.5) and a "reasonable" value for the parameter f (e.g., $f=g_1$). The second is to use historical data to estimate all weights occurring in an estimator such as

$$\hat{\sigma}^2_{H} = a\overline{O^{*2}} + b\overline{R^2} + c\overline{\left|O^*\right|}\,\overline{R}\,. \tag{15}$$

3 Empirical Results

To assess the performance of the volatility estimators discussed in Section 2, the daily opening, high, low and closing prices of those components of the Dow Jones Industrial Average (DJI) were downloaded from Yahoo!Finance which have the longest history. The selected stocks are Alcoa (AA), Boeing (BA), Caterpillar (CAT), Du Pont (DD), Walt Disney (DIS), General Electric (GE), Hewlett-Packard (HPQ), IBM (IBM), and Coca-Cola (KO). Their prices are available since January 2, 1962. The sample period ends on March 13, 2013. Figure 3 shows the (log) absolute returns (a), the squared returns (b), and the fourth powers of the returns (c). Obviously, any statistical inference based on the fourth powers would be dubious at best. The outcome would depend only on a very small number of extreme returns. The MSE of the competing estimators was therefore estimated in a nonstandard way. Statistics of squared returns were averaged before they were squared again. More precisely, the variance V of an estimator $\hat{\sigma}^2(i)$, which uses only information from the ith trading day, was estimated by $\hat{V}(N)$, where:

$$\hat{V}(m) = \tfrac{1}{2}\left(\sqrt{\tfrac{\pi}{2}}\,\tfrac{1}{N-1}\sum_{i=2}^{m}\left|\hat{\sigma}^2(i) - \hat{\sigma}^2(i-1)\right|\right)^2$$

if $m>1$ and $\hat{V}(1) = 0$. Note that if $\hat{\sigma}^2(i-1)$, $\hat{\sigma}^2(i)$ are i.i.d. $N(\sigma^2 + B, V)$, then:

$\hat{\sigma}^2(i) - \hat{\sigma}^2(i-1) \sim N(0, 2V)$

and $E\left|\hat{\sigma}^2(i) - \hat{\sigma}^2(i-1)\right| = \sqrt{\tfrac{2}{\pi}}\sqrt{2V}$.

The bias B was estimated by $\hat{B}(N)$, where:

$$\hat{B}(m) = \tfrac{1}{N}\sum_{i=1}^{m}\left(\hat{\sigma}^2(i) - (C_i - C_{i-1})^2\right)$$

(note that the missing value C_0 was replaced by O_1).

In general, a rolling window is more appropriate for bias estimation than an expanding window. However, since all competing estimators use only the latest information, the bias is mainly caused by the use of wrong weights for the different intraday statistics. There is no classical bias-variance trade-off.

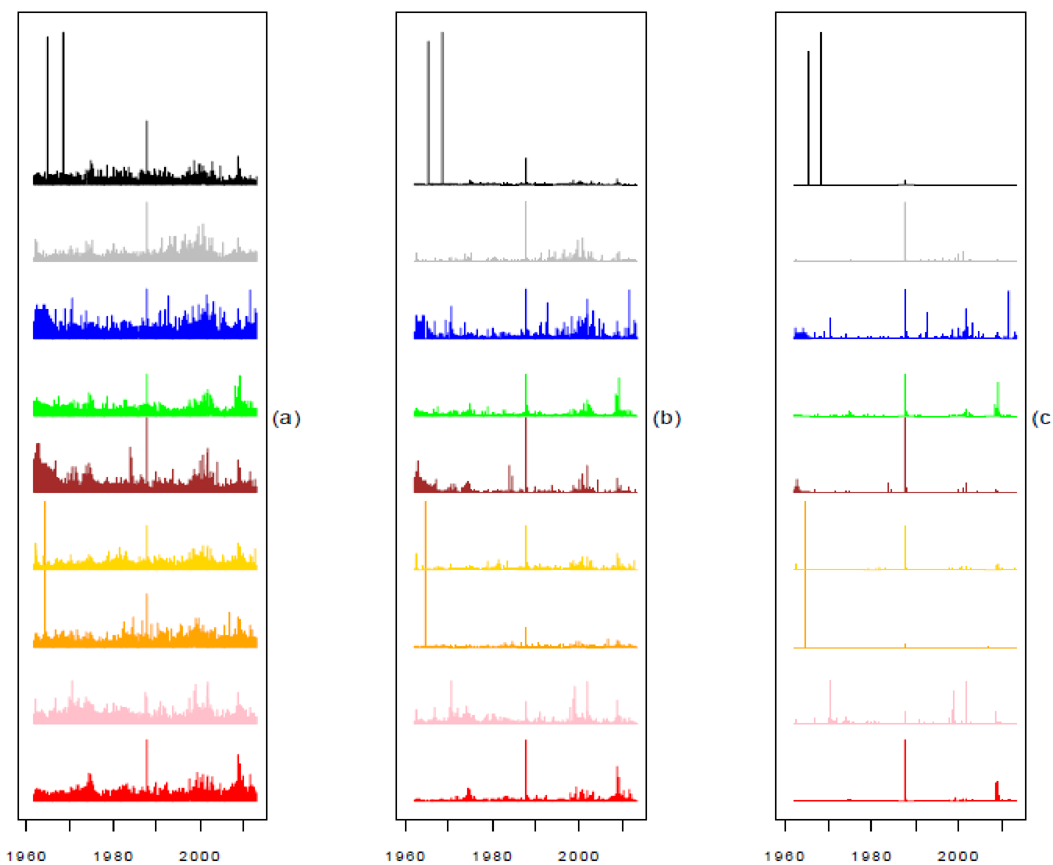

Figure 3: Plots of (log) absolute returns (a), squared returns (b), and fourth powers of returns (c). Data: AA (red), BA (pink), CAT (orange), DD (gold), DIS (brown), GE (green), HPQ (blue), IBM (gray), KO (black)

To detect possible changes over time, the increasing sums $\hat{V}(m)+\hat{B}^2(m)$, m=1,...,N, were plotted against time. Figure 4 shows that the standard estimator (1) and the simplest estimator based on intraday statistics (2) are consistently outperformed by the range based estimator (4) and the drift-robust estimator (6). Further improvements can be obtained by using the more sophisticated estimators depending on the unknown parameter f. Figure 5 shows that the choice of f is not critical. Any value below 0.5 but not too close to zero yields a very competitive version of the estimator (9). To avoid any suspicion of data mining, f=g_1 will be used in the following because this choice is based on theoretical arguments only. Figure 6 shows that in this case (10) is indeed an improvement over (9). The performance is increased if joint effects between the different intraday statistics are taken into account. However, using in addition also $|O_i^*|R_i$ leads to a further improvement. The new estimators (14) (with a=0.5 and f=g_1) and (15) have the smallest MSE. For (15), a=0 was used. The first term was excluded because it is far less reliable than the other two. The weights b=0.33 and c=36 were found from historical data. However, the danger of a data-mining bias is relatively small because the same weights were used for all stocks and all time periods.

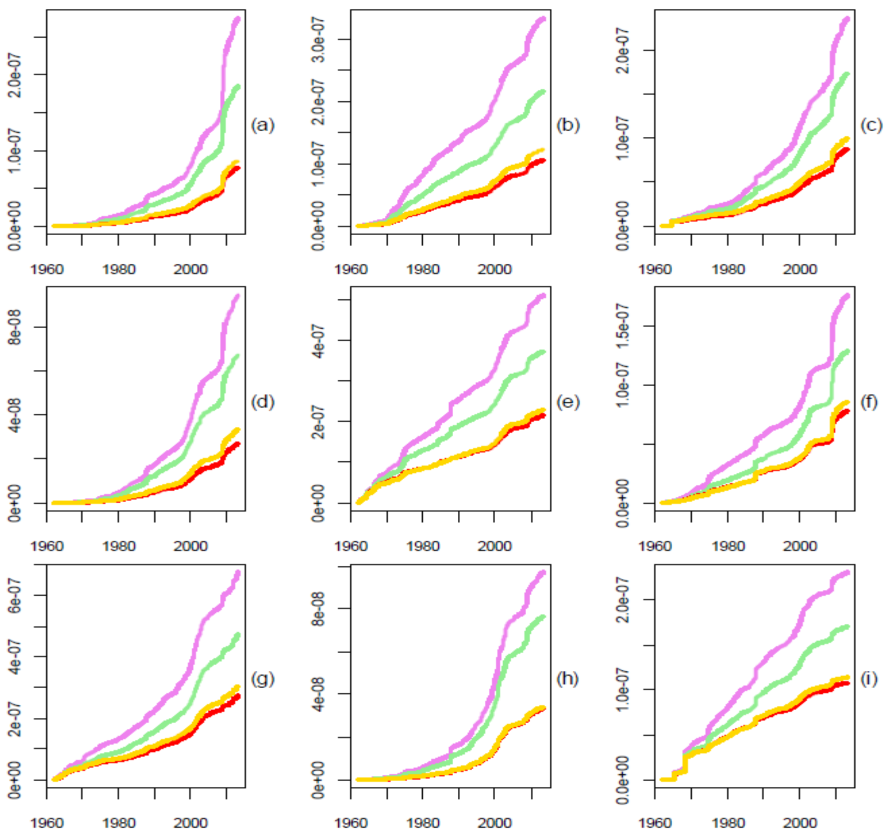

Figure 4: Comparison of different return variance estimators with respect to the cumulative MSE (violet: standard estimator (1), lightgreen: simplest estimator based on intraday statistics (2), red: range based estimator (4), gold: drift-robust estimator (6)). Data: (a) AA, (b) BA, (c) CAT, (d) DD, (e) DIS, (f) GE, (g) HPQ, (h) IBM, (i) KO

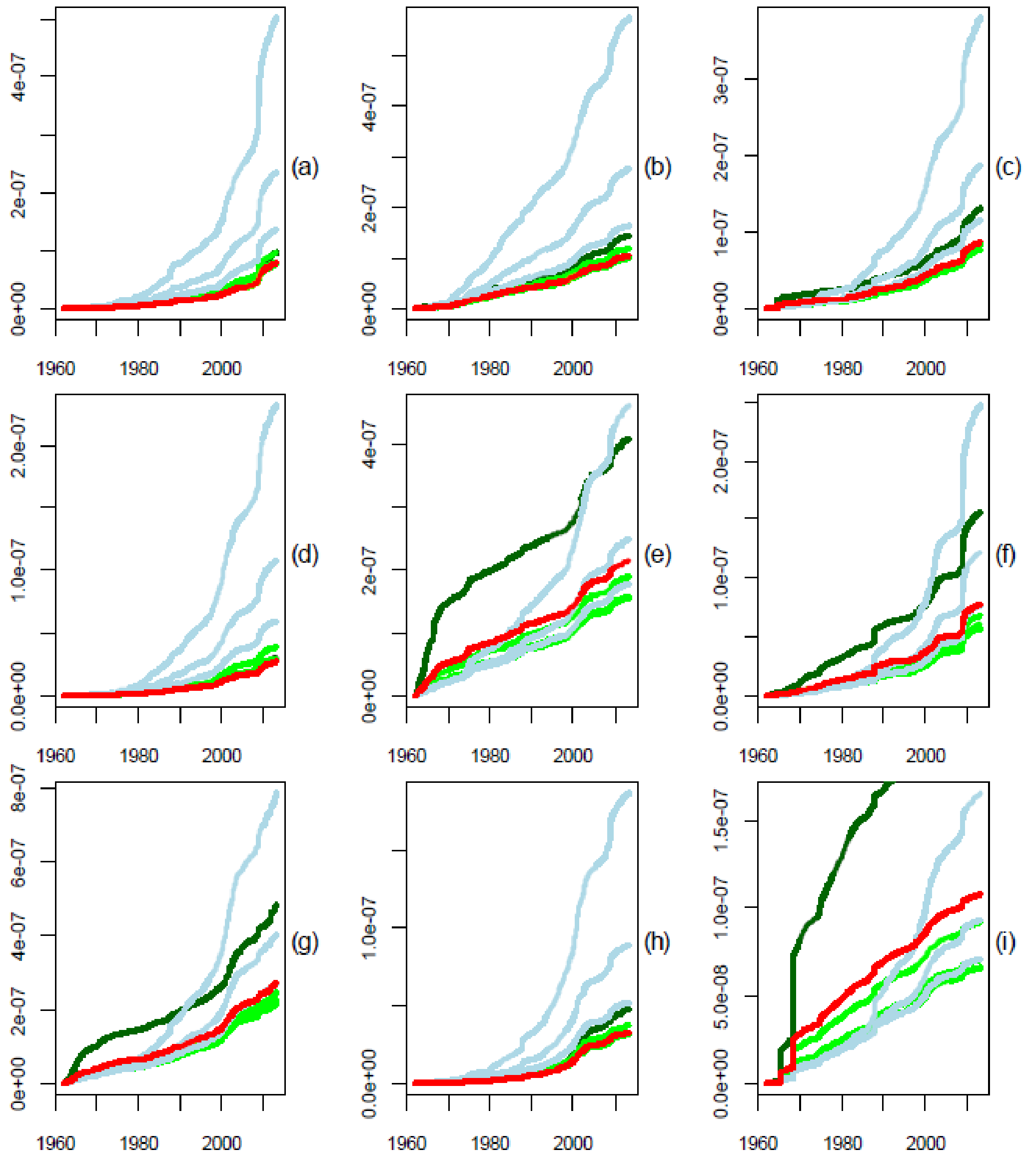

Figure 5: Comparison of the simple range based return variance estimator (4) (red) with different versions of the optimized estimator (9) (*f*=0.1: darkgreen, *f*=0.2, 0.3, 0.4: green, *f*=0.5, 0.6., 0.7: lightblue) with respect to the cumulative MSE.
Data: (a) AA, (b) BA, (c) CAT, (d) DD, (e) DIS, (f) GE, (g) HPQ, (h) IBM, (i) KO

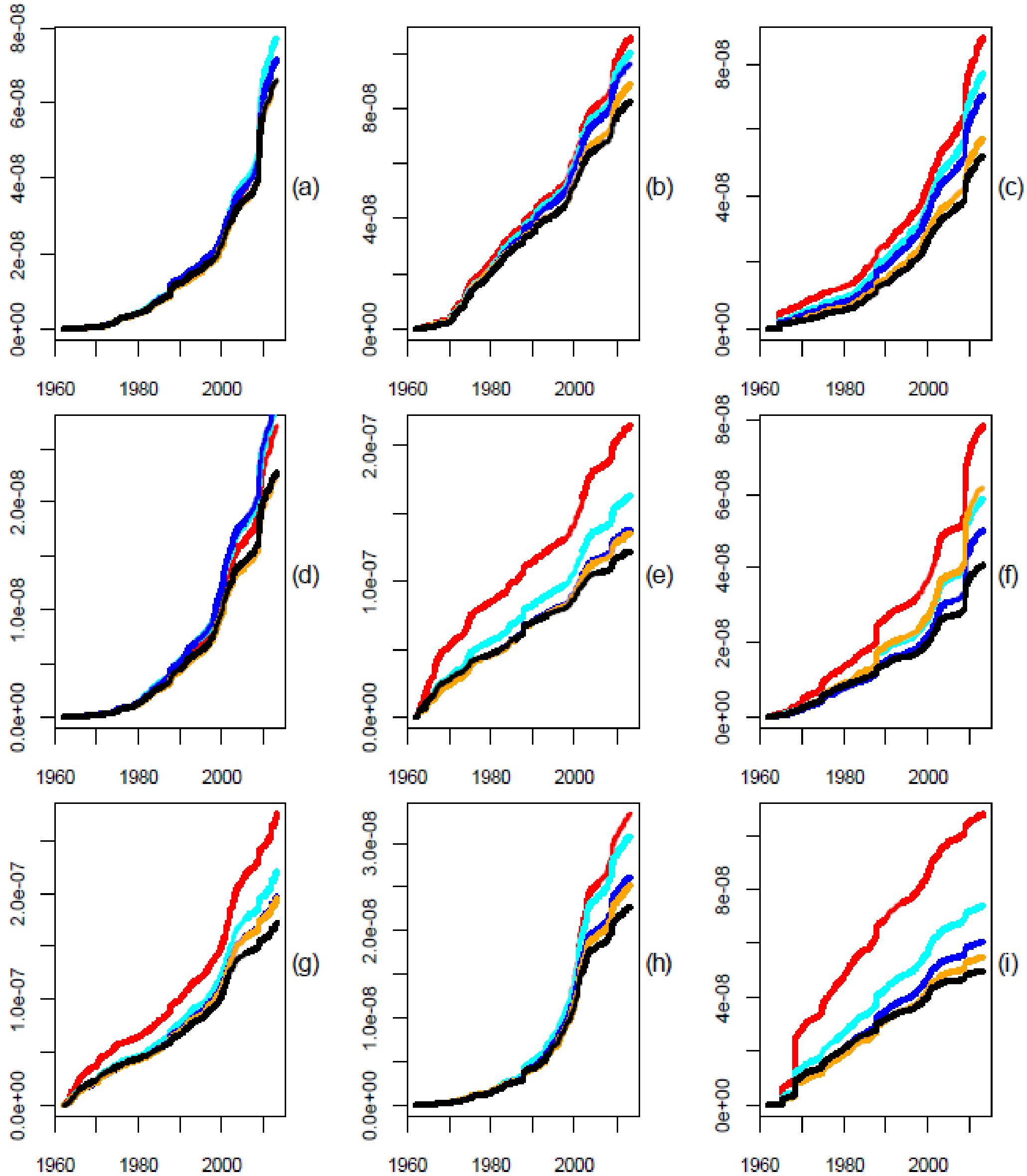

Figure 6: Comparison of different return variance estimators with respect to the cumulative MSE (red: range based estimator (4), cyan: optimized estimator (9), blue: improved optimized estimator (10), orange: composite estimator (14), black: pragmatic estimator (15)). Data: (a) AA, (b) BA, (c) CAT, (d) DD, (e) DIS, (f) GE, (g) HPQ, (h) IBM, (i) KO

4 Conclusion

The empirical study presented in Section 3 showed that the estimators for the daily return variances proposed in Section 2 outperform the existing estimators consistently over all stocks and all time periods. The competing estimators were compared with respect to the MSE which is the usual way to optimize the trade-off between bias and variance. The problem in the case of financial applications is that the fourth powers of stock returns are extremely volatile. The variances of the variance estimators were therefore not obtained directly from fourth powers but rather from squared averages of squares. The stability of the results shows that this approach is appropriate. Thus, it seems that the new estimators are indeed the methods of choice for the estimation of the daily return variances based on opening, high, low, and closing prices.

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