Enhancement of the bond portfolio

Immunization under a

parallel shift of the yield curve

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Abstract

Hedging under a parallel shift of the interest rate curve is well-known for a long date in finance literature. It is based on the use of a duration-convexity approximation essentially pioneered by Fisher-Weil [2]. However the situation is inaccurately formulated such that the obtained result is very questionable.

Motivations and enhancement of such approximation have been performed in our recent working paper [5],"Enhancement of the Fisher-Weil bond technique immunization". So it is seen that the introduction of a term measuring the passage of time and high order sensitivities lead to very accurate approximation of the zero-coupon price change. As a result, the immunization of a portfolio made by

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coupon-bearing bonds may be reduced to a non-linear and integer minimization problem.

In the present work, we show that actually a mixed-integer linear programming is needed to be considered. This last can be handled by making use of standard solvers as the CPLEX software.

1   Introduction

Our main purpose in this paper is to provide an enhancement of the bond portfolio immunization pioneered by F. Macaulay [8], F. Redington [11], and also extended by L. Fisher & R. Weil [2]. Parts of the present results are drawn from our recent working paper [5].Here we provide a clarification about the non-linear and integer optimization problem left non-analyzed in full generality in this last work. Though it is well-known that all points of the interest rate curve do not really move in a parallel fashion, there are at least three reasons which motivate us to reconsider here the bond immunization problem. First this particular situation is (and continues to be) used by many people as a benchmark framework for the bond immunization. Moreover some empirical results [12] tend to state that using a stochastic model for the interest rate does not provide a remarkable immunization out-performance compared with the one obtained from the simple Fisher-Weil technique. Second, considering a parallel shift of the interest rate curve is among the standard mean to stress test the financial position.

So having the behavior immunization in such an extreme case is helpful for the investor on a bond portfolio. Third, the accurate analysis, as we perform in this

particular case, may clarify well the situation under a stochastic model driven by a one-uncertainty factor as it is recently developed in [10].

The immunization idea relies on matching the first (and probably the second) order sensitivities of the position to hedge and the hedging instrument with respect to a parallel shift of the yield curve. Therefore the task is essentially based on an approximation of the zero-coupon bond change. It means that to ensure a reliable immunization, rather than to question about the appropriateness of the interest rate curve parallel shift assumption (as done by various authors), we think there is also a room on the exploration of the validity of the approximation to use. In any case, the recourse to a given a model always spans an incorrect view of the reality. However when a model is chosen, it becomes crucial to analyze about the consistency and correctness of its use. As presented in our previous work [7] and [5], the classical Fisher-Weil immunization technique suffers from at least five drawbacks:

the time-passage is neglected, contrarily as one can easily observed in reality;

the shift is assumed to be infinitesimal, but the sense of this last is not clear;

short positions are considered without taking into account the associated managing costs;

the hedging allocation is given in term of bond proportions rather than in term of security numbers as is really required in trading;

bounds for the hedging error are unknown and sometimes the variance information is used, however this last is not economically sounding for the investor's perspective.

In [5], the classical Fisher-Weil bond change approximation and the associated bond hedging technique are enhanced such that we are able to solve simultaneously all of these five issues.

For the convenience, some of our main previous results are reported here.

Essentially we have seen that the immunization problem is reduced to some

non-linear and integer optimization. Numerical examples, limited to a hedging portfolio made by one type of bonds in long positions and one type of bonds in short positions, are given in [5]. In this particular case, the optimization to perform may be solved by an enumerative method. The case of a portfolio hedging made by more types of bonds remains practically unsolved at this stage. So our new contribution in this paper is to show that really the problem may be solved by running a mixed-integer-linear-programming. The numerical examples, we consider in this paper, put in perspective that the use of various types of bonds lead to reduce considerably the possible maximum loss related to the hedging operation.

We emphasize that our main concern here is on the correctness and accurateness of the immunization approach as in [7] and [5]. This is performed by a technical consideration and does not lean on any particular financial data. The validity of a parallel shift assumption for the given interest rate curve is of few importance. In our illustrative examples, we have considered possible shift size values up to order 2.5 % to test the limit of our approach.

This paper is organized as follows. Our main results are stated in Section 2. After recalling primer notions on bonds and interest rate curve, we present in Proposition 2.1 of Subsection 2.1 the basic identity decomposition of portfolio change which is the main key for the immunization. The hedging formulation is performed in Subsection 2.2. Particularly in Theorem 2, we present the expression of the overall hedged portfolio change. This last enables us to state, in Theorem 3, that the considered bond portfolio immunization is reduced to a non-linear and integer optimization problem. In our working paper [5], we have seen that this last can be solved with an enumerative method whenever the hedging portfolio is made just by one type of bonds in long positions and one type of bonds on short positions. However this last paper remains silent about the approach to use facing such a non-linear and integer optimization problem, which is non-tractable in general. Therefore in Proposition3.4 of Subsection 2.3, we show that this problem

can be reduced to a mixed integer linear problem which may be handled by several solvers as the commercial solver CPLEX. This last is used in our numerical examples, displayed in Section 3. We conclude in Section 4.

2   Main Results

Our result on hedging is based on the decomposition of a bond portfolio change which is presented in Subsection 2.1. Then we can apply this finding to formulate the hedging problem in Subsection 2.2. Finally in Subsection 2.3, the optimization problem linked to such a hedging operation is analyzed.

2.1.   Portfolio change

A bond is a debt security such that the issuer owes to the holder a debt and, depending on the terms of the considered bond, is obliged to pay interest (often named coupon) and/or repay the principal at a later date, called maturity. In this work, we consider vanilla bonds and assume that the issuer may not default until the maturity. The time-t value of a coupon-bearing bond is defined by

$$B_t = \sum_{k=1}^{M} C_k \, P\big(t; \tau_k(t)\big) \tag{1}$$

Where

$$\tau_k(t) = t_k - t = \frac{\text{number days between } t_k \text{ and } t}{base}$$

$$C_k = N \times c \times \tau(t_{k-1}, t_k) \quad and \quad C_M = N\{1 + c \times \tau(t_{M-1}, t_M)\}$$

are the coupons paid respectively at time t_k and t_M . Here N denotes the bond face value and c is the annual coupon rate. The base is 360 or 365 or other number depending on the contract nature. The time-t-value of the zero-coupon bond $P\big(t; \tau_k(t)\big)$ for a time-to-maturity $\tau_k(t) > 0$ is

$$P\big(t; \tau_k(t)\big) = \exp\big(-y\big(t; \tau_k(t)\big)\tau_k(t)\big) \tag{2}$$

It is set that $P\big(t; \tau_k(t)\big) \equiv 1$. The quantity $y\big(t; \tau_k(t)\big)$ is refereed as a yield and corresponds to the continuous interest rate which applies at time t for the period $[t; t + \tau_k(t)]$. The yield curve or time-t zero-coupon rates is defined by the map

$$\tau \in (0, \infty) \longrightarrow y(t; \tau) \tag{3}$$

In reality $y(t; \tau)$ is not completely given for the whole points in the semi-real axis $(0, \infty)$ .We have only at our disposal some (discrete) yield-to-maturities $y(t; \tilde{\tau}_1), \dots, y\big(t; \tilde{\tau}_j\big), \dots. y(t; \tilde{\tau}_m)$. Facing to this lack of data, any yield-to-maturity $y(t; \tau)$ with $\tilde{\tau}_j < t < \tilde{\tau}_{j+1}$ is for instance defined via a linear interpolation. This linear interpolation represent an acceptable approximation of the function $y(t; \tau)$.

To understand the dynamic of the zero interest rate curve determined by $y(t + s; \tau)(\cdot) - y(t; \tau)$ is a main issue in management of instruments linked to interest rates. For this purpose, sometimes it is reasonable to assume that

$$y(t + s; \tau)(\cdot) = y(t; \tau) + \varepsilon(\cdot\,; t, s) \tag{4}$$

for a future time $t + s$ not too far from time t in the sense that $t + s < t_1 < .. < t_M$. The point here is that the real number $\varepsilon(\cdot)$ does not depend on the time-to-maturity τ. The assumption (4), referred as a parallel shift of the yield-curve, has been introduced and used in finance literature [2].

Let us denote by $\mathcal{V}_t$ the present time-t-value of a portfolio made by coupon-bearing bonds in long and/or short positions. So we assume that there are I^{**} types of bonds $B^{**}_{.;i^{**}}$ in long positions and I^* types of bonds $B^*_{.;i^*}$ in short positions inside the considered portfolio. Of course I^{**} and I^* stand for positive integer numbers. Each bond $B^{**}_{.;i^{**}}$, $i^{**} \in \{1, \dots , I^{**} \}$, is assumed to have a maturity $T^{**}_{M^{**}_{i^{**}}}$ and a first coupon paid at time $t^{**}_{1;i^{**}}$. Similarly each bond $B^*_{.;i^*}$, $i^* \in \{1, \dots, I^* \}$ is characterized by a maturity $T^*_{M^*_{i^*}}$ and a first coupon paid at time $t^*_{1;i^*}$.

The time-t- value of such a portfolio may be written as

$$\mathcal{V}_t = \sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**} B_{t;i^{**}}^{**} - \sum_{i^*=1}^{I^*} n_{i^*}^{*} B_{t;i^*}^{*} \tag{5}$$

Therefore there are $n_{i^{**}}^{**}$ bonds of type i^{**} each worth $B_{.;i^{**}}^{**}$ and $n_{i^*}^{*}$ bonds of type i^* each worth $B_{.;i^*}^{*}$.

The manager of such a portfolio has an issue to maintain a level value at least approximately to $\mathcal{V}_t$ at the future time $t + s$ where s some nonnegative real number is. In practice s corresponds to the horizon for which she has a more and less clear view about a possible movement of the market. To simplify the situation, in this paper we will just focus on the case where s is sufficiently close to t such that no coupon from all the bonds is paid during the time-period $(t, t + s)$ that is

$$t < t + s < \min \{ t_{1;B_{.;1}^{**}}, \ldots t_{1;B_{.;I^{**}}^{**}}; \ t_{1;B_{.;1}^{*}}, \ldots t_{1;B_{.;I^*}^{*}} \} \tag{6}$$

The future portfolio value $\mathcal{V}_{t+s}(\cdot)$ is not known at time t and should depend on the structure taken by the interest rate curve at time $+s$. A full description of the portfolio value $\mathcal{V}_{t+s}(\cdot) - \mathcal{V}_t$, in the case of a parallel shift of the interest rate curve, is given by the following.

Proposition 2.1 *Let us consider a time horizon s not too large in the sense of (6). Assume that the interest rate curve has done any arbitrary parallel shift $\varepsilon(\cdot)$ not equal to zero and not too negative, satisfying*

$$- \varepsilon^\circ < \varepsilon(\cdot) < \varepsilon^{\circ\circ} \tag{7}$$

for some ε° and $\varepsilon^{\circ\circ}$ with $0 < \varepsilon^\circ, \varepsilon^{\circ\circ}$ and

$$\varepsilon^\circ < \min \{ \left(y\left(t + \tau, t_{k^{**};i^{**}}^{**}\right) \right)_{k^{**} \in \{1,\ldots,M_{i^{**}}^{**}\}; \ i^{**} \in \{1,\ldots,I^{**}\}}$$

$$\left(y\left(t + \tau, t_{k^*;i^*}^{*}\right) \right)_{k^* \in \{1,\ldots,M_{i^*}^{*}\}; \ i^* \in \{1,\ldots,I^*\}} \} \tag{8}$$

and let p be a nonnegative integer number. Then there is some real number $\rho(\cdot) = \rho(\varepsilon; p)$ with $0 < \rho(\cdot) < \varepsilon(\cdot)$ or $\varepsilon(\cdot) < \rho(\cdot) < 0$, such that we have the

following portfolio change decomposition

$$\mathcal{V}_{t+s}(\cdot) - \mathcal{V}_t$$

$$= \mathbf{Res}(t, s, \mathcal{V})$$

$$+ \sum_{l=1}^{p} \frac{(-1)^l}{l!} \mathbf{Sens}(l; t, s, \mathcal{V}) \, \varepsilon^l(\cdot)$$

$$+ \frac{(-1)^{p+1}}{(p+1)!} \mathbf{Sens}(p+1; t, s, \mathcal{V}; \rho(\cdot)) \varepsilon^{p+1}(\cdot) \tag{9}$$

where

$$\mathbf{Res}(t, s, \mathcal{V}) = \sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**} \mathbf{Res}(t, s, B_{\cdot; i^{**}}^{**}) - \sum_{i^{*}=1}^{I^{*}} n_{i^{*}}^{*} \mathbf{Res}(t, s, B_{\cdot; i^{*}}^{*}) \tag{10}$$

$$\mathbf{Sens}(l; t, s, \mathcal{V}) = \sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**} \mathbf{Sens}(l; t, s, B_{\cdot; i^{**}}^{**}) - \sum_{i^{*}=1}^{I^{*}} n_{i^{*}}^{*} \mathbf{Sens}(l; t, s, B_{\cdot; i^{*}}^{*}) \tag{11}$$

$$\mathbf{Sens}(p+1; t, s, \mathcal{V}; \rho(\cdot)) = \sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**} \mathbf{Sens}\left(p+1; t, s, B_{\cdot; i^{**}}^{**}; \rho(\cdot)\right)$$

$$- \sum_{i^{*}=1}^{I^{*}} n_{i^{*}}^{*} \mathbf{Sens}\left(p+1; t, s, B_{\cdot; i^{*}}^{*}; \rho(\cdot)\right) \tag{12}$$

$$\mathbf{Res}(t, s, B) = \sum_{k=1}^{M} C_k \left\{ \exp\left(- y(t; \tau_k(t,s))\tau_k(t,s)\right) - \exp\left(- y(t; \tau_k(t))\tau_k(t)\right) \right\}$$

$$\tag{13}$$

$$\mathbf{Sens}(l; t, s, B) = \sum_{k=1}^{M} \{\tau_k(t,s)\}^l C_k \, \exp\{- y(t; \tau_k(t,s))\tau_k(t,s)\} \tag{14}$$

$$\mathbf{Sens}(p+1; t, s, B; \rho(\cdot))$$

$$= \sum_{k=1}^{M} \{\tau_k(t,s)\}^{p+1} C_k \, \exp\left\{-\left(y(t; \tau_k(t,s)) + \rho(\cdot)\right)\tau_k(t,s)\right\}$$

$$\tag{15}$$

For $\varepsilon(\cdot) = 0$ then it is clear that

$$\mathcal{V}_{t+s}(\cdot) - \mathcal{V}_t = \mathbf{Res}(t, s, \mathcal{V})$$

This case corresponds to the situation where the interest rate curve at the

horizon $t + s$ remains the same as in time t, which corresponds to the case $y(t; \tau)(\cdot) = y(t; \tau)$. It means that the term residual $\mathbf{Res}(t, s, \mathcal{V})$ in (9), corresponds to the time-passage effect on the bond portfolio when the interest rate curve remains unchanged.

The term $\mathbf{Sens}(l; t, s, \mathcal{V})$, with $l \in \{1, 2, \dots p\}$ should be viewed as the bond portfolio *l-th* order sensitivity with respect to the parallel shift of the interest rate curve. By extension, for $l = 0$, we can set

$$\mathbf{Sens}(0; t, s, \mathcal{V}) = \mathcal{V}_{t+s|\varepsilon=0}$$

This corresponds to the (deterministic) bond portfolio price at time $t + s$ if the interest rate curve remains unchanged.

In the direction of identity (9) and for any arbitrary shift $\varepsilon(\cdot)$ consistent with the view (7), we can recast the portfolio change into

$$\{\mathcal{V}_{t+s}(\cdot) - \mathcal{V}_t\} - \left\{\mathbf{Res}(t, s, \mathcal{V}) + \sum_{l=1}^{p} \frac{(-1)^l}{l!} \mathbf{Sens}(l; t, s, \mathcal{V})\, \varepsilon^l(\cdot)\right\}$$

$$= \mathbf{Remainder}\big(p + 1; t, s, \mathcal{V}; \rho(\cdot)\big)$$

(16)

If the remainder term defined by

$$\mathbf{Remainder}\big(p + 1; t, s, \mathcal{V}; \rho(\cdot)\big) = \frac{(-1)^{p+1}}{(p+1)!} \mathbf{Sens}\big(p + 1; t, s, B; \rho(\cdot)\big)\varepsilon^{p+1}(\cdot)$$

(17)

may be neglected following the perspective of the investor in this portfolio bond $\mathcal{V}_t$, then the bond portfolio relative change during the time-period $(t, t + s)$ is approximated by the sum of two terms as

$$\{\mathcal{V}_{t+s}(\cdot) - \mathcal{V}_t\} \approx \left\{\mathbf{Res}(t, s, \mathcal{V}) + \sum_{l=1}^{p} \frac{(-1)^l}{l!} \mathbf{Sens}(l; t, s, \mathcal{V})\, \varepsilon^l(\cdot)\right\} \quad (18)$$

The first deterministic term (i.e. well-known at time t) $\mathbf{Res}(t, s, \mathcal{V})$ corresponds to the passage of time. Such a term, introduced in [7], has never been considered

in literature dealing with bond change approximations. As depending on the future shift $\varepsilon(\cdot)$ of the interest rate curve, the second term in the right side of (18) is stochastic and appears to be a sum of various sensitivities of the bond portfolio weighted by power values of $-\varepsilon(\cdot)$. It should be noted that the classical duration and convexity are given respectively by

$$\mathbf{Dur}(t, \mathcal{V}) = \frac{1}{\mathcal{V}_t} \mathbf{Sens}(1; t, 0, \mathcal{V}) \quad \text{and} \quad \mathbf{Conv}(t, \mathcal{V}) = \frac{1}{\mathcal{V}_t} \mathbf{Sens}(2; t, 0, \mathcal{V})$$

such that instead of (18) with $p = 2$ and for any arbitrary s satisfying (6) the classical Fisher-Weil (1971) bond relative change approximation is

$$\{ \mathcal{V}_{t+s}(\cdot) - \mathcal{V}_t \} \approx \mathcal{V}_t \times \{\mathbf{Dur}(t, \mathcal{V})\varepsilon(\cdot) + \mathbf{Conv}(t, \mathcal{V})\varepsilon(\cdot)\} \tag{19}$$

The superiority of our approximation (18) in comparison with (19) is largely analyzed and illustrated in our working paper [5].

2.2. Hedging formulation

From now let us denote by V_t the present time t value of the portfolio made by long and short positions

$$V_t = \sum_{i^{**}=1}^{I^{**}} \tilde{n}_{i^{**}}^{**} \, \tilde{B}_{t;i^{**}}^{**} - \sum_{i^*=1}^{I^*} \tilde{n}_{i^*}^{*} \, \tilde{B}_{t;i^*}^{*} \tag{20}$$

we would like to immunize against the parallel shift of the interest rate. To try to maintain the value V_t at the future time horizon $t + s$, where s>0 is some nonnegative real number, the portfolio manager has to put in place a given hedging technique. Various approaches are known in theory and used in practice(see for instance [1] and the references therein). Here we slightly differ with the standard asset-liability management for which the liability and asset are taken to have the same value at the starting time immunization. The hedging idea relies on using another bond portfolio, referred here as the hedging instrument. This last would lead to a profit compensating the portfolio loss in case of adverse shift of the interest rate curve. As a consequence, instead of the absolute change

$V_{t+s}(\cdot) - V_t$ associated with the naked portfolio, at the horizon $t + s$ the change for the covered portfolio is given by

$$W_{t;t+s}(\cdot) = \{V_{t+s}(\cdot) - V_t\} + \text{P\&L\_Hedging\_instruments}_{t,t+s}(\cdot) \quad (21)$$

The hedging instrument has a similar formula as the portfolio given in (5) that is

$$\text{P\&L\_Hedging\_instruments}_t \equiv H_t = \sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**} B_{t;i^{**}}^{**} - \sum_{i^*=1}^{I^*} n_{i^*}^* B_{t;i^*}^* \quad (22)$$

For convenience, we consider the same notations as (5) where the $n_{i^{**}}^{**}$ bonds $B_{t;i^{**}}^{**}$ are in long position and the $n_{i^*}^*$ bonds $B_{t;i^*}^*$ are in short position[4]. The associated profit and loss during the time-period $(t, t + s)$ is given by

$$\text{P\&L\_Hedging\_instruments}_{t,t+s}(\cdot) =$$

$$\sum_{i^{**}=1}^{I^{**}} \{B_{t+s;i^{**}}^{**}(\cdot) - B_{t;i^{**}}^{**}\} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \{B_{t+s;i^*}^*(\cdot) - B_{t;i^*}^*\} n_{i^*}^*$$

$$-\left\{\frac{1}{P(t,s)} - 1\right\}\left\{\sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**} B_{t+s;i^{**}}^{**} + \left(\lambda + \frac{\eta s}{1 - P(t,s)}\right) \sum_{i^*=1}^{I^*} n_{i^*}^* B_{t;i^*}^*\right\}$$

$$(23)$$

where $0 < \eta, \lambda < 1$ as $\eta = 0.1\%$ and $\lambda = 25\%$ for example. In one hand η represents the interest rate associated with borrowing of bond securities in order to perform the short sell operation. Indeed to partially prevent counter-party risk, it is required an amount of deposits $\lambda\left(\sum_{i^*=1}^{I^*} n_{i^*}^* B_{t;i^*}^*\right)$. It should be emphasized that in this article we do not take into consideration the transaction costs. The amount borrowed and required to realize the hedge is

$$\left\{\sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**} B_{t;i^{**}}^{**} + \left(\lambda + \frac{\eta s}{1 - P(t,s)}\right) \sum_{i^*=1}^{I^*} n_{i^*}^* B_{t;i^*}^*\right\}$$

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[4] Observe that they are different from those forming the portfolio in (5).

which corresponds, at time $t + s$ a payment of the interest with the deterministic level

$$\left\{\frac{1}{P(t,s)} - 1\right\}\left\{\sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**} B_{t;i^{**}}^{**} + \left(\lambda + \frac{\eta s}{1 - P(t,s)}\right) \sum_{i^{*}=1}^{I^{*}} n_{i^{*}}^{*} B_{t;i^{*}}^{*}\right\}$$

Technically the offsetting effect resulting from the consideration of the hedged portfolio, as characterized by the profit and loss $W_{t,t+s}(.)$ defined in (21), is performed by matching the various sensitivities associated to the portfolio to hedge and the hedging instrument.

When considering the portfolio and the hedging instrument, as respectively defined in (20) and (22), the time-horizon s is said to be not too large whenever $t < t + s$

$$< min\left\{t_{1;B_{.;1^{**}}^{**}}, \dots t_{1;B_{.;I^{**}}^{**}}; \ t_{1;B_{.;1^{*}}^{*}}, \dots t_{1;B_{.;I^{*}}^{*}}; \ t_{1;B_{.;1^{**}}^{**}}, \dots t_{1;B_{.;I^{**}}^{**}}; \ t_{1;B_{.;1^{*}}^{*}}, \dots t_{1;B_{.;I^{*}}^{*}}\right\}$$

$$(24)$$

We also consider admissible parallel shift $\varepsilon(\cdot)$ in the sense that

$$-\varepsilon(\cdot) < min\begin{cases} \left(y\left(t + \tau, \tilde{t}_{k^{**};i^{**}}^{**}\right)\right)_{k^{**} \, \epsilon \, \{1,\dots,\tilde{M}_{i^{**}}^{**}\}; \ i^{**} \, \epsilon \, \{1,\dots,\tilde{I}^{**}\}}; \\ \left(y\left(t + \tau, \tilde{t}_{k^{*};i^{*}}^{*}\right)\right)_{k^{*} \, \epsilon \, \{1,\dots,\tilde{M}_{i^{*}}^{*}\}; \ i^{*} \, \epsilon \, \{1,\dots,\tilde{I}^{*}\}}; \\ \left(y\left(t + \tau, t_{k^{*};i^{*}}^{*}\right)\right)_{k^{*} \, \epsilon \, \{1,\dots,M_{i^{*}}^{*}\}; \ i^{*} \, \epsilon \, \{1,\dots,I^{*}\}}; \\ \left(y\left(t + \tau, t_{k^{**};i^{**}}^{**}\right)\right)_{k^{**} \, \epsilon \, \{1,\dots,M_{i^{**}}^{**}\}; \ i^{**} \, \epsilon \, \{1,\dots,I^{**}\}} \end{cases}$$

Here $\tilde{t}_{k^{**};\,i^{**}}^{**}$ denotes the time where the k^{**}-th coupon of the i^{**} bond $\tilde{B}_{t;i^{**}}^{**}$ (in long position) maturing at $t_{\tilde{M}^{**}{}_{i^{**}}}$ is paid. The quantities $\tilde{t}_{k^{*};i^{*}}^{*}, t_{k^{**};i^{**}}^{**}, t_{k^{*};i^{*}}^{*}$ may be defined similarly as $\tilde{t}_{k^{**};i^{**}}^{**}$.

All of these considerations lead us now to write the full expression of the covered portfolio P &L level in (23) by putting in evidence the possible compensation between the various values of the involved coupon-bearing bonds.

Theorem 2.2 *Consider a bond portfolio as in (20) and a hedging instrument as in (22). Assume that s is not large as in (24) and the parallel shift admissible in the sense of (25). Let p be a nonnegative integer number. Then real numbers* $\rho^V(\cdot), \rho^{**}(\cdot), \rho^*(\cdot)$ *(depending on* $\varepsilon(\cdot)$ *and p) exist such that the profit and loss* $W_{t;t+s}(\cdot)$ *for the overall portfolio is given by the following expansion:*

$$
W_{t;t+s}(\cdot) = \left\{ \Theta_0^V + \sum_{i^{**}=1}^{I^{**}} \Theta_{0;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \Theta_{0;i^*}^* n_{i^*}^{**} \right\}
$$

$$
+ \sum_{l=1}^{p} \frac{(-1)^l}{l!} \left\{ \Theta_l^V + \sum_{i^{**}=1}^{I^{**}} \Theta_{l;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \Theta_{l;i^*}^* n_{i^*}^{*} \right\} \varepsilon^l(\cdot)
$$

$$
+ \frac{(-1)^{(p+1)}}{(p+1)!} \left\{ \Theta_{p+1}^V(\rho^V) + \sum_{i^{**}=1}^{I^{**}} \Theta_{p+1;i^{**}}^{**}(\rho^{**}) n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \Theta_{p+1;i^*}^*(\rho^*) n_{i^*}^{*} \right\} \varepsilon^{p+1}(\cdot)
$$

$$(26)$$

where

$$\Theta_0^V = \mathbf{Res}(t, s, V) \tag{27}$$

$$\Theta_{0;i^{**}}^{**} = \mathbf{Res}\left(t, s, B_{.;i^{**}}^{**}\right) - \left\{ \frac{1}{P(t,s)} - 1 \right\} B_{.;i^{**}}^{**} \tag{28}$$

$$\Theta_{0;i^*}^* = \mathbf{Res}\left(t, s, B_{.;i^*}^*\right) - \left\{ \frac{1}{P(t,s)} - 1 \right\} \left(\lambda + \frac{\eta s}{1 - P(t,s)} \right) B_{.;i^*}^* \tag{29}$$

$$\Theta_l^V = \mathbf{Sens}(l; t, s, \mathcal{V}); \quad \Theta_{l;i^{**}}^{**} = \mathbf{Sens}\left(l; t, s, B_{.;i^{**}}^{**}\right) \tag{30}$$

$$\Theta_{l;i^*}^* = \mathbf{Sens}\left(l; t, s, B_{.;i^*}^*\right)$$

$$\Theta_{p+1}^V(\rho^V) = \mathbf{Sens}(p+1; t, s, V; \rho^V); \tag{31}$$

$$\Theta_{p+1;i^{**}}^{**}(\rho^{**}) = \mathbf{Sens}\left(p+1; t, s, B_{.;i^{**}}^{**}; \rho^{**}\right); \tag{32}$$

$$\Theta_{p+1;i^*}^*(\rho^*) = \mathbf{Sens}\left(p+1; t, s, B_{.;i^*}^*; \rho^*\right)$$

Here Θ_0^V, Θ_1^V and Θ_{p+1}^V are similarly defined as in (10), (11) and (12) as functions of $\tilde{B}_{t;i^{**}}^{**}, \tilde{B}_{t;i^*}^*, \tilde{n}_{i^{**}}^{**}, \tilde{n}_{i^*}^*, \tilde{I}^{**}$ and $\tilde{I}^*$.

This result clearly explicits the offsetting effect which arises between the portfolio
V_t to hedge and the hedging portfolio H_t. At the present time t the value

of $W_{t;t+s}(\cdot)$ remains unknown, but we hope that it should be a small quantity when its value is revealed at the horizon $t+s$. It means that the issue for the hedging is to be able to suitably choose the bond security numbers $n_{i^{**}}^{**}$ and $n_{i^*}^*$ in order to satisfy this requirement.

We need to introduce more restriction on the parallel shift satisfying $\varepsilon = \varepsilon(\cdot)$, satisfying (7) for some nonnegative ε° and $\varepsilon^{\circ\circ}$ such that

$$
\varepsilon^\circ < \min \left\{
\begin{array}{l}
\left(y\left(t+\tau, \tilde{t}_{k^{**};i^{**}}^{**}\right) \right)_{k^{**} \in \{1,\dots,\tilde{M}_{i^{**}}^{**}\};\ i^{**} \in \{1,\dots,\tilde{I}^{**}\}}; \\[2mm]
\left(y\left(t+\tau, \tilde{t}_{k^*;i^*}^{*}\right) \right)_{k^{**} \in \{1,\dots,\tilde{M}_{i^*}^{*}\};\ i^{**} \in \{1,\dots,\tilde{I}^{*}\}}; \\[2mm]
\left(y\left(t+\tau, t_{k^{**};i^{**}}^{**}\right) \right)_{k^{**} \in \{1,\dots,M_{i^{**}}^{**}\};\ i^{**} \in \{1,\dots,I^{**}\}}; \\[2mm]
\left(y\left(t+\tau, t_{k^*;i^*}^{*}\right) \right)_{k^* \in \{1,\dots,M_{i^*}^{*}\};\ i^* \in \{1,\dots,I^{*}\}}
\end{array}
\right\}
$$

Then referring to (7), (24) and (32) , we have

$$
\Theta_{p+1;\ i^{**}}^{**} \le \exp\left[\varepsilon^\circ\left(T_{\tilde{M}_{i^{**}}^{**}} - s\right)\right] \mathbf{Sens}\left(p+1;t,s,B_{\cdot;i^{**}}^{**};\rho^{**}\right) \equiv \Upsilon_{p+1;i^{**}}^{**} \quad (33)
$$

and similarly

$$
\Theta_{p+1;\ i^{*}}^{*} \le \exp\left[\varepsilon^\circ\left(T_{M_{i^{*}}^{*}} - s\right)\right] \mathbf{Sens}\left(p+1;t,s,B_{\cdot;i^{*}}^{*};\rho^{*}\right) \equiv \Upsilon_{p+1;i^{*}}^{*} \quad (34)
$$

Similarly $|\Theta_{p+1}^V|$ may be estimated as follows

$$
|\Theta_{p+1}^V| \le \max\left\{ \sum_{i^{**}=1}^{\tilde{I}^{**}} \exp\left[\varepsilon^\circ\left(T_{\tilde{M}_{i^{**}}^{**}} - s\right)\right] \mathbf{Sens}\left(p+1;t,s,B_{\cdot;i^{**}}^{**};\rho^{V}\right)\tilde{n}_{i^{**}}^{**}; \right.
$$

$$
\left. \sum_{i^{*}=1}^{\tilde{I}^{*}} \exp\left[\varepsilon^\circ\left(T_{\tilde{M}_{i^{*}}^{*}} - s\right)\right] \mathbf{Sens}\left(p+1;t,s,B_{\cdot;i^{*}}^{*};\rho^{V}\right)\tilde{n}_{i^{*}}^{*}\right\} \equiv \Upsilon_{p+1}^{V}
$$

$$(35)$$

For the convenience we introduce the following I^{**}-dimensional vectors

$$
n^{**} = (n_1^{**}, \dots, n_I^{**})', \qquad B_t^{**} = \left(B_{t;1^{**}}^{**}, \dots, B_{t;I^{**}}^{**}\right)',
$$

$$
\Theta_0^{**} = \left(\Theta_{0;1}^{**}, \dots, \Theta_{0;I^{**}}^{**}\right)', \Theta_1^{**} = \left(\Theta_{1;1}^{**}, \dots, \Theta_{1;I^{**}}^{**}\right)' \quad (36)
$$

$$
\Theta_{p+1}^{**} = \left(\Theta_{p+1;1}^{**}, \dots, \Theta_{p+1;I^{**}}^{**}\right)'
$$

Analogously we can introduce and define the I^*-dimensional vectors n^*, $B_t^*, \Theta_0^*, \Theta_l^*$ and Θ_{p+1}^*. With all of these notations, we are now in position to state the suitable (robust) allocation problem to determine the vectors n^{**}, n^* corresponding to the various numbers of bonds used in the hedging instrument.

Theorem 2.3 *Consider a bond portfolio as in (20) and a hedging instrument as in (22). Assume that* s *is a time-horizon not large as in (24) and the yield curve has done an admissible parallel shift in the sense of (25). Let* p *be a nonnegative integer number. Then the (vector) numbers* n^{**} *and* n^* *of bond defining the hedging instrument as in (36) may be found as a solution to the integer optimization problem:*

$$(P_0) \qquad (n^{**}, n^*) = argmin \{F(n^{**}, n^*)| \ (n^{**}, n^*) \ \epsilon D_0\} \qquad (37)$$

where the constraint D_0 *is defined as the set of couples* (n^{**}, n^*) *satisfying*

$$a^{**} n^{**} + a^* n^* \leq D$$

for all $n^{**} \epsilon \ \mathbb{N}^{I^{**}}, n^* \epsilon \ \mathbb{N}^{I^*}$, *with*

$$a^{**} = \left\{\frac{1}{P(t,s)} - 1\right\} B_t^{**}$$

$$a^* = \left\{\frac{1}{P(t,s)} - 1\right\} \left(\lambda + \frac{\eta s}{1 - P(t,s)}\right) B_t^* \qquad (38)$$

Here D *is the amount allowed by the investor not to be exceeded in the hedging operation. The objective function* $F(n^{**}, n^*)$ *is defined by*

$$F(n^{**}, n^*) = \sum_{l=0}^{p} \frac{1}{l!} \left| \Theta_l^V + \sum_{i^{**}=1}^{I^{**}} \Theta_{l;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \Theta_{l;i^*}^* n_{i^*}^* \right| \ \epsilon^l$$

$$+ \frac{1}{(p+1)!} \left\{ \Upsilon_{p+1}^V + \sum_{i^{**}=1}^{I^{**}} \Upsilon_{p+1;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \Upsilon_{p+1;i^*}^* n_{i^*}^* \right\} \epsilon^{p+1} \qquad (39)$$

Here $\epsilon = max\{ \varepsilon^\circ, \varepsilon^{\circ\circ}\}$ and the quantities $\Theta_0^V, \Theta_0^{**}, \Theta_0^*, \Theta_l^V, \Theta_l^{**}, \Theta_l^*, \Upsilon_{p+1}^V, \Upsilon_{p+1}^{**}, \Upsilon_{p+1}^V$ B_t^{**}, B_t^* are given respectively above in (28),(29),(30),(31),(33),(34),(35) and (36). In the standard immunization framework, the idea remains to match the portfolio sensitivities with those of the corresponding hedging instrument which,

By (39) can be performed by taking

$$\Theta_l^V + \Theta_l^{**} \cdot n^{**} - \Theta_l^* \cdot n^* = 0 \quad for \quad all \quad l \in \{0,1\dots,p\} \qquad (40)$$

Here (40) may be seen as a linear system of $p+1$ -equations with $I^{**} + I^*$ unknowns. Typically we are in the situation where $p + 1 \leq I^{**} + I^*$.

Even for the particular case $p + 1 = I^{**} + I^*$ and if the system admits a solution, a difficult arises since the variables defined by n^{**} and n^* are restricted to the integer numbers.

There is also the usual approach used by people by considering all n^{**} and n^* which minimize the square sum

$$\sum_{l=0}^{p} \left(\Theta_l^V + \Theta_l^{**} \cdot n^{**} - \Theta_l^* \cdot n^* \right)^2$$

However this should not the right way to follow, since not only we lose both the control of the size of the maximum hedging loss, and the attenuator effect brought by the term $\frac{1}{l!}\epsilon^l$. Therefore we have to afford the minimization problem raised in this Theorem. This is our object in the next Subsection 2.3.

2.3. Optimization problem

According to Theorem 2.3, the bond portfolio hedging is reduced to the minimization   problem (P_0) defined by integer linear constraints, such that we are face with an integer optimization problem. Observe that the objective function is both non-linear, non-convex and non-differentiable at the origin. To overcome these difficulties we make use of a known linearization technique as in [2],[6],[9] which consists to replace the initial problem (P_0) by an equivalent linear problem (P_1).

Recall that a Linear Programming (LP) problem has the form

$$(LP): x = argmin \{C^t x \mid \quad Ax \leq b , x \geq 0\} \qquad (41)$$

where $C \in \mathbb{R}^n , b \in \mathbb{R}^m , A \in \mathbb{R}^{m \times n}$ are given and x is a $\mathbb{R}^n$ unknown vector .

Here m and n are integer numbers which represent respectively the number of constraints and the number of variables. LP is well-known and may be solved either by an exact methods, referred as the simplex method as pioneered by Dantzig 1974, or interior point methods   as introduced by Dinkin 1967 and also by the Karmarkar algorithm 1984, [6] .

Coming back to our problem (P_0) and assuming that $\varepsilon(\cdot)$ is a fixed given constant, then we are lead to introduce the following function

$$G(x, n^{**}, n^{*}) = \sum_{l=0}^{p} x_l + \frac{1}{(p+1)!} \left\{ \Upsilon_{p+1}^{V} + \sum_{i^{**}=1}^{I^{**}} \Upsilon_{p+1;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^{*}=1}^{I^{*}} \Upsilon_{p+1;i^{*}}^{*} n_{i^{*}}^{*} \right\} \epsilon^{p+1}$$

$$(42)$$

where the components $x_l's$ of x are real variables. Actually, our motivation here is to remove the non-linearity by setting

$$x_l = \frac{1}{l!} \left| \Theta_l^{V} + \sum_{i^{**}=1}^{I^{**}} \Theta_{l;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^{*}=1}^{I^{*}} \Theta_{l;i^{*}}^{*} n_{i^{*}}^{*} \right| \epsilon^l \text{ for all } l \in \{0,1 \dots p\} \qquad (43)$$

Therefore we obtain the following result.

Proposition 3.4 .*The problem (P_0) is equivalent to the following minimization Problem*

$$(P_1) : \boldsymbol{x} = \boldsymbol{argmin} \{G(x, n^{**}, n^{*}) \mid (x, n^{**}, n^{*}) \in D_1 \} \qquad (44)$$

where D_1 is defined as the set of triplets (x, n^{**}, n^{*}) satisfying the constraints

$$a^{**} n^{**} + a^{*} n^{*} \leq D$$

$$0 \leq x_l + \frac{1}{l!} \left| \Theta_l^{V} + \sum_{i^{**}=1}^{I^{**}} \Theta_{l;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^{*}=1}^{I^{*}} \Theta_{l;i^{*}}^{*} n_{i^{*}}^{*} \right| \epsilon^l \text{ for all } l \in \{0,1 \dots p\}$$

$$(45)$$

$$0 \leq x_l - \frac{1}{l!} \left| \Theta_l^{V} + \sum_{i^{**}=1}^{I^{**}} \Theta_{l;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^{*}=1}^{I^{*}} \Theta_{l;i^{*}}^{*} n_{i^{*}}^{*} \right| \epsilon^l \text{ for all } l \in \{0,1 \dots p\}$$

$$(46)$$

with the restrictions that $x = (x_l)_l$, $n^{**} \in \mathbb{N}^{I^{**}}$ and $n^* \in \mathbb{N}^{I^*}$. The objective function G is defined in (42).

In this Proposition, by the equivalence between (P_0) *and* (P_1) we mean that if an optimal solution (n^{**}, n^*) to (P_0) does exist, then (P_1) admits an optimal solution (x, n^{**}, n^*)[5] , and conversely if (x, n^{**}, n^*) is an optimal solution to (P_1) then (P_0) admits (n^{**}, n^*) as an optimal solution.

Therefore, with the above result, we are lead to solve problem (P_1) instead of (P_0) .

Observe that both the objective function and constraints associated with (P_1) are given by linear transformations, with mixed integer and real coefficients.Such a problem (P_1) is commonly referred as a Mixed Integer Linear Problem (MILP).

MILP is recognized as an NP-hard problem because of the non-convexity of the domain and the number of possible combinations of the variables. For small dimensions, MILP can be solved by exact methods that provide an exact optimal solutions. In this case the most of available exact methods are Branch and Bound, Branch and Cut, Branch and Price [5].

However the complexity of MILP exponentially increases with the number of variables and the above quoted methods can fail. To overcome this inconvenience, meta-heuristics methods (as Genetic Algorithm and Ant Colony Optimization [3]).

Usually there are various solvers which may be used to derive exact solution to the MILP. One modern commercial solver we make use here is the CPLEX solver 9.0.Details and references related to such an application are freely available on the web as for instance :

http://www.iro.umontreal.ca/~gendron/IFT6551/CPLEX/HTML/

---

[5] With $x = (x_l)_l$ and x_l defined from n^{**} *and* n^*

Proof of Proposition 3.4

Let us prove the equivalence between the two problems (P_0) and (P_1) In the one-hand, assume that (P_1) admits (x, n^{**}, n^*) as an optimal solution. It means that

$$G(x, n^{**}, n^*) \leq G(x, n^{**}, n^*) \quad for \ all \quad (x, n^{**}, n^*) \ \epsilon \ D_1 \tag{48}$$

Then we get that (n^{**}, n^*) is an optimal solution to problem (P_0) since for any $(n^{**}, n^*) \ \epsilon \ D_0$.

$F(n^{**}, n^*) = G(x, n^{**}, n^*)$ for $x = (x_l)_l$ and with x_l defined from in (43).

$\leq G(x, n^{**}, n^*)$ for any $(x, n^{**}, n^*) \epsilon \ D_1$ due to (48).

$= F(n^{**}, n^*)$ by taking $x = (x_l)_l$ with x_l defined from n^{**} and n^* as in (43).

Conversely in the other-hand, assume that P_0 admits (n^{**}, n^*) as an optimal solution, which means that

$$F(n^{**}, n^*) \leq F(n^{**}, n^*) \quad for \ all \quad (n^{**}, n^*) \ \epsilon \ D_0 \tag{49}$$

Let us define $x = (x_l)_l$ with x_l defined from n^{**} and n^* as in (43). Then we get that (x, n^{**}, n^*) is an optimal solution to problem (P_1) since for any $(x, n^{**}, n^*) \ \epsilon \ D_1$.

$G(x, n^{**}, n^*) = F(n^{**}, n^*)$

$\leq F(n^{**}, n^*)$ for any $(n^{**}, n^*) \epsilon \ D_0$ due to (49).

$= G(x, n^{**}, n^*)$ since $\frac{1}{l!} |\Theta_l^V + \sum_{i^{**}=1}^{I^{**}} \Theta_{l;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \Theta_{l;i^*}^* n_{i^*}^*| \epsilon^l \leq x_l$

due to (46) and (47).

3   Numerical illustrations

To illustrate our results, we consider the following yield curve (Table 1).

Since here only the yields for the 1,2,…11 year maturities are given, then we make use of a linear interpolation to get the yields for the other maturities. A plot

of this curve is displayed in Figure 1.

Table 1 : Annual yield curve

Maturity τ	1	2	3	4	5	6	7	8	9	10	11
$y(t; \tau)$	4.35	4.79	6.07	6.4	6.66	6.88	7.02	7.13	7.23	7.30	7.53

The characteristics of the bonds inside the portfolio to hedge are as Table 2. Therefore the portfolio to hedge has the value 96 911.2050 Euro, and we are in the case of the portfolio V_t, as defined in (5), such that $\tilde{I}^{**} = 4, \tilde{I}^* = 3$,

$\tilde{n}_1^{**} = 1\,000, \tilde{n}_2^{**} = 1\,500, \tilde{n}_3^{**} = 500, \tilde{n}_4^{**} = 750, \tilde{n}_5^{**} = 500, \tilde{n}_1^* = 1\,000,$

$\tilde{n}_2^* = 900 \ and \ \tilde{n}_3^* = 1\,000.$ For simplicity, all of these bonds are supposed to have the same coupon payment dates.
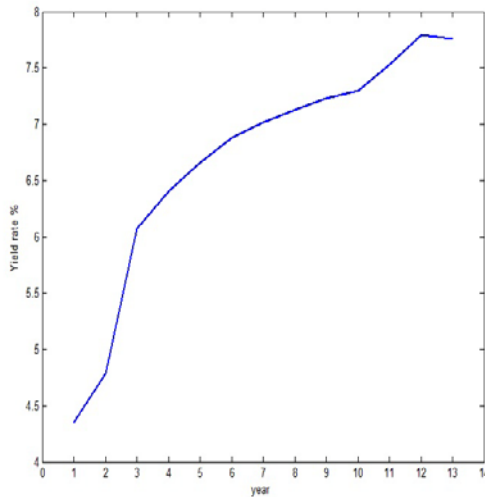

Figure 1: Annual yield curve

We work under the situation as Table 3.

The hedging operation is assumed to be done with the constraint of a maximal allowed amount $D = 9\,468.1\ Euro$ (which is roughly equal to 10% of the initial portfolio value V_t). The deposit and security borrowing rates are respectively $\lambda = 25\%$ and $\eta = 0.1\%$ The hedging horizon is 90days which corresponds to $s = 0.25$.   To simplify we take $\varepsilon^{\circ\circ} = \varepsilon^{\circ} = 25\%$ which corresponds to a large shift of the curve interest rate. For the sensitivities computations, we will restrict to the maximum order of $p = 5$. The hedging portfolio we will make use is built on various bonds whose the characteristics are summarized by the Table 4.

Table 2 : Portfolio Characteristics

Bond type	number	Coupon rate (%)	Maturity (years)	Price (Euro)
$\tilde{B}_1^{**}$	1000	3	3	91.4506
$\tilde{B}_2^{**}$	1500	5	4	94.7829
$\tilde{B}_3^{**}$	500	7	5	101.0106
$\tilde{B}_4^{**}$	750	4	10	76.3227
$\tilde{B}_5^{**}$	500	5	12	78.5785
$\tilde{B}_1^{*}$	1000	4	2	98.3289
$\tilde{B}_2^{*}$	900	5	3	96.8498
$\tilde{B}_3^{*}$	1000	6	4	98.256

Table 3: Situation under consideration

s	90 days
$P(t,s) = e^{-y(t,s)*s}$	0.9973
$\left(\dfrac{1}{P(t,s)} - 1\right)$	0.0027
λ	25%
η	0.1%
$\left(\lambda + \dfrac{\eta s}{P(t,s)}\right)$	0.342
$\tau(t_{k-1}, t_k)$	1 year
$\varepsilon°$	2.5%
$\varepsilon°°$	2.5%
$\epsilon = \max(\varepsilon°°, \varepsilon°)$	2.5%

Table 4 : Characteristics of the bonds used for the hedging

Bond type	B_1^{**}	B_2^{**}	B_1^{*}	B_2^{*}	B_3^{*}	B_4^{*}
Number bond	n_1^{**}	n_2^{**}	n_1^{*}	n_2^{*}	n_3^{*}	n_4^{*}
Coupon rate (%)	6.5	4.75	3.5	7	6.25	5
Maturity (years)	5	8	2	4	5	10
Price (Euro)	98.9153	85.1694	97.3958	101.7304	97.8677	83.3557

The sensitivities for all of these bonds may be summarized by the following:

Table 5 : Sensitivities of the bond used to hedging

l	0	1	2	3	4	5
$\Theta_{l;1}^{**}$	1.6830	419.5557	1 892.6648	8 764.8048	41 024.5417	193 041.0256
$\Theta_{l;2}^{**}$	1.4674	559.4606	4 047.1645	303 204.363	230 368.7089	1 762 505.555
$\Theta_{l;1}^{*}$	1.2900	169.4436	293.9647	512.5172	835.4642	1 565.9817
$\Theta_{l;2}^{*}$	1.8614	349.4567	1 256.0857	4 614.0959	17 100.7973	63 658.7391
$\Theta_{l;3}^{*}$	1.8410	416.7741	1 883.3104	8 723.0387	40878.0480	192 418.39408
$\Theta_{l;4}^{*}$	1.5850	645.0346	5 725.0584	53 287.2283	505 591.3337	4842855.4011

It should be observed here that for each bond B more the order considered is high, more the corresponding sensitivity has a high value. In contrast, the normalized term as $\frac{1}{l!} \mathbf{Sens}(l, t, s; B)\epsilon^l$, tends to have small values for large orders l. A similar observation may be made for the remainder terms introduced in Theorem2.2 . Indeed for $p = 5$ and $\epsilon = 2.5\%$ then we get the following

Table 6: Remainder term of the bonds used for the hedging

	Υ_{p+1}^i	$\dfrac{\Upsilon_{p+1}^i * \epsilon^{p+1}}{(p+1)!}$
$\Upsilon_{p+1;1}^{**}$	1 025 973.3781	$3.5e^{-007}$
$\Upsilon_{p+1;2}^{**}$	1 431 858.0459	$5.62e^{-006}$
$\Upsilon_{p+1;1}^{*}$	2 862.1783	$9.7e^{-010}$
$\Upsilon_{p+1;2}^{*}$	260 912.7687	$8.8e^{-008}$
$\Upsilon_{p+1;3}^{*}$	1 022 880.2930	$3.5e^{-007}$
$\Upsilon_{p+1;4}^{*}$	59 509 913.8555	$2.0e^{-005}$

The sensitivities and remainder term corresponding to the portfolio to hedge are summarized by the following:

Table 7 : Sensitivities of the portfolio to hedge

l	0	1	2	3	4	5
Θ_l^V	2 653.97	1 020 499.06	9 011 651.04	84 643 343.53	847 635 181.58	8 842 848 568.71

and

Table 8 : Remainder term of the portfolio to hedge

Υ_{p+1}^V	$\dfrac{1}{(p+1)!}\Upsilon_{p+1}^V * \epsilon^{p+1}$
124 775 708 343.03	$4.23e^{-002}$

Three hedging situations are now considered for comparisons.

First, we examine the hedging situation using with two types of bonds: 'one in long and one in short position' made respectively by bonds B_1^{**} and B_1^{*} as described in Table 4. The numbers $\boldsymbol{n_1^{**}}$ and $\boldsymbol{n_1^{*}}$ of bonds B_1^{**} and B_1^{*} respectively required for the hedging, as described in the above Theorem 2.2 and Proposition3. 4, may be determined by using the IBM ILOG CPLEX' solver. After 0.08 second running time (in our computer processor :AMD Sempron(tm) M120 2.10 GHz)), we get the following result:

Table 9 : Result hedging instruments with two bonds

n_1^{**}	n_1^*	$F(n^{**}, n^*)$
0	6 023	Loss=7 607.09 Euro

Here the number $7\,607.087\ Euro$ represents the (possible) maximum loss corresponding to the shift $\epsilon = 2.5\%$. Of course the loss should be less than this value for any shift $\varepsilon \in (-2.5\%, 2.5\%)$. It may be observed that this maximum loss represents around 7.85% of the portfolio initial value V_t.

Second, we consider the hedging situation when using four bonds made by B_1^{**}, B_2^{**} in long position and B_1^*, B_2^* in short position. After a running time of 0.09 second we get the values of $n_1^{**}, n_2^{**}, n_1^*\ and\ n_2^*$ as follows:

Table 10 : Result hedging instruments four bonds

n_1^{**}	n_2^{**}	n_1^*	n_2^*	$F(n^{**}, n^*)$
1	0	1	2 921	Loss=4 652.36 Euro

It appears that the possible maximum loss value is 4.8% of the portfolio value. This loss is less than the one which possibly arises in the case where just two types of bonds are used for the hedging.

Finally we assume the hedging situation under six types of bonds made by B_1^{**}, B_2^{**} in long position and $B_1^*, B_2^*, B_3^*, B_4^*$ in short position. The values of n_1^{**}, n_2^{**}, $n_1^*, n_2^*, n_3^*\ and\ n_4^*$ obtained after a running time of 0.08 second are summarized by

Table 11: Result hedging instrument bond with six bonds

n_1^{**}	n_2^{**}	n_1^*	n_2^*	n_3^*	n_4^*	$F(n^{**}, n^*)$
97	336	3	2	289	1748	Loss=0.68

It appears clearly here that the possible maximum loss is strongly reduced and takes the insignificant value of $0.6760\ Euro$. Therefore the hedging operation seems to be theoretically and practically perfect. However we should be aware that for the considered example the transaction costs are not considered. Moreover in many interest rate markets, we can have just very few available (risk-free) bonds for the trading and hedging perspectives.

4   Conclusion

The present paper is devoted to the hedging of a bond portfolio under a parallel shift of the interest rate curve, as pioneered by various authors as F. Macaulay (1938), F. Redington (1952) and L. Fisher & R. Weil [2]. Their approach is enhanced here, in the sense that we take into account the passage of time and deal with non-infinitesimal yield shift. As presented in the above development, and proved in our previous working paper [5], from our approach it becomes possible to accurately monitor the hedging error. Actually for this last quantity, a deterministic and pointwise estimate may be obtained when using the hedger's view about the curve shift. Moreover in this work, we have taken into account some facts that are less considered in literature related to hedging of bond portfolios. For instance it is common to consider short positions without taking care of the associated transaction fees. Moreover the hedging allocation is solved in term of bond proportions rather than in term of security numbers. Taking these

issues into account, it is displayed here that a non-linear and integer optimization problem has to be considered. Then we have proved that   this last is equivalent to a Mixed-Integer-Linear Problem, which can be solved by making use of various solvers as the CPLEX-software. With the illustrative examples introduced above, it appears that the hedge quality may be improved when the hedging portfolio is made by a large number of bonds in long and short positions.

The main assumption, about the parallel shift of the interest rate curve, underlying the results of this paper is too restrictive and has a few probabilities to happen in real market. However with the fact that our approach allows a shift of arbitrary size, then the result obtained here may be of interest in the perspective of obtaining useful indication for a stressed and/or extreme situation.

The particular situation of a parallel shift of the curve remains also to be a practical and theoretical benchmark for any comparison of a bond portfolio immunization under a more realistic term structure of the interest rate.

Finally, the approach and idea introduced in this paper appear to be useful when tackling the bond portfolio immunization under an assumption of an interest rate curve moving in a non-parallel fashion. As an example, the case of a term structure driven by a one-uncertainty factor is recently performed by the third author in [10].

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