Journal of Finance and Investment Analysis, vol.1, no.2, 2012, 157-199
ISSN: 2241-0988 (print version), 2241-0996 (online)
International Scientific Press, 2012

Project selection of robust portfolio models with incomplete information

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Abstract

Robust Portfolio Modeling (RPM) Theory is a decision-support methodology to analyze multiple criteria project portfolio problems. Liesioa et al [17] generalized RPM based on the appendix information, and studied the characteristics of non-inferior solution sets, but they did not compare the portfolio of each other in this non-inferior solution sets, and could not offer a precise decision. This paper considers this question, and gives a positive answer. The results posed in this paper can be regarded as a natural generalization of the work [17].

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Article Info: *Received* : March 20, 2012. *Revised* : April 11, 2012
Published online : May 31, 2012

JEL classification numbers: G11, M48

Keywords: Preference programming, Incomplete information, Interval number, Portfolio

1 Introduction

Portfolio selection theory by H. Markowitz [19] initiated a new epoch for the mathematical tool to studying the financial issues; F. W. McFarlan [21] first applied this modern portfolio theory to study the project selection and risk management.

It is well known that the core idea of portfolio theory is to balance risks and returns, and it requires investors to make a trade-off between the high risk-high return and the low risk-low return.

For the research headway of should be the of portfolio selection theory, we now first take an overview of this theory as follows.

(A) The kind of return-risk models represented by Markowitz [19]; The single-index model by Sharpe [31]; The mean-lower-semi variance model by Markowitz [20] and Mao [23]; The mean-variance skewness model by Konno and Suzuki [16];

(B) The model of controlling the probability of losses represented by VaR: safety first model by Roy [27]; VaR-model by Philippe [26]; CVaR -model by Rockafenar and Uryasev [29];

(C) The selection model of dynamic portfolios: Mossin [24] used the dynamic programming method to extend the single-term model to the multi-term situation; Yin and Zhou [36] proposed a discrete-time mean-variance portfolio selection model; Zhou and Yin [38] studied the continuous-time mean-variance portfolio theory.

There are many other methods of portfolio managements as follows:

1) Financial models: The net present value (NPV), the method of internal should be intrinsic rates of returns (IRR), the method of financial ratios J. F. Bard [3];

2) Stochastic financial models: Monte Carlo method, Decision tree method, Option pricing theory, etc., T. Luehrman [18];

3) Scoring models: score and grade should be grade approaches according to the quantified problems by D.L.Hall and A.Naudia [12];

4) AHP methods: the multi-objective decision-making approach which was most commonly used and usually compared based on the project, such as the scoring method;

5) Behavioral approaches: decision-makers being involved in decision-making models, such as the Delphi method and Q-sort Ulrich K. Eppinger S.[33]; Beaujon et al [4] proposed a mixed integer programming model to solve the project portfolio issues and obtained an optimal solution; Dickinson et al [8] proposed the concept of the trust matrix to describe the relationship among projects, and presented a multi-stage model of optimal portfolios of projects; Golabi et al.[10] took the preference in the solar energy project portfolio selection into account; Kleinmuntz and Kleinmuntz [15] adopted the strategy of approaching the valuation of health care funds; Stummer and Heidenberger [32] considered the interactivity of the project with multi-attribute parameters; G. Iyengar and D. Goldfarb [14] established a robust portfolio investment model with regard to the case of the uncertain information.

On the other hand, the multi-criteria decision was from the concept of Pareto optimal proposed by Pareto, it wasn't until 1960s that the multi-criteria decision really were turned into a normative decision-making approach by Charnes and Cooper [6], and the Electre method by Roy [28]; Hwang and Yoon [13] divided the multi-criteria decision problems into the multi-attribute decision making and the multi-objective decision making, and discussed them separately. Mac Crimmon [22] summarized the methods and applications of the multi-attribute decision making, and presented many potential conceptual methods, etc.

At the same time, Von Neumann and Morgenstern [25] proposed the axiom system of the expected utility, and obtained the conclusion that the decision-makers could maximize the effectiveness; Savage [30] carried out a further expansion to expected utility model with the subjective probability factor; Zadeh [37] proposed the concept of fuzzy sets, and distinguished between vague and random, and applied it to a decision making. Since 1970s, S. Greco et al [11] and others compared the fuzzy sets, and studied the sorting methods; Bellman and Zadeh [5] proposed a basic model of fuzzy decisions based on the multi-objective decision; Danzig [7] studied the stochastic programming; The typical representation of description targets were rough sets and multivariable statistical methods, S. Greco et al. [11] and others carried out some remarkable researches on rough sets.

For the theory of RPM, Liesiöa et al [17] made a further expansion of RPM and studied the properties of non-inferior solution sets based on the impact of the additional information, but they did not make further comparisons on the programs in non-inferior solution sets, and thus they failed to provide an accurate decision which can make a program to decision makers.

Motivated by these statements above, it is interesting to study and provide an optimal program to a decision-maker in non inferior solution sets. We in this paper wish to study it and present an accurate decision-making program in the case when the decision maker's preference is considered.

The arrangement of this paper is as follows. In the first-two Sections, we state some necessary notations and terminologies. In Section 3, we first discuss the narrowing role of the preference in non-inferior solution sets, and describe that any form of preference plans would not lead any new element into the non-inferior solution set. Next, by using a few decision-making criteria related to the stability, we present the construction of ordering relations under the case of the complete uncertainty information with a priori probability, and give the algorithm of optimal solutions based on the ordering relation. Section 4 considers the case of projects with variable scales. For the case of variable scales, it poses a method of stable optimal solutions considering the risk preference of decision makers. In this setting, it gives the algorithm process to solve the stable optimal solution with the risk preference of decision makers.

The results obtained in this paper are interesting and can be regarded as a natural generalization of the classical conclusions for project decisions.

2 Preliminary Notes

2.1 The robust portfolio model with incomplete information

J. Liesiöa et al [17] described the extended RPM as follows:

$$Max \quad V = \sum_{i=1}^{n} x_i v_i$$
$$s.t. \begin{cases} C(P) \le B \\ x_i \in [0,1] \cap N \end{cases}$$

where V is the total revenue of a portfolio program, x_i is the i-th program in the alternative project library, and takes only 0 or 1; v_i is the comprehensive income of the i-th program in the alternative project library, C indicates the consumed resource, B indicates the constrained resource.

In the evaluation of multi-indexes, it is easy to see that the return of a portfolio p is given as follows

$$V(p,w,v) = \sum_{x^j \in p} V(x^j) = \sum_{x^j \in p} \sum_{i=1}^{n} w_i v_i^j \tag{1}$$

where v_i^j is the return score of the alternative project x^j with respect to the i-th indicator, $v_i^j \in [0,1]$, $v^j = [v_1^j, \cdots, v_n^j]$ means the return score vector of the j-th alternative project; $w=(w_1,\cdots,w_n)$, $w\in S_w^0=\{w\in R^n \mid w_i\ge 0,\sum_{i=1}^{n} w_i=1\}$, where w_i is used to measure the relative importance of the i-th evaluated parameter; $V(p,w,v)$ is the comprehensive income of the portfolio p; $p\subseteq X$ is a subset of viable projects and indicates a feasible portfolio. Rewrite the formula (1) and get:

$$V(p,w,v)=\sum_{i=1}^{n} w_i \sum_{x^j\in p} v_i^j \tag{2}$$

where $\sum_{x^j\in p} v_i^j$ is the return score of the portfolio p with the i-th indicator.

The portfolio collection satisfying some constraint conditions can be denoted by:

$$P_F=\{p\in P \mid C(p)\le B\} \tag{3}$$

Considering the incomplete information, the upper and lower bounds of return scores v_i^j of the alternative project x^j with respect to the i-th indicator are $\overline{v}_i^j$ and $\underline{v}_i^j$, respectively, so, for the indicators of program properties, the incomplete information set can be described as an interval $[\underline{v}_i^j,\overline{v}_i^j]$ containing the return score v_i^j. The weight set S_w is a convex set which is limited by the decision maker's preference and expressed by a linear inequality. Then, the feasible solution set S_v can be expressed as $S_v=\{v\in R_+^{m\times n} \mid v_i^j\in[\underline{v}_i^j,\overline{v}_i^j]\}$.

For a given portfolio *p*, there is $V(p,w,v)\in[\min_{w\in S_w}\underline{V}(p,w),\max_{w\in S_w}\overline{V}(p,w)]$, $w\in S_w$, $v\in S_v$, where $V(p,w,v)$ is the comprehensive income of the portfolio *p*.

Considering two mappings $\overline{V},\underline{V}: P\times S_w^0\to R_+$ given as the following

$$\overline{V}(p,w)=\sum_{x^j\in p}\sum_{i=1}^{n} w_i\overline{v}_i^j \tag{4}$$

$$\underline{V}(p,w)=\sum_{x^j\in p}\sum_{i=1}^{n} w_i\underline{v}_i^j \tag{5}$$

The formulae (4) and (5) indicate the upper and lower bounds of comprehensive incomes of the portfolio P. We denote by $S \equiv S_w \times S_v$ the information set. Then, $s(w,v) \in S$ means $w \in S_w, v \in S_v$. Thus, S is a completely information set with respect to weights and the parameter value.

Referring to (2) and (3), the project portfolio model with incomplete information can be expressed as follows:

$$Max \quad V(p,w,v) = \sum_{i=1}^{n} w_i \sum_{x^j \in p} v_i^j$$

$$\text{s.t.} \quad \begin{cases} C(p) \le B \\ x_i \in [0,1] \cap N \end{cases}$$

where $v_i^j \in [\underline{v}_i^j, \overline{v}_i^j]$, $w \in S_w \subseteq S_w^0 = \{w \in R^n \mid w_i \ge 0, \sum_{i=1}^{n} w_i = 1\}$.

2.2 The order relations and non-inferior set

Definition 2.2.1 [17] *If*

$$\begin{cases} \mathrm{V}(p,w,v) \ge \mathrm{V}(p',w,v), \forall (w,v) \in S \\ \mathrm{V}(p,w,v) > \mathrm{V}(p',w,v),\ \exists (w,v) \in S,\ p,p' \in P \end{cases},$$

then we call $p \succ p'$. For two portfolios $p, p' \in P$, $p \succ p'$ means that the portfolio p is superior than p' in S.

The following Theorem shows the comparison of two portfolios.

Definition 2.2.2 *The non-inferior set referring to information set S is denoted by $\mathrm{P}_\mathrm{N}(S)$, and defined as $P_\mathrm{N}(S) = \{p \in P_F \mid p \not\succ_s p', \forall p' \in P_F\}$.*

If there's no disagreement about the information of S, we mark that $P_\mathrm{N} \equiv P_\mathrm{N}(S)$.

2.3 The calculation of non-inferior sets

If the score of characteristic assessments is completely determined, the calculation of non-inferior solution sets is easy and achieved should be easily achieved easy and achieved through the gradual reduction algorithm of the multi-objective multi-attribute (see Ehrgott and Gandibleux [9]), but up to now, the gradual reduction algorithm of multi-objectives under the incomplete information has not been solved. J. Liesiöa [17] proposed a dynamic programming algorithm to calculate the non- inferior set.

To facilitate the narrative, let us define the sets of two portfolios below:

$P_F^k = \{p \in P_F \mid p \subseteq \{x^1, \ldots, x^k\}\}$, $P_N^k = \{p \in P_F^k \mid \nexists p' \in P_F^k, s.t\ p' \succ p, C(p') \leq C(p)\}$

$k = 1, \cdots, m$, where P_N^k is a non-inferior solution set of k alternative programs.

3 Project Selection Problems of Robust Portfolio Models Based on Incomplete Information

For a given non-inferior set $P_N(S)$, it is naturally to ask a question that for the element of solutions in an non-inferior solution set, how to compare it with others, how to find out an optimal solution in so many non-inferior solutions without undermining the stability of the model.

For the stability of solutions, we think that the pessimistic person should use the pessimistic criterion (max-min criterion) to get the best choice under the worst situation, the program selected at this point is recorded as A_{worst}. Taking an acceptable possible value P_{worst}, and fitting the project with the possibility being larger than P_{worst} into the alternative set. Next, considering the regret criterion (min-max criterion), and the maximum possible loss, in this setting, one can take the maximum possible loss that the decision maker accepted as L_{worst}, if there is any element in the alternative set with the maximum possible loss which is less than L_{worst} comparing with program A_{worst}, then one can select the one with the largest expectation satisfying the claim. Then, the optimal solution A^* is obtained.

The optimal solution A^* in general is stable even though it may take the one which is poorer than the lower bound of program A_{worst}, and this loss-possibility is less than P_{worst}.

3.1 The form-preference of a information set S

An information set with preferences is denoted by $\widetilde{S}$, this information set contains the no-weight information S_w and the property indicator information S_v. The preference is expressed by the weight w and the assessment score v of index returns. Set $S \equiv S_w \times S_v$, it is not hard to show that there holds the following.

Lemma 3.1.1 *If* $\widetilde{S}_w \subset S_w, \widetilde{S}_v \subset S_v$, *then* $\widetilde{S} \subset S$.

Lemma 3.1.2 *If* $\widetilde{S}_w \subseteq S_w, \widetilde{S}_v \subset S_v$, *then* $\widetilde{S} \subset S$.

Theorem 3.1.1 *For the information set* $\widetilde{S}_w$ *and* $\widetilde{S}_v$, *if one of them at least is really included in* S, *where* $S \equiv S_w \times S_v$, $\operatorname{int}(S) \cap \widetilde{S} \neq \varnothing$, *then there holds* $P_N(\widetilde{S}) \subseteq P_N(S)$.

Proof. By Lemma 3.1.1 and Lemma 3.1.2 and the work of J. Liesiöa et al [17], it is not hard to show that Theorem 3.1.1 is tenable. This ends the proof of Theorem 3.1.1. □

It is well known from Theorem 3.1.1 that the weight w and the additional preference information of the assessment score v do not add any new element to the non-inferior solution set. Thus, we can present the optimal solution of non-inferior solution sets and narrow the non-inferior solution set based on the earlier works.

The form preference of the order relationship $\succ$

In order to narrow the non-inferior set $P_N(S)$, we need the more strict partial order relation to make the comparing project be clearer. Here's the definition of the partial order structure in order relations.

Definition 3.1.1 *If* $\begin{cases} \forall \quad p, p' \in P, p \succ_1 p' \Rightarrow p' \succ_2 p \\ \exists \quad p, p' \in P, p' \succ_2 p \ , \ p \not\succ_1 p' \end{cases}$, *it is called that* $\succ_2$ *is stronger than* $\succ_1$ *defined on P.*

Theorem 3.1.2 *Suppose that* $\succ_2$ *is stronger than* $\succ_1$, *then* $P_N(\succ_2) \subseteq P_N(\succ_1)$.

Proof. Assume that $P_N(\succ_2) \subseteq P_N(\succ_1)$ is not tenable, then there is at least a $p_0 \in P_N(\succ_2)$, but $p_0 \notin P_N(\succ_1)$, thus $p_0 \in P_F \setminus P_N(\succ_1)$. By Definition of the non-inferior set, we know that

$$\begin{aligned} &\forall p' \in P_F \setminus P_N(\succ_2), \textit{then } p_0 \succ_2 p \\ &\forall p \in P_N(\succ_1), p_0 \in P_F \setminus P_N(\succ_1), \textit{then } p \succ_1 p_0 \end{aligned} \tag{6}$$

Since $\succ_2$ is stronger than $\succ_1$, then there is

$$\forall p \in P_N(\succ_1), p_0 \in P_F \setminus P_N(\succ_1), \textit{then } p \succ_2 p_0 \tag{7}$$

Formulas (6) and (7) are contradictory $\therefore P_N(\succ_2) \subseteq P_N(\succ_1)$. This ends the proof of Theorem 3.1.2. □

3.2 The determination of objective weights

Definition 3.2.1 [35] *Define* $\tilde{a}$ *as* $\tilde{a} = [\underline{a}, \overline{a}] = \{a \mid \underline{a} \le a \le \overline{a}, \underline{a}, \overline{a} \in R\}$. $\tilde{a}$ *is called an interval number. If* $\overline{a} = \underline{a}$, $\tilde{a}$ *degenerates into a real number. The center and width of the interval number* $\tilde{a}$ *are defined as*

$$m(a) = \frac{1}{2}[\underline{a} + \overline{a}], \quad w(a) = \frac{1}{2}[\overline{a} - \underline{a}],$$

respectively.

Definition 3.2.2 [1] *Let* $\circ \in \{+, -, \times, \div\}$ *be a binary operation on a real space, we call* $\tilde{a} \circ \tilde{b} = \{x \circ y : x \in \tilde{a}, y \in \tilde{b}\}$ *a binary operation of all closed interval sets. In the setting of division operation,* $\tilde{b}$ *can not be equal to 0. The other operations can be stated as*

$$\tilde{a} + \tilde{b} = [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \quad \tilde{a} - \tilde{b} = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$$

$$k\tilde{a} = k[\underline{a}, \overline{a}] = \begin{cases} [k\underline{a}, k\overline{a}], & k \ge 0 \\ [k\overline{a}, k\underline{a}], & k < 0 \end{cases}$$

Definition 3.2.3 *Define*

$$P(\tilde{a} \ge \tilde{b}) = \frac{(\overline{a} - \underline{a}) + (\overline{b} - \underline{b}) - \min\{\overline{b} - \underline{a}, 0\}}{(\overline{a} - \underline{a}) + (\overline{b} - \underline{b})} = 1 - \frac{\min\{\overline{b} - \underline{a}, 0\}}{(\overline{a} - \underline{a}) + (\overline{b} - \underline{b})}$$

as an approximate estimated value $P(\tilde{a} \ge \tilde{b})$ *of interval numbers.*

It is easy to see that $\Pr ob(\tilde{a} \ge \tilde{b}) + \Pr ob(\tilde{b} \ge \tilde{a}) = 1$.

Suppose $\sum_{i=1}^{n} w_i^2 = 1$ and $w = (w_1, w_2, \cdots w_n)$. The decision-making matrix is

$$\tilde{v} = [\tilde{v}^1, \cdots, \tilde{v}^m] = \begin{bmatrix} \tilde{v}_1^1 & \tilde{v}_1^2 & \cdots & \tilde{v}_1^m \\ \tilde{v}_2^1 & \tilde{v}_2^2 & \cdots & \tilde{v}_2^m \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{v}_n^1 & \tilde{v}_n^2 & \cdots & \tilde{v}_n^m \end{bmatrix}.$$

Making a standardized treatment on $\tilde{v}$, then one arrives at a standardized matrix $\widetilde{R} = (\widetilde{r}_i^{\,j})_{m\times n}$.

The normalization formula is

$$\widetilde{r}_i^{\,j} = \frac{\widetilde{v}_i^{\,j}}{\max\limits_{i\in\{1,2,\cdots,n\}} \overline{v}_i^{\,j}} \tag{8}$$

From the algorithm of interval numbers, it is easy to know that r_i^j is an interval number, and denoted by $\widetilde{r}_i^{\,j} = [\underline{r}_i^{\,j}, \overline{r}_i^{\,j}]$. It is obvious that the consolidated revenue is

$$V(p,w,v) = \sum_{i=1}^{n} w_i \sum_{x^j\in p} v_i^j .$$

Xu [34] thinks that the departure $d(\tilde{a},\tilde{b}) = \left\|\tilde{a}-\tilde{b}\right\| = \left|\overline{b}-\overline{a}\right| + \left|\underline{b}-\underline{a}\right|$ between interval numbers $\tilde{a}$ and $\tilde{b}$ reflects the ease-level of comparing programs of each other. In this article, a different view is presented. The departure-degree is an absolute quantity, but the ease-level to compare two intervals should be reflected by the possibility of comparing their sizes, and should be a relative number.

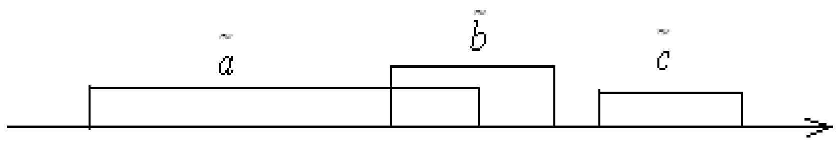

Figure 1: The simple case: a, b, c are of comparable

As shown in Figure 1, it is easy to see that $d(\tilde{a},\tilde{b}) > d(\tilde{c},\tilde{b})$ and $\tilde{c} > \tilde{b}$. The relationship between interval numbers $\tilde{a}$ and $\tilde{b}$ is not very clear, but it is very easy by Definition 3.2.3 to reflect it.

Determination of weights with subjective preferences on projects

Definition 3.2.4 $\alpha(\tilde{a},\tilde{b}) = \dfrac{\max(\underline{a},\underline{b}) - \min(\overline{a},\overline{b})}{(\overline{a}-\underline{a})+(\overline{b}-\underline{b})}$ *is called the coincidence degree of the interval numbers* $\tilde{a}$ *and* $\tilde{b}$, *where* $\tilde{a}=[\underline{a},\overline{a}]$, $\tilde{b}=[\underline{b},\overline{b}]$.

Assume the preference of normalized alternative projects x^i is $\tilde{\theta}_i = [\underline{\theta}_i, \overline{\theta}_i] \subseteq [0,1]$, and the attribute weight satisfying the unit constraint $\sum_{i=1}^{n} w_i^2 = 1$ is $w = (w_1, w_2, \cdots w_n)$.

The objective function is as follows:

$$\max \alpha_V(w) = \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha(\tilde{r}_i^{\,j},\tilde{\theta}_i)\cdot w_j = \sum_{i=1}^{n}\sum_{j=1}^{m}\frac{\max\{\underline{r}_i^{\,j},\underline{\theta}_i\}-\min\{\overline{r}_i^{\,j},\overline{\theta}_i\}}{(\overline{r}_i^{\,j}-\underline{r}_i^{\,j})-(\overline{\theta}_i-\underline{\theta}_i)}\cdot w_j$$

Then we can translate it into the following optimization problem:

$$\begin{cases} \max \alpha_V(w) = \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha(\tilde{r}_i^{\,j},\tilde{\theta}_i)\cdot w_j = \sum_{i=1}^{n}\sum_{j=1}^{m}\dfrac{\max\{\underline{r}_i^{\,j},\underline{\theta}_i\}-\min\{\overline{r}_i^{\,j},\overline{\theta}_i\}}{(\overline{r}_i^{\,j}-\underline{r}_i^{\,j})-(\overline{\theta}_i-\underline{\theta}_i)}\cdot w_j \\ s.t. \quad w_j \geq 0, j=1,2,\cdots,n, \ \sum_{j=1}^{n} w_j^2 = 1 \end{cases}$$

Considering a Lagrange function below

$$L(w,\zeta) = \sum_{i=1}^{n}\sum_{j=1}^{m}\frac{\max(\underline{r_i^{\,j}},\underline{\vartheta_i})-\min(\overline{r_i^{\,j}},\overline{\vartheta_i})}{(\overline{r_i^{\,j}}-\underline{r_i^{\,j}})+(\overline{\vartheta_i}-\underline{\vartheta_i})}\cdot w_j + \frac{1}{2}\zeta\Big(\sum_{j=1}^{n} w_j^2 - 1\Big).$$

Take the partial derivative to this Lagrange function, and make it be equal to 0, then we have

$$\begin{cases} \dfrac{\partial L}{\partial w_j} = \sum_{i=1}^{n}\dfrac{\max(\underline{r_i^{\,j}},\underline{\vartheta_i})-\min(\overline{r_i^{\,j}},\overline{\vartheta_i})}{(\overline{r_i^{\,j}}-\underline{r_i^{\,j}})+(\overline{\vartheta_i}-\underline{\vartheta_i})} + \zeta w_j = 0. \\ \dfrac{\partial L}{\partial \zeta} = \sum_{j=1}^{m} w_j^2 - 1 = 0 \end{cases}$$

The optimal solution is

$$w_j^0 = \frac{\sum_{i=1}^{n} \frac{\max(\underline{r}_i^j, \underline{\vartheta}_i) - \min(\overline{r}_i^j, \overline{\vartheta}_i)}{(\overline{r}_i^j - \underline{r}_i^j) + (\overline{\vartheta}_i - \underline{\vartheta}_i)}}{\sqrt{\sum_{j=1}^{m} \left(\sum_{i=1}^{n} \frac{\max(\underline{r}_i^j, \underline{\vartheta}_i) - \min(\overline{r}_i^j, \overline{\vartheta}_i)}{(\overline{r}_i^j - \underline{r}_i^j) + (\overline{\vartheta}_i - \underline{\vartheta}_i)} \right)^2}}.$$

Normalize w^* by $w_j^* = \frac{w_j^0}{\sum_{j=1}^{m} w_j^0}$, we arrive at

$$w_j^* = \frac{\sum_{i=1}^{n} \frac{\max(\underline{r}_i^j, \underline{\vartheta}_i) - \min(\overline{r}_i^j, \overline{\vartheta}_i)}{(\overline{r}_i^j - \underline{r}_i^j) + (\overline{\vartheta}_i - \underline{\vartheta}_i)}}{\sum_{j=1}^{m} \sum_{i=1}^{n} \frac{\max(\underline{r}_i^j, \underline{\vartheta}_i) - \min(\overline{r}_i^j, \overline{\vartheta}_i)}{(\overline{r}_i^j - \underline{r}_i^j) + (\overline{\vartheta}_i - \underline{\vartheta}_i)}} \tag{9}$$

That's the objective weight required to determine.

3.3 The optimal solution and the algorithm

Definition 3.3.1 *Let* $\tilde{a}$ *and* $\tilde{b}$ *be two interval numbers or one of them be an interval number, where* $\tilde{a} = [\underline{a}, \overline{a}]$, $\tilde{b} = [\underline{b}, \overline{b}]$, *we call*

$$p(\tilde{a} \ge \tilde{b}) = \begin{cases} 1, & \overline{a} \ge \overline{b} \\ \frac{1}{2} \cdot \frac{\overline{b} - \underline{b}}{\overline{a} - \underline{a}} + \frac{\overline{a} - \overline{b}}{\overline{a} - \underline{a}} & \underline{a} \le \underline{b} \le \overline{b} \le \overline{a} \\ \frac{\underline{a} - \underline{b}}{\overline{b} - \underline{b}} + \frac{\overline{b} - \underline{a}}{\overline{b} - \underline{b}} \cdot \frac{\overline{a} - \overline{b}}{\overline{a} - \underline{a}} + \frac{1}{2} \cdot \frac{\overline{b} - \underline{a}}{\overline{a} - \underline{a}} \cdot \frac{\overline{b} - \underline{a}}{\overline{b} - \underline{b}} & \underline{a} \le \underline{b} \le \overline{b} \le \overline{a} \end{cases} \tag{10}$$

the possibility of $\tilde{a} \ge \tilde{b}$, *and denote by the order relationship of* $\tilde{a}$ *and* $\tilde{b}$ *as* $\tilde{a} \ge_p \tilde{b}$.

The calculation of the possibility of $\tilde{a} \ge \tilde{b}$ is divided into three cases as the following figures:

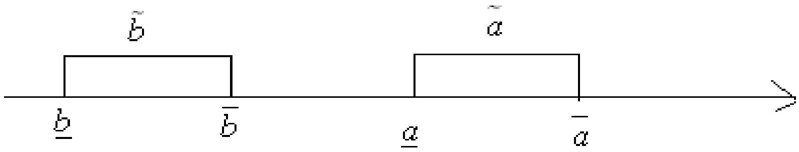

Figure 2: a and b have no overlapped, and a>b

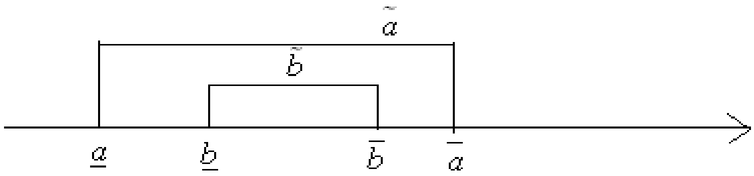

Figure 3: b is included in a.

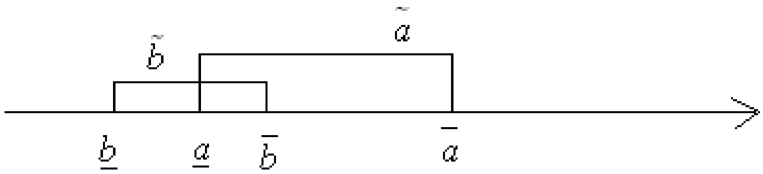

Figure 4: a and b have part overlapped

From Figures 2,3,4, it is easy to see that the possibility of $\tilde{a} \geq \tilde{b}$ is

$$\begin{cases} 1, & \bar{a} \geq \bar{b} \\ \frac{1}{2}\cdot\frac{\bar{b}-\underline{b}}{\bar{a}-\underline{a}}+\frac{\bar{a}-\bar{b}}{\bar{a}-\underline{a}} & \underline{a} \leq \underline{b} \leq \bar{b} \leq \bar{a} \\ \frac{\underline{a}-\underline{b}}{\bar{b}-\underline{b}}+\frac{\bar{b}-\underline{a}}{\bar{b}-\underline{b}}\cdot\frac{\bar{a}-\bar{b}}{\bar{a}-\underline{a}}+\frac{1}{2}\cdot\frac{\bar{b}-\underline{a}}{\bar{a}-\underline{a}}\cdot\frac{\bar{b}-\underline{a}}{\bar{b}-\underline{b}} & \underline{b} \leq \underline{a} \leq \bar{b} \leq \bar{a} \end{cases}$$

Based on the regret criteria (min-max criteria), one can define the concept of the maximum loss miscarriage of justices as follows.

Definition 3.3.2 *Suppose that the correct order of portfolio investments is " p is superior than p' ", and the maximum loss miscarriage of justices for investing* *portfolios* p *and* p' *in the order-relationship is* $\overline{V}(p,w,v)-\underline{V}(p',w,v)$ *, and denote it by* $l_{p\backslash p'}$.

1. The definition of the order relationship based on a median

Definition 3.3.3 *Define a relationship* $\succ_1$ *on P as follows:*

If $\begin{cases} \mathrm{m}(\mathrm{V}(p,w,v)) \geq \mathrm{m}(\mathrm{V}(p',w,v)), & \forall (w,v) \in S \\ \mathrm{m}(\mathrm{V}(p,w,v)) > \mathrm{m}(\mathrm{V}(p',w,v)), & \exists (w,v) \in S \end{cases}$ *, then we say* $p \succ_1 p'$ *.*

The expression of $p \succ_1 p'$ can be obtained by Definition 3.2.1, and the calculation is as follows:

$$
\begin{aligned}
& p \succ_1 p' \\
\Leftrightarrow & \begin{cases} \mathrm{m}(\mathrm{V}(p,w,v)) \geq \mathrm{m}(\mathrm{V}(p',w,v)), & \forall (w,v) \in S \\ \mathrm{m}(\mathrm{V}(p,w,v)) > \mathrm{m}(\mathrm{V}(p',w,v)), & \exists (w,v) \in S \end{cases} \\
\Leftrightarrow & \frac{1}{2}(\overline{V}(p,w)+\underline{V}(p,w)) > \frac{1}{2}(\overline{V}(p',w)+\underline{V}(p',w)), \forall (w,v) \in S \\
\Leftrightarrow & \frac{1}{2}(\sum_{\mathrm{x}^{\mathrm{j}} \in p} \sum_{i=1}^{n} w_i \overline{v}_i^j + \sum_{\mathrm{x}^{\mathrm{j}} \in p} \sum_{i=1}^{n} w_i \underline{v}_i^j) > \frac{1}{2}(\sum_{\mathrm{x}^{\mathrm{j}} \in p'} \sum_{i=1}^{n} w_i \overline{v}_i^j + \sum_{\mathrm{x}^{\mathrm{j}} \in p'} \sum_{i=1}^{n} w_i \underline{v}_i^j), \forall (w,v) \in S
\end{aligned}
\tag{11}
$$

2. Construction of the order relationship based on the pessimistic criteria

Definition 3.3.4 *Construct the relationship* $\succ_2$ *on* P *as follows:*

If $\underline{V}(p,w) > \underline{V}(p',w), \exists (w,v) \in S \quad p, p' \in P$ *, then we say* $p \succ_2 p'$ *.*

The expression of $p \succ_2 p'$ can be obtained by Definition 3.2.1, and the description is as follows:

$$
\begin{aligned}
& p \succ_2 p' \\
\Leftrightarrow & \underline{V}(p,w) > \underline{V}(p',w), \quad (w,v) \in S \quad p, p' \in P \\
\Leftrightarrow & \sum_{\mathrm{x}^{\mathrm{j}} \in p} \sum_{i=1}^{n} w_i \underline{v}_i^j > \sum_{\mathrm{x}^{\mathrm{j}} \in p'} \sum_{i=1}^{n} w_i \underline{v}_i^j \quad p, p' \in P
\end{aligned}
\tag{12}
$$

3. Construction of the order relationship based on the probability of misjudgments

We now consider the acceptance of decision makers on the investment of misjudgment as a parameter of a decision rule. Because the probability of $\tilde{a} \geq \tilde{b}$ is $p(\tilde{a} \geq \tilde{b})$, then the probability of misjudgments on the investment is $1 - p(\tilde{a} \geq \tilde{b})$.

Suppose that the highest probability of the miscarriage of justices that can be accepted by decision makers is $p_{false} \in [0, 0.5]$. Define the order relationship $\succ_p$ as below.

Definition 3.3.5. *Define the order relationship* $\succ_p$ *on* P *as follows: for the given* $p_{false} \in [0, 0.5]$ *: there is* $V(p', w, v) \geq_p V(p', w, v), \forall (w, v) \in S, p, p' \in P,$ *if* $p \geq 1 - p_{false}$*, then we say that* $p \succ_p p'$.

The expression of $p \succ_p p'$ can be obtained by Definition 3.2.1, the calculation is as follows:

$$\begin{aligned} & p \succ_{\mathrm{p}} p' \\ \Leftrightarrow\ & \mathrm{V}(p, w, v) \geq_{\mathrm{p}} \mathrm{V}(p', w, v) \\ \Leftrightarrow\ & P(\mathrm{V}(p, w, v) \geq \mathrm{V}(p', w, v)) \geq 1 - p_{false} \end{aligned}$$

$$\Leftrightarrow \begin{cases} A \geq 1 - p_{false}, & \text{if} \quad \underline{V}(p, w) \leq \underline{V}(p', w) \leq \overline{V}(p', w) \leq \overline{V}(p, w), \\ B + C \geq 1 - p_{false}, & \text{if} \quad \underline{V}(p', w) \leq \underline{V}(p, w) \leq \overline{V}(p', w) \leq \overline{V}(p, w) \ . \end{cases}$$

$$\Leftrightarrow \begin{cases} \bar{A} \geq 1 - p_{false}, \text{if } \ \underline{V}(p) \leq \underline{V}(p') \leq \overline{V}(p') \leq \overline{V}(p), \\ \bar{B} + \bar{C} \geq 1 - p_{false}, \text{if } \underline{V}(p') \leq \underline{V}(p) \leq \overline{V}(p') \leq \overline{V}(p). \end{cases} \tag{13}$$

where

$$A := \frac{1}{2} \cdot \frac{\overline{V}(p', w) - \underline{V}(p', w)}{\overline{V}(p, w) - \underline{V}(p, w)} + \frac{\overline{V}(p, w) - \overline{V}(p', w)}{\overline{V}(p, w) - \underline{V}(p, w)} \quad ,$$

$$B:=\frac{\underline{V}(p,w)-\underline{V}(p',w)}{\overline{V}(p',w)-\underline{V}(p',w)}+\frac{\overline{V}(p',w)-\underline{V}(p,w)}{\overline{V}(p',w)-\underline{V}(p',w)}\cdot\frac{\overline{V}(p,w)-\overline{V}(p',w)}{\overline{V}(p,w)-\underline{V}(p,w)},$$

$$C:=\frac{1}{2}\cdot\frac{\overline{V}(p',w)-\underline{V}(p,w)}{\overline{V}(p,w)-\underline{V}(p,w)}\cdot\frac{\overline{V}(p',w)-\underline{V}(p,w)}{\overline{V}(p',w)-\underline{V}(p',w)}$$

$$\overline{A}:=\frac{1}{2}\cdot\frac{\sum_{x^j\in p'}\sum_{i=1}^{n}w_i\overline{v}_i^j-\sum_{x^j\in p'}\sum_{i=1}^{n}w_i\underline{v}_i^j}{\sum_{x^j\in p}\sum_{i=1}^{n}w_i\overline{v}_i^j-\sum_{x^j\in p}\sum_{i=1}^{n}w_i\underline{v}_i^j}+\frac{\sum_{x^j\in p}\sum_{i=1}^{n}w_i\overline{v}_i^j-\sum_{x^j\in p'}\sum_{i=1}^{n}w_i\overline{v}_i^j}{\sum_{x^j\in p}\sum_{i=1}^{n}w_i\overline{v}_i^j-\sum_{x^j\in p}\sum_{i=1}^{n}w_i\underline{v}_i^j},$$

$$\overline{B}:=\frac{\sum_{x^j\in p}\sum_{i=1}^{n}w_i\underline{v}_i^j-\sum_{x^j\in p'}\sum_{i=1}^{n}w_i\underline{v}_i^j}{\sum_{x^j\in p'}\sum_{i=1}^{n}w_i\overline{v}_i^j-\sum_{x^j\in p'}\sum_{i=1}^{n}w_i\underline{v}_i^j}+\frac{\sum_{x^j\in p'}\sum_{i=1}^{n}w_i\overline{v}_i^j-\sum_{x^j\in p}\sum_{i=1}^{n}w_i\underline{v}_i^j}{\sum_{x^j\in p'}\sum_{i=1}^{n}w_i\overline{v}_i^j-\sum_{x^j\in p'}\sum_{i=1}^{n}w_i\underline{v}_i^j}\cdot\frac{\sum_{x^j\in p}\sum_{i=1}^{n}w_i\overline{v}_i^j-\sum_{x^j\in p'}\sum_{i=1}^{n}w_i\overline{v}_i^j}{\sum_{x^j\in p}\sum_{i=1}^{n}w_i\overline{v}_i^j-\sum_{x^j\in p}\sum_{i=1}^{n}w_i\underline{v}_i^j}$$

$$\overline{C}:=\frac{1}{2}\cdot\frac{\sum_{x^j\in p'}\sum_{i=1}^{n}w_i\overline{v}_i^j-\sum_{x^j\in p}\sum_{i=1}^{n}w_i\underline{v}_i^j}{\sum_{x^j\in p}\sum_{i=1}^{n}w_i\overline{v}_i^j-\sum_{x^j\in p}\sum_{i=1}^{n}w_i\underline{v}_i^j}\cdot\frac{\sum_{x^j\in p'}\sum_{i=1}^{n}w_i\overline{v}_i^j-\sum_{x^j\in p}\sum_{i=1}^{n}w_i\underline{v}_i^j}{\sum_{x^j\in p'}\sum_{i=1}^{n}w_i\overline{v}_i^j-\sum_{x^j\in p'}\sum_{i=1}^{n}w_i\underline{v}_i^j},$$

where $\succ_p$ only describes the probability of losses, but it does not describe the value of losses, so it is still not reasonable. Thus we should further consider the losses value of misjudgments.

4. Construction of the order relationship based on the given order relationship

Suppose the biggest misjudgment losses is l_0, and define the order relationship as follows.

Definition 3.3.6 *For a given order relationship* $\succ_0$ *and the biggest misjudgment loss* l_0, *we define the order relationship* $\succ_{0,l}$ *on P as follows:*

$$\text{If }\begin{cases} p\succ_0 p' \\ l_{p\backslash p'}\le l_0 \end{cases} p,p'\in P\text{, then we say that } p\succ_{0,l} p'.$$

where $\succ_{0,l}$ *can be regarded as the value of losses based on* $\succ_0$, *it indicates that the order relationship can be established only when the order relationship* $\succ_0$

can be established and the misjudgment loss value is less than l_0.

A specific expression of Definition 3.3.6 can be obtained from the expression related to the portfolio in Definition 3.3.2, and this description is just as follows:

$$\begin{cases} p \succ_0 p' \\ l_{p \backslash p'} \le l_0 \end{cases}, \ p, p' \in P$$

$$\Leftrightarrow \begin{cases} p \succ_0 p' \\ \overline{V}(p,w) - \underline{V}(p,w) \le l_0 \end{cases}, p, p' \in P$$

$$\Leftrightarrow \begin{cases} p \succ_0 p' \\ \overline{V}(p,w) - \underline{V}(p',w) \le l_0 \end{cases}, p, p' \in P$$

$$\Leftrightarrow \begin{cases} p \succ_0 p' & p, p' \in P \\ \sum_{x^j \in p} \sum_{i=1}^{n} w_i \overline{v}_i^j - \sum_{x^j \in p'} \sum_{i=1}^{n} w_i \underline{v}_i^j \le l_0, \end{cases} \tag{14}$$

Definition 3.3.7 *For the biggest misjudgment loss* l_0, *we can define the order relationship* $\succ_{0,l}$ *on* P *as follows:*

If $\begin{cases} \mathrm{V}(p,w,v) \ge_{\mathrm{p}} \mathrm{V}(p',w,v), & \forall (w,v) \in S \\ l_{p \backslash p'} \le l_0 \end{cases}$ $p, p' \in P$, *then we say* $p \succ_{0,l} p'$

Property 3.3.1 *The order relationship* $\succ_1$ *reflects the expected benefits of decision makers with the principle of the same possibility by the expectations of programs. The author thinks the solution is worse, in fact, it's a special case of* $\succ_{p_{false}}$ *when* $p_{false} = 0.5$.

Property 3.3.2 *The order relationship* $\succ_2$ *reflects the expected preferences of decision makers with the principle of the same possibility by the worst condition. From a certain point of view, this order relationship is stable.*

Property 3.3.3 *The order relationship* $\succ_p$ *reflects the relationship between the sizes of two portfolios when the highest probability of miscarriage of justices is*

p_{false}*, it also reflects the control of the possibility of losses. When the interval number p is bigger than* p'*, the probability is bigger than* $1-p_{false}$*, and the sizes of two programs can be compared. The order relationship* $\succ_p$ *is a relaxation on the order relationship that only depends on the median to judge the size, where* $\succ_p$ *presents only the condition that the misjudgment probability is less than* p_{false} *when there are enough advantages to satisfy the order relation, so* p_{false} *can be used as a criteria to measure the stable solution*.

Property 3.3.4. *Given an order relationship* $\succ_0$*, it reflects the relationship between the sizes of two portfolios when the highest losses of miscarriage of justice is* l_0*, it also reflects the control of the loss value. The order relationship* $\succ_{0,l}$ *implies a relaxation on the order relationship* l_0 *(it's easy to prove, the order relationship* $\succ_0$ *is a special case when the order relationship* $\succ_{0,l}$ *achieves* $l_0 = +\infty$*). The order relationship* $\succ_{0,l}$ *is tenable only when the condition that the loss of miscarriage of justices is less than* l_0 *holds. Thus this order relationship can be used as an criteria to measure the stability of solutions*.

The order relationships defined above can be combined according to the actual situation, it is comprehensive to be considered that, they are strict or relaxed on the original order relationship according to the relationship of "and" and "or" of the multiply constrained order relationship. Then it presents the process of constructing a stable optimal solution A^* based on the given order relationship. Note the number of elements in the non-inferior set P_N as N, and all the elements are denoted by $\{p_1, p_2, \cdots p_N\}$. Let the highest probability of misjudgments be $p_{false} = P_{worst}$ ($P_{worst} \geq 0.5$), and the highest loss of misjudgments be $l_0 = L_{worst}$, and $N(S)$ be the number of elements in the set S. Then, one can give a special algorithm as follows:

Algorithm 3.3.1

Step1 a) $A^* \leftarrow p_1$, $T = 1$

b) for i=2, $\cdots$, N , if $p_i \succ_2 A^*$, $A^* \leftarrow p_i$, $T = i$

c) Note , $A_{worst} = p_T$ $\underline{A_{worst}} = \underline{V}(p_T)$, $\overline{A_{worst}} = \overline{V}(p_T)$

Step2 for i=1, $\cdots$, N , if $p_i \succ_{p,l} A^*$, $P_N(S_1) \leftarrow p_i$,

Step3 if $P_N(S_1) \neq \varnothing$, the elements in its are $\{p_1', p_2', \cdots p_N'\}$ $A^* \leftarrow p_1'$

for i=1, $\cdots$, N($P_N(S_1)$) , if $p_i' \succ_1 A^*$, $A^* \leftarrow p_i'$

The optimal solution A^* satisfies the following conditions:

1) The proceed obtained within the probability which is not less than $1 - P_{worst}$ is bigger than that of the optimal solution A_{worst} under the pessimistic criterion.
2) Even the proceed given within the less probability P_{worst} is smaller than A_{worst}, corresponding to the proceed of A_{worst}, its gap would not be larger than L_{worst}, and the proceed under the worst condition would not be less than $\underline{V}(A_{worst}) - L_{worst}$.

In fact, by step 2, the relationship between the elements in alternative set and the optimal solution A_{worst} with the pessimistic criterion can only be shown as in Figure 5;

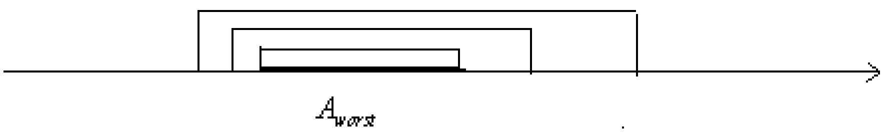

Figure 5: The relationship of the relative position between the interval of elements in the alternative set and PT

The lower bound of p_i' is lower than that of A_{worst} which meets step1; its upper bound is higher than the upper bound of A_{worst}, the distance between the upper bound of p_i' and the upper bound of A_{worst} is larger than the distance between the lower bound of p_i' and the lower bound of A_{worst}.

Theorem 3.3.1 *Let* $\tilde{a}=[\underline{a},\overline{a}]$, $\tilde{b}=[\underline{b},\overline{b}]$ *satisfy* $\underline{a}\le\underline{b}\le\ \overline{b}\le\overline{a}$, *then there holds* $p(\tilde{a}\ge\tilde{b})=p(\tilde{a}\ge m(b))$.

Proof. As shown in Figure 6, and from Definition 3.3.1 it is not hard to show that

$$p(\tilde{a}\ge\tilde{b})=\frac{1}{2}\cdot\frac{\overline{b}-\underline{b}}{\overline{a}-\underline{a}}+\frac{\overline{a}-\overline{b}}{\overline{a}-\underline{a}}=\frac{\overline{a}-\frac{1}{2}\overline{b}-\frac{1}{2}\underline{b}}{\overline{a}-\underline{a}}=\frac{\overline{a}-\frac{1}{2}\left(\overline{b}+\underline{b}\right)}{\overline{a}-\underline{a}} \tag{15}$$

But $$\frac{\overline{a}-m(b)}{\overline{a}-\underline{a}}=\frac{\overline{a}-\frac{1}{2}\left(\overline{b}+\underline{b}\right)}{\overline{a}-\underline{a}} \qquad \text{(Definition 3.3.1)} \tag{16}$$

Integrate the formula (15) and (16), it follows immediately that there holds $p(\tilde{a}\ge\tilde{b})=p(\tilde{a}\ge m(b))$.

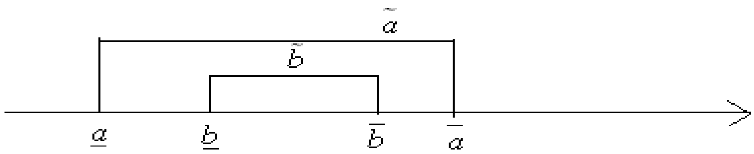

Figure 6: The research on the relationship between the probability of a>b and the probability of a>m(b).

Conclusion 3.3.1 *The optimal solution* A^* *implies that one can get the return* $\frac{1}{2}\left(\overline{V}(A_{worst})+\underline{V}(A_{worst})\right)$ *with the probability being at least* $1-P_{worst}$, *and the minimum return being not less than* $\overline{V}(A_{worst})-L_{worst}$. *Its expected profit is equal to* $m(V(A^*))$.

The values of P_{worst} and L_{worst} depend on the risk acceptance of decision makers. Especially when $P_{worst} = 0$ and $L_{worst} = [\underline{V}(A_{worst}), \overline{V}(A_{worst})]$, the optimal solution is exactly A_{worst} under the pessimistic criteria.

At this point, the optimal solution A^* has been constructed.

Construction of a stable solution with the priori probability

Suppose the probability density in the interval $[\underline{a}, \overline{a}]$ is $f_{\tilde{a}}(x)$, the probability density in the interval $[\underline{b}, \overline{b}]$ is $f_{\tilde{b}}(x)$. If $\tilde{a}$ and $\tilde{b}$ do not overlap, their sizes can be compared easily and the probability is 1. Now should be 1. Now we discuss the condition when $\tilde{a}$ and $\tilde{b}$ have the part overlapped, as shown in the following Figure 7:

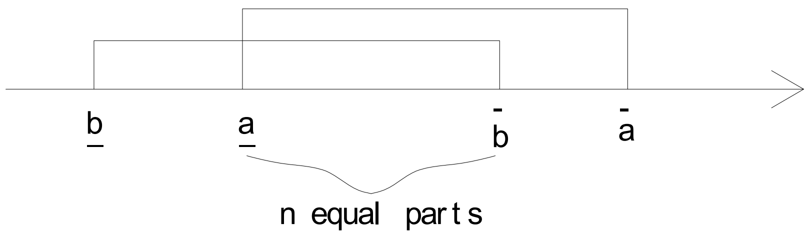

Figure 7: The coincidence of $\tilde{a}$ and $\tilde{b}$ with a prior probability

Suppose the overlapped part of $\tilde{a}$ and $\tilde{b}$ is $[\underline{c}, \overline{c}]$, and $[\underline{c}, \overline{c}]$ is divided into n equal portions. If the true values of $\tilde{a}$ and $\tilde{b}$ are in the same range, and the true value of $\tilde{a}$ is equal to that of $\tilde{b}$. Then the probability $P(\tilde{b} < \tilde{a})$ can be expressed as (17) below

$$P(\tilde{b}<\tilde{a})=\int_{\underline{b}}^{\underline{c}} f_{\tilde{b}}(x)dx+\int_{\underline{c}}^{\overline{c}} f_{\tilde{b}}(x)dx\cdot\int_{\overline{c}}^{\overline{a}} f_{\tilde{a}}(x)dx$$
$$+\frac{\overline{c}-\underline{c}}{\overline{b}-\underline{b}}\cdot\frac{\overline{c}-\underline{c}}{\overline{a}-\underline{a}}\cdot\frac{1}{n}\cdot f_{\tilde{b}}(\underline{c}+\frac{\overline{c}-\underline{c}}{n})(\frac{1}{2}\cdot\frac{1}{n}\cdot f_{\tilde{a}}(\underline{c}+\frac{\overline{c}-\underline{c}}{n})+\frac{1}{n}(f_{\tilde{a}}(\underline{c}+2\frac{\overline{c}-\underline{c}}{n})+\cdots+f_{\tilde{a}}(\overline{c})))$$
$$+\frac{\overline{c}-\underline{c}}{\overline{b}-\underline{b}}\cdot\frac{\overline{c}-\underline{c}}{\overline{a}-\underline{a}}\cdot\frac{1}{n}\cdot f_{\tilde{b}}(\underline{c}+2\frac{\overline{c}-\underline{c}}{n})(\frac{1}{2}\cdot\frac{1}{n}\cdot f_{\tilde{a}}(\underline{c}+2\frac{\overline{c}-\underline{c}}{n})+\frac{1}{n}(f_{\tilde{a}}(\underline{c}+3\frac{\overline{c}-\underline{c}}{n})+\cdots+f_{\tilde{a}}(\overline{c})))$$
$$+\cdots+\frac{\overline{c}-\underline{c}}{\overline{b}-\underline{b}}\cdot\frac{\overline{c}-\underline{c}}{\overline{a}-\underline{a}}\cdot\frac{1}{n}\cdot f_{\tilde{b}}(\overline{c}-\frac{\overline{c}-\underline{c}}{n})(\frac{1}{2}\cdot\frac{1}{n}\cdot f_{\tilde{a}}(\overline{c}-\frac{\overline{c}-\underline{c}}{n})+\frac{1}{n}f_{\tilde{a}}(\overline{c}))$$
$$=\int_{\underline{b}}^{\underline{c}} f_{\tilde{b}}(x)dx+\int_{\underline{c}}^{\overline{c}} f_{\tilde{b}}(x)dx\cdot\int_{\overline{c}}^{\overline{a}} f_{\tilde{a}}(x)dx$$
$$+\frac{\overline{c}-\underline{c}}{\overline{b}-\underline{b}}\cdot\frac{\overline{c}-\underline{c}}{\overline{a}-\underline{a}}\frac{1}{n^2}\sum_{i=1}^{n}(f_{\tilde{b}}(\underline{c}+\frac{i(\overline{c}-\underline{c})}{n})(\sum_{j=i}^{n} f_{\tilde{a}}(\underline{c}+\frac{j(\overline{c}-\underline{c})}{n})+\frac{1}{2}\cdot f_{\tilde{a}}(\underline{c}+i\frac{\overline{c}-\underline{c}}{n})))\quad(17)$$

When n is large enough, then the formula (16) can be simplified as

$$P(\tilde{b}<\tilde{a})=\int_{\underline{b}}^{\underline{c}} f_{\tilde{b}}(x)dx+\int_{\underline{c}}^{\overline{c}} f_{\tilde{b}}(x)dx\cdot\int_{\overline{c}}^{\overline{a}} f_{\tilde{a}}(x)dx$$
$$+\frac{\overline{c}-\underline{c}}{\overline{b}-\underline{b}}\cdot\frac{\overline{c}-\underline{c}}{\overline{a}-\underline{a}}\frac{1}{n^2}\sum_{i=1}^{n}(f_{\tilde{b}}(\underline{c}+\frac{i(\overline{c}-\underline{c})}{n})\cdot\sum_{j=i}^{n} f_{\tilde{a}}(\underline{c}+\frac{j(\overline{c}-\underline{c})}{n}))\quad(18)$$

When n goes to infinity, and the limit of type (17) does exist, then one gets

$$P(\tilde{b}<\tilde{a})=\int_{\underline{b}}^{\underline{c}} f_{\tilde{b}}(x)dx+\int_{\underline{c}}^{\overline{c}} f_{\tilde{b}}(x)dx\cdot\int_{\overline{c}}^{\overline{a}} f_{\tilde{a}}(x)dx$$
$$-\lim_{n\to\infty}\frac{\overline{c}-\underline{c}}{\overline{b}-\underline{b}}\cdot\frac{\overline{c}-\underline{c}}{\overline{a}-\underline{a}}\frac{1}{n^2}\sum_{i=1}^{n}(f_{\tilde{b}}(\underline{c}+\frac{i(\overline{c}-\underline{c})}{n})\cdot\sum_{j=i}^{n} f_{\tilde{a}}(\underline{c}+\frac{j(\overline{c}-\underline{c})}{n}))\quad(19)$$

Therefore, one can define the possibility of $\tilde{a}\geq\tilde{b}$.

Definition 3.3.8 *For the intervals* $\tilde{a}=[\underline{a},\overline{a}]$ *and* $\tilde{b}=[\underline{b},\overline{b}]$, *the overlapped part of* $\tilde{a}$ *and* $\tilde{b}$ *is* $[\underline{c},\overline{c}]$, *hence we call*

$$P(\tilde{b}<\tilde{a})=\int_{\underline{b}}^{\underline{c}} f_{\tilde{b}}(x)dx+\int_{\underline{c}}^{\bar{c}} f_{\tilde{b}}(x)dx\cdot\int_{\bar{c}}^{\bar{a}} f_{\tilde{a}}(x)dx$$

$$-\lim_{n\to\infty}\frac{\bar{c}-\underline{c}}{\bar{b}-\underline{b}}\cdot\frac{\bar{c}-\underline{c}}{\bar{a}-\underline{a}}\frac{1}{n^2}\sum_{i=1}^{n}(f_{\tilde{b}}(\underline{c}+\frac{i(\bar{c}-\underline{c})}{n})\cdot\sum_{j=i}^{n} f_{\tilde{a}}(\underline{c}+\frac{j(\bar{c}-\underline{c})}{n}))$$

the possibility of $\tilde{a}\geq\tilde{b}$*, and denote the order relationship of* $\tilde{a}$ *and* $\tilde{b}$ *by* $\tilde{a}\geq_p\tilde{b}$.

Definition 3.3.9 *Suppose the distribution function of the interval number* $\tilde{a}$ *is* $F_{\tilde{a}}(x)$*, and its density function is* $f_{\tilde{a}}(x)$*, we define the expectation of the interval number* $\tilde{a}$ *as* $E(\tilde{a})=\int_{\underline{a}}^{\bar{a}} xf_{\tilde{a}}(x)dx$.

When the random variable p moves in the range $V(p,w,v)$, we can calculate the loss value of each point and get the result by integration. If the order of integrals may change, then one can give the expression of the expected losses of misjudgments as $E(V(p,w,v))-E(V(p',w,v))$.

Definition 3.3.10 *Suppose the correct order is the portfolio* p *being superior than* p'*, we define the expected losses of misjudgments on portfolios* p,p' *as* $E(V(p,w,v))-E(V(p',w,v))$*, and denote by* $E_l(p\setminus p')$.

Specifically it can be expressed as:

$$\begin{aligned}&E(V(p,w,v))-E(V(p',w,v))\\&=E(\sum_{i=1}^{n} w_i\sum_{x^j\in p} v_i^j)-E(\sum_{i=1}^{n} w_i\sum_{x^j\in p'} v_i^j)\\&=E(\sum_{i=1}^{n} w_i\sum_{x^j\in p}\int_{\underline{v_i^j}}^{\overline{v_i^j}} xf_{v_i^j}(x)dx)-E(\sum_{i=1}^{n} w_i\sum_{x^j\in p'}\int_{\underline{v_i^j}}^{\overline{v_i^j}} xf_{v_i^j}(x)dx)\end{aligned}\tag{20}$$

Definition 3.3.11 *Define VaR of misjudgments on portfolios* p,p' *as the maximum loss of misjudgments within the total probability of occurrences being less than* α*, and denote it by*

$$\Pr ob\{\mathrm{V}(p,w,v)-\mathrm{V}(p',w,v)>VaR\}=\alpha \tag{21}$$

Typically, for a given confidence level $\alpha\in(0,1)$*, and the decision variable* x*, we call*

$$y_\alpha(x)=\min\{y\in R:\Psi(x,y)\ge\alpha\}$$

a loss of α*-VaR of the decision variable* x *with the confidence level* α*.*

Introduce the following function:

$$\phi_\alpha(x)=(1-\alpha)^{-1}\int_{f(x,z)\ge y_\alpha(x)} f(x,z)p(z)dz$$

For a given confidence level $\alpha\in(0,1)$ and the decision variable x, we call $\phi_\alpha(x)$ as a loss of α-VaR of the decision variable x with the confidence level α.

1. Construction of the order relationship based on the expected return

Definition 3.3.12 *Define the order relationship* $\succ_3$ *on P as follows:*

If $\mathrm{E}(\mathrm{V}(p,w,v))>\mathrm{E}(\mathrm{V}(p',w,v)),\quad \exists(w,v)\in S$*, then* $p\succ_3 p'$.

The expression of $p\succ_1 p'$ can be given by Definition 3.3.9 as follows:

$$\begin{aligned}
&p\succ_1 p'\\
&\Leftrightarrow \mathrm{E}(\mathrm{V}(p,w,v))>\mathrm{E}(\mathrm{V}(p',w,v)),\quad \exists(w,v)\in S\\
&\Leftrightarrow E(\sum_{i=1}^n w_i\sum_{x^j\in p} v_i^j)>E(\sum_{i=1}^n w_i\sum_{x^j\in p'} v_i^j)\\
&\Leftrightarrow E(\sum_{i=1}^n w_i\sum_{x^j\in p}\int_{\underline{v_i^j}}^{\overline{v_i^j}} xf_{v_i^j}(x)dx)>E(\sum_{i=1}^n w_i\sum_{x^j\in p'}\int_{\underline{v_i^j}}^{\overline{v_i^j}} xf_{v_i^j}(x)dx)
\end{aligned} \tag{22}$$

Construction of the order relationship based on the probability of misjudgments

The expression of $p\succ_p p'$ can be obtained by Definition 3.3.8 as follows:

Suppose the overlapped part of $\mathrm{V}(p,w,v)$ and $\mathrm{V}(p',w,v)$ is $[\underline{c},\overline{c}]$, then there holds

$$
\begin{aligned}
& p \succ_{\mathrm{p}} p' \\
\Leftrightarrow\ & \mathrm{V}(p,w,v) \geq_{\mathrm{p}} \mathrm{V}(p',w,v) \\
\Leftrightarrow\ & P(\mathrm{V}(p,w,v) \geq \mathrm{V}(p',w,v)) \geq 1 - p_{false} \\
\Leftrightarrow\ & \begin{cases} 1, \text{ if } \overline{V}(p',w) \leq \underline{V}(p,w) \\ E - F \geq 1 - p_{false}, \text{if } \underline{V}(p,w) \leq \overline{V}(p',w) \end{cases}
\end{aligned}
\tag{23}
$$

where $E := \int_{\underline{V}(p',w)}^{\underline{c}} f_{\tilde{b}}(x)dx + \int_{\underline{c}}^{\overline{c}} f_{\tilde{p}'}(x)dx \cdot \int_{\underline{c}}^{\overline{a}} f_{\tilde{p}}(x)dx$,

$$F := \lim_{n\to\infty} \frac{\overline{c}-\underline{c}}{\overline{V}(p',w)-\underline{V}(p',w)} \cdot \frac{\overline{c}-\underline{c}}{\overline{V}(p,w)-\underline{V}(p,w)} \frac{1}{n^2} \sum_{i=1}^{n} (f_{\tilde{p}'}(\underline{c}+\frac{i(\overline{c}-\underline{c})}{n})) \sum_{j=1}^{i} f_{\tilde{p}}(\underline{c}+\frac{j(\overline{c}-\underline{c})}{n})$$

where $\succ_p$ only describes the probability of losses, but does not describe the loss value, so it is still not reasonable. Then we should consider the biggest misjudgment loss based on the order relationship $\succ_p$ with the probability of misjudgments.

2. Construction of the order relationship with α -CVaR based on the original order relationship

Definition 3.3.13 *For a given order relationship* $\succ_0$ *and the maximum CVaR,* $CVaR_0$ *of misjudgments with a confidence level* α *on the set* P*, we define the order relationship* $\succ_{0,c\mathrm{var}}$ *as follows:*

$$If \begin{cases} p \succ_0 p' \\ CVaR \leq CVaR_0 \end{cases} p, p' \in P, \text{ then } p \succ_{0,c\mathrm{var}} p'.$$

where α *-CvaR is the expectation value of the losse which is higher than* α *-VaR when the confidence level is* α*. It also presents the acceptance of decision makers on the loss when the condition is worse.*

The expression of $\succ_{0,c\mathrm{var}}$ can be obatined by Definition 3.3.11 as follows:

$$\begin{cases} p \succ_0 p', p, p' \in P \\ CVaR \leq CVaR_0 \end{cases}$$

$$\Leftrightarrow \begin{cases} p \succ_0 p' \\ (1-\alpha)^{-1} \int_{f(x,z) \geq y_\alpha(x)} f(x,z) p(z) dz, p \succ_{0p'} \end{cases}$$

$$\Leftrightarrow \begin{cases} p \succ_0 p' \\ (1-\alpha)^{-1} \int_{f(x,z) \geq \min\{y \in R, \Psi(x,y) \geq \alpha\}} f(x,z) \; p(z) dz, p \succ_{0p'} \end{cases}$$

$$\Leftrightarrow \begin{cases} p \succ_0 p' \\ (1-\alpha)^{-1} \int_{f(x,z) \geq \min\{y \in R, F(V(p,w,v)) \geq \alpha\}} x f_V(x) dx \;, p \succ_{0p'} \end{cases}$$

Definition 3.3.14 *For the maximum CVaR,* $CVaR_0$ *of misjudgment with a confidence level* α *on the set* P*, we can define the order relationship* $\succ_{0,c\text{var}}$ *as follows:*

If $\begin{cases} \mathrm{V}(p,w,v) \geq_{\mathrm{p}} \mathrm{V}(p',w,v), \forall (w,v) \in S \\ l_{p \setminus p'} \leq l_0, \; p, p' \in P \end{cases}$ *, then we say* $p \succ_{p,c\text{var}} p'$.

Property 3.3.6 *The order relationship* $\succ_3$ *reflects the expected return of decision makers by the expected return of the program when the distribution of a priori probability is known.*

Property 3.3.7 *The order relationship* $\succ_{0,c\text{var}}$ *reflects the sizes of two portfolios and the control of expected losses based on* $\succ_0$ *when the maximum CVaR of misjudgments that can be accepted with a confidence level* α *is* $CVaR_0$.

Property 3.3.8 *The order relationship* $\succ_{p,c\text{var}}$ *reflects the sizes of two portfolios when the highest probability of misjudgments that can be accepted by decision makers is* p_{false} *and the maximum CVaR of misjudgments that can be accepted with a confidence level* α *is* $CVaR_0$. *The order relationship* $\succ_{p,c\text{var}}$ *presents only the condition that the misjudgment probability is less than* p_{false} *and the maximum CVaR of misjudgments is* $CVaR_0$ *, which has enough advantages to satisfy the* *order relation, so it can be used as a criteria to measure the stability of solutions.*

Example 3.3.1

The indicators investigated by projects are: the cash flow u_1, the number u_2 of new customers, the amount u_3 of tax-free concessions, the long-term development u_4, the corporate image u_5, the good response u_6 of employees. After the calculation of the non-inferior solution set, there are remained 5 portfolios to be studied. The comprehensive scores of each program are in the tables:

Table 1: The comprehensive scores

	U_1(ten thousand)	U_2(people)	U_3(ten thousand)	U_4	U_5	U_6
X_1	[8.5, 9.0]	[90,92]	[0.91,0.94]	[0.92,0.95]	[0.89,0.91]	[0.92,0.97]
X_2	[9.1, 9.4]	[81,96]	[0.83,0.99]	[0.87,0.96]	[0.86,0.98]	[0.86,0.97]
X_3	[8.8, 9.1]	[82,85]	[0.90,0.93]	[0.90,0.93]	[0.85,0.89]	[0.90,0.92]
X_4	[9.2, 9.6]	[91,94]	[0.85,0.88]	[0.85,0.89]	[0.84,0.90]	[0.91,0.94]
X_5	[8.6, 8.9]	[89,92]	[0.91,0.95]	[0.92,0.93]	[0.91,0.93]	[0.85,0.88]

Use the method in Section 3.2.1 to calculate w. Concrete steps are as follows:

Step 1: First, use the formula (8) to normalize it and get

Table 2: Normalized the decision matrix

	U_1	U_2	U_3	U_4	U_5	U_6
X_1	[0.377,0.401]	[0.344,0.411]	[0.387,0.423]	[0.402,0.422]	[0.384,0.410]	[0.415,0.427]
X_2	[0.379,0.416]	[0.381,0.423]	[0.385,0.425]	[0.391,0.429]	[0.416,0.438]	[0.384,0.439]
X_3	[0.384,0.408]	[0.367,0.376]	[0.401,0.433]	[0.397,0.434]	[0.386,0.410]	[0.398,0.434]
X_4	[0.381,0.401]	[0.412,0.429]	[0.377,0.433]	[0.407,0.424]	[0.395,0.425	[0.407,0.427]
X_5	[0.382,0.413]	[0.412,0.413]	[0.397,0.414]	[0.401,0.413]	[0.412,0.414]	[0.380,0.397]

Step 2: Use the formula (8) to calculate the weight

$$w=\left(0.1662, 0.1691, 0.1554, 0.1582, 0.1936, 0.1579\right).$$

Step 3: Use Algorithm 3.1.1 to solve.

Step 1 Using the formula (2) to calculate the lower bound of x_1 to x_5, the comparison shows that x_5 has the highest lower bound, $\underline{x_5}=0.393$, set $A_{worst}=x_5$, calculate $\overline{x_5}=0.409$.

Step 2 Take $p_{false}=0.4$ and the loss of misjudgment is less than $l_0=0.03$. Use type (13) and type (14) to calculate and compare the order relationship defined in Definition 3.3.7 with A_{worst}, it will be found that x_4 is in the alternative set.

Step 3. Because there's only one element in the alternative set and its expectation must be higher than A_{worst}, so at the end, the optimal solution is x_4.

The optimal solution $A^*=x_4$ satisfies: the probability of getting a gain 0.399 is not less than 0.4; the minimum income would not be less than 0.379.Its expected income is 0.406.

Without the stability problems, only from the perspective of the possible relationship among the sizes, then the sort relationship is given as: $x_2 \underset{0.5150}{\succ} x_4 \underset{0.6072}{\succ} x_5 \underset{0.5098}{\succ} x_3 \underset{0.6120}{\succ} x_1$

The optimal solution is x_2. Although, the average rate of the comprehensive return is higher, and its stability is less than x_4, so it could not be selected.

4 Stable Solutions of Portfolios Containing Size-Variable Projects With Risk Preferences

4.1 Modeling and Analysis

For convenience, we suppose that the alternative projects 1 to l have non-fixed sizes, the symbols are defined as follows:

$X=\{x^1,\ldots,x^m\}$: The vectors of m alternative projects that depend on the n parameter indexes;

$X_D=\{x^1,\ldots,x^{m-l}\}$: The $m-l$ alternative projects with fixed sizes depended on the n parameter indexes;

$X_B=\{x^1,\ldots,x^l\}$: The l alternative projects; v_i^j: The return score of alternative project x^j with the indicator i, satisfying $v_i^j\in[0,1]$; w_i: the relative importance of evaluation parameter i;

$w=(w_1,\cdots,w_n)^T$: Weight vector that satisfies $w\in S_w^0=\{w\in R^n \mid w_i\geq 0,\sum_{i=1}^n w_i=1\}$;

$V(p,w,v)$: The comprehensive income of portfolio p;

c_k^j: The number of the k kinds of resources consumed by alternative projects x^j, $c_k^j\geq 0$;

$C(x^j)=[c_1^j,\cdots,c_q^j]^T$: The vector of resources consumed by alternative projects x^j;

B_k: The limit of the first k kinds of resources ($k=1,\cdots,q$);

$B=[B_1,\cdots,B_q]^T$: The vector of limited resources, $B\in R_+^q$;

B_D: The vector of resources consumed by all the alternative projects;

P_F: The portfolio set satisfies the constraint condition $P_F=\{p\in P \mid C(p)\leq B\}$;

$\bar{v}_i^j$: The upper bound of v_i^j; $\underline{v}_i^j$: The lower bound of v_i^j;

S_w: The set of the weights of property indicators with incomplete information, $S_w\in S_w^0$;

S_v: The set of practical value:

Suppose the income is proportional to the size of projects, then the income of the j-th project with controllable-scale is

$$V(x^j) = x^j \sum_{i=1}^{n} w_i v_i^j = \sum_{i=1}^{n} x^j w_i v_i^j$$

The income of the j-th project with the fixed size is: $V(p,w,v) = \sum_{i=1}^{n} w_i \sum_{x^j \in p} v_i^j$.

Then the comprehensive income of the portfolio with a controllable-scale is:

$$V(p,w,v) = \sum_{x^j \in p} V(x^j) = \sum_{x^j \in p} \left(\sum_{t=1}^{n} w_t v_t^j + x^j \sum_{i=1}^{n} w_i v_i^j \right).$$

The portfolio model including the projects which have controllable-scales is as the follows

$$Max \quad V(p,w,v) = \sum_{x^j \in p} V(x^j) = \sum_{x^j \in p} \left(\sum_{t=1}^{n} w_t v_t^j + x^j \sum_{i=1}^{n} w_i v_i^j \right)$$

$$\text{s.t.:} \begin{cases} C(p) \le B \\ x_i \in [0,1] \cap N & i = 1,\cdots m-l \\ x_i \ge 0 & i = 1,\cdots l \end{cases}$$

where $v_i^j \in [\underline{v}_i^j, \overline{v}_i^j], w \in S_w \subseteq S_w^0 = \{w \in R^n \mid w_i \ge 0, \sum_{i=1}^{n} w_i = 1\}$.

$$Max \quad V(p,w,v) = \sum_{x^j \in p} V(x^j) = \sum_{x^j \in p \cap X_D} V(x^j) + \sum_{x^j \in p \cap X_B} x^j V(x^j)$$

$$\text{s.t.:} \begin{cases} B_D(p) + B_B(p) \le B \\ x_i \in [0,1] \cap N & i = 1,\cdots m-l \\ x_j \ge 0 & \mathrm{j} = 1,\cdots l \end{cases} \tag{24}$$

The objective function consists of two parts; one is an 0-1 programming problem of interval parameters forms, the other is a linear programming problem of interval parameters forms. The two parts contact each other through the resource constraints in formula (24).

For the 0-1 programming problem, the most common approach is the brute-force. Section 3 has presented how to get the optimal solution with a given risk preference of decision makers. That is to say, the project selection with fixed scale has been made, and hence we have only to solve the linear programming problem of interval parameters forms.

If it has been solved, one can take the solution back to original problem, and set it as the element in alternative set.

Clearly, there is a solution as $\{x^1, x^2, \cdots, x^l, x^{l+1}, \cdots, x^m \mid x^i \in \{0,1\}, i = l+1, \cdots m\}$, then all the solutions are the elements in the set $2^{|m|}$. Then, the residue work is to find the maximum from these elements.

We now solve the following problems step by step:

1. For the portfolio with fixed scale, one can solve the linear programming problem, can we further narrow their scopes?
2. For a fixed asset project, how to solve the optimal solution of the linear programming problem?
3. After solving the project with variable scale, we can give the whole process of solving the optimal solution.

4.2 The determine of primary set of parts of the fixed size

Let now consider the ratio of the benefit on the cost, and define the original interest rate below.

Definition 4.2.1. *Denote by* $\dfrac{\sum_{i=1}^{n} w_i v_i^j}{c_i^j}$ *the original interest rate of* x^j *with the resource* c_i.

Definition 4.2.2. *We say that* $x^i \prec_c x^j$, *if* $\dfrac{\sum_{k=1}^{n} w_k \overline{v_k^j}}{c_k^j} < \dfrac{\sum_{k=1}^{n} w_k \underline{v_k^i}}{c_k^i}$, $\forall c_i \in C$, $x^j \in X_B = \{x^1, \ldots, x^l\}$, $x^i \in X_D = \{x^1, \ldots, x^{m-l}\}$.

It is easy to see that if $x^i \prec_c x^j$, the income with any resource x^i is less than x^j. Then the same resource x^j consumed implies the more benefits. The final set of the optimal solution must not contain x^i. So, the primary item set can be denoted by $X_c = \{x_i \mid x_i \not\prec x_j, \forall x_j \in X_B; x_i \in X_D\}$.

4.3 Stable solution of risk-free portfolio problem with interval form

For a fixed portfolio p_{D0}, the objective function becomes:

$$Max \quad V(p,w,v) = \sum_{x^j \in p_{D0}} V(x^j) + \sum_{x^j \in p \cap X_B} x^j V(x^j)$$

The constraints become the following forms: $B_B(p) \le B \text{ -} B_D(p)$

$$\begin{cases} \text{the objective function:} & Max \quad V(p,w,v) = \sum_{i=1}^{l} x^j V(x^j) \\ \text{constraint conditions:} & \begin{cases} \sum_{j=1}^{l} x^j C(x^j) = B \text{ -} B_D(p) \\ x^j \ge 0 \qquad\qquad j = 1, \cdots l \end{cases} \end{cases}$$

$V(x^j)$ is actually an interval number. Considering the stability of solutions, the decision-makers will have their own risk preferences which are actually a kind of strong constrain condition.

Similar to Section 3.3, the optimal solution x_{worst} based on the pessimistic criteria is solved, then in this setting the probability of the misjudgment on comprehensive income $V(x)$ is less than $p_{false} = P_{worst}$ and the loss of the misjudgment is less than $l_0 = L_{worst}$. Then we keep these solutions in the alternative solution set.

1. The optimal solution based on the pessimistic criteria

When the decision makers think the result is not too bad with the worst case, it’s clearly the pessimistic criteria (max-min criteria). The interval number linear programming problem mentioned above is transformed as follows:

$$\begin{cases} \text{the objective function:} & Max \quad V(x,v)=\sum_{i=1}^{l} x^j \underline{V}(x^j) \\ \text{constraint conditions:} & \begin{cases} \sum_{x^j\in p} x^j C(x^j) \le B \text{ -} B_D(p_{D0}) \\ x_i \ge 0 \qquad\qquad i=1,\cdots l \end{cases} \end{cases}$$

This problem can be solved with the ordinary linear programming simplex method.

Assume the solution is $x_{worst}=(x_{w_1},\cdots,x_{w_l})^T$, denote the comprehensive income corresponding to x_{worst} by $V(x_{worst})$.

2. The limit of the misjudgment probability

It is not hard to see that the probability of $\tilde{a}\ge\tilde{b}$ is the centre $m(b)$ of $\tilde{b}$ on the interval $\tilde{a}$, when the interval is completely contained and the information is unknown.

Theorem 4.3.1 *If the two internal numbers* $\tilde{a}=[\underline{a},\overline{a}]$ *and* $\tilde{b}=[\underline{b},\overline{b}]$ *meet* $\underline{a}\le\underline{b}\le\ \overline{b}\le\overline{a}$, *then* $p(\tilde{a}\ge\tilde{b})=p(\tilde{a}\ge m(b))=\dfrac{\overline{a}-m(b)}{\overline{a}-\underline{a}}$.

Proof. As shown in Figure 8, $p(\tilde{a}\ge\tilde{b})=\dfrac{\overline{a}-\frac{1}{2}\left(\overline{b}+\underline{b}\right)}{\overline{a}-\underline{a}}=\dfrac{\overline{a}-m(b)}{\overline{a}-\underline{a}}$

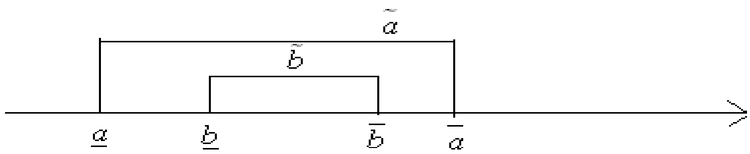

Figure 8: The figure of Theorem 4.3.1

As the probability of misjudgment on comprehensive income $V(x)$ relative to x_{worst} is less than $p_{false} = P_{worst}$, then

$$\frac{\sum_{i=1}^{l} x^j \overline{V}(x^j) - \frac{1}{2}V(x_{worst})}{\sum_{i=1}^{l} x^j \overline{V}(x^j) - \sum_{i=1}^{l} x^j \underline{V}(x^j)} \geq 1 - P_{worst} .$$

Then, the following formulae are obtained:

$$\frac{\sum_{i=1}^{l} x^j \overline{V}(x^j) - \frac{1}{2}V(x_{worst})}{\sum_{i=1}^{l} x^j \overline{V}(x^j) - \sum_{i=1}^{l} x^j \underline{V}(x^j)} \geq 1 - P_{worst}$$

$$\Leftrightarrow \sum_{i=1}^{l} x^j \overline{V}(x^j) - \frac{1}{2}V(x_{worst}) \geq (1 - P_{worst})(\sum_{i=1}^{l} x^j \overline{V}(x^j) - \sum_{i=1}^{l} x^j \underline{V}(x^j))$$

$$\Leftrightarrow P_{worst} \sum_{i=1}^{l} x^j \overline{V}(x^j) + (1 - P_{worst}) \sum_{i=1}^{l} x^j \underline{V}(x^j) \leq \frac{1}{2}V(x_{worst})$$

$$\Leftrightarrow \sum_{i=1}^{l} \left(P_{worst} \cdot \overline{V}(x^j)\right) \cdot x^j + \sum_{i=1}^{l} \left((1 - P_{worst}) \cdot \underline{V}(x^j)\right) \cdot x^j \leq \frac{1}{2}V(x_{worst})$$

$$\Leftrightarrow \sum_{i=1}^{l} \left(P_{worst} \cdot \overline{V}(x^j) + (1 - P_{worst}) \cdot \underline{V}(x^j)\right) \cdot x^j \leq \frac{1}{2}V(x_{worst})$$

3. The loss of the misjudgment

When the loss of misjudgments is less than $l_0 = L_{worst}$, and one can add a constraint condition:

$$\overline{V}(x_{worst}) - \sum_{i=1}^{l} x^j \underline{V}(x^j) \leq L_{worst}$$

By a direct computation, one arrives at:

$$\overline{V}(x_{worst}) - \sum_{i=1}^{l} x^j \underline{V}(x^j) \leq L_{worst} \Leftrightarrow \sum_{i=1}^{l} \underline{V}(x^j) x^j \geq \overline{V}(x_{worst}) - L_{worst} \tag{25}$$

It is easy to see that the solutions meeting only the above conditions are stable. Now we select the one with the highest expectation among them. As the information in intervals is unknown, the expectation is exactly the median of interval numbers. Hence, the objective function is:

$$Max \quad V(x,v) = \frac{1}{2}\sum_{i=1}^{l}\left(\underline{V}(x^j) + \overline{V}(x^j)\right)x^j$$

With the two constraint conditions above, the solution is actually the maximum expectation that meets the probability of misjudgments and the loss of misjudgments. The linear programming problem is transformed as follows:

$$Max \quad V(x,v) = \frac{1}{2}\sum_{i=1}^{l}\left(\underline{V}(x^j) + \overline{V}(x^j)\right)x^j$$

$$\begin{cases} \sum_{x^j \in p} x^j C(x^j) \le B \text{ -} B_D(p_{D0}) \\ x_i \ge 0 \qquad\qquad i = 1, \cdots l \\ \sum_{i=1}^{l}\left(P_{worst} \cdot \overline{V}(x^j) + (1-P_{worst}) \cdot \underline{V}(x^j)\right) \cdot x^j \le \frac{1}{2} V(x_{worst}) \\ \sum_{i=1}^{l} \underline{V}(x^j) x^j \ge \overline{V}(x_{worst}) - L_{worst} \end{cases} \tag{26}$$

This is a simple linear programming problem and it can be easily solved.

4.4 Decision making process of portfolios with the variable scale

By integrating 0-1 interval parametric programming and the interval parameter linear programming, we can present the decision making processes in the following algorithm form.

Algorithm 4.4.1

Step1 Use Formula (9) to calculate weight;

Step2 Solve the original interest rate according to Definition 4.2.1. Remove the inferior alternative projects according to Definition 4.2.2. Denote the residue set rejected the inferior alternative projects by X_c , and denote the set meeting resource constraints in X_c by $P_C(S)$.

Step3 Make Cycles. While $P_C(S) \neq \varnothing$

a) Take $p \in P_C(S)$, solve the linear programming problem (25), and denote the solution by x_p.

b) Add x_p into the empty set $P_0(S)$.

c) $P_C(S) = P_C(S) \setminus p$

Step 4 Use Algorithm 3.3.1 to get the stable optimal solution of $P_0(S)$.

4.5 Application Analysis

The indicators investigated by projects are: the cash flows u_1, the number u_2 of new customers, the duty-free amount u_3, the long-term development u_4, the corporate image u_5, the good response u_6 of employees. The fund used by the companies is limited to 500,000, HR is limited to 500,000.

A set of alternative projects is { x_1, x_2, x_3, x_4, x_5, x_6, x_7}, where x_1, x_2, x_3, x_4, x_5 are projects with the fixed scale, x_6, x_7 are projects with the variable scale. The comprehensive scores of each project are shown in Table 3 (x_6 and x_7 indicate the values corresponding to the capital investment of 10000).

Table 3: The comprehensive scores

	U_1(10000)	U_2	U_3(10000)	U_4	U_5	U_6
x1	[85, 90]	[90,92]	[0.91,0.94]	[0.92,0.95]	[0.89,0.91]	[0.92,0.97]
x2	[91, 94]	[81,96]	[0.83,0.99]	[0.87,0.96]	[0.86,0.98]	[0.86,0.97]
x3	[88, 91]	[82,85]	[0.90,0.93]	[0.90,0.93]	[0.85,0.89]	[0.90,0.92]
x4	[92, 96]	[91,94]	[0.85,0.88]	[0.85,0.89]	[0.84,0.90]	[0.91,0.94]
x5	[86, 89]	[89,92]	[0.91,0.95]	[0.92,0.93]	[0.91,0.93]	[0.85,0.88]
x6	[29, 33]	[29,34]	[0.24,0.27]	[0.27,0.28]	[0.27,0.32]	[0.29,0.33]
x7	[29, 31]	[27,31]	[0.29,0.32]	[0.31,0.34]	[0.30,0.33]	[0.27,0.29]

The resource constraints of each project are shown in Table 4 (x_6 and x_7 indicate the values corresponding to the capital investment of 10000).

Table 4: The resource consumptions

	C_1（10000）	C_2（number of people）
X_1	35	30
X_2	27	26
X_3	32	32
X_4	28	29
X_5	34	32
X_6	1.0	1
X_7	1.0	1.2

Solve the optimal program according to Algorithm 4.4.1 described in Section 4.4.
Step1: First normalize it:

Table 5: Normalized decision matrix

	U_1	U_2	U_3	U_4	U_5	U_6
X_1	[0.377,0.401]	[0.344,0.411]	[0.387,0.423]	[0.402,0.422]	[0.384,0.410]	[0.415,0.427]
X_2	[0.379,0.416]	[0.381,0.423]	[0.385,0.425]	[0.391,0.429]	[0.416,0.438]	[0.384,0.439]
X_3	[0.384,0.408]	[0.367,0.376]	[0.401,0.433]	[0.397,0.434]	[0.386,0.410]	[0.398,0.434]
X_4	[0.381,0.401]	[0.412,0.429]	[0.377,0.433]	[0.407,0.424]	[0.395,0.425	[0.407,0.427]
X_5	[0.382,0.413]	[0.412,0.413]	[0.397,0.414]	[0.401,0.413]	[0.412,0.414]	[0.380,0.397]

Then by using formula (9), we can calculate the weight as follows:

$$w = (0.1662, 0.1691, 0.1554, 0.1582, 0.1936, 0.1579)$$

Step 2: According to Definition 4.2.1 and Definition 4.2.2, we calculate the original interest rate of each project and compare them, and obtain the results as follows:

After screening, the projects x_1, x_3, x_5 with fixed scales are screened out, x_2, x_4 are left. So the primary project set is $X_c = \{x_2, x_4, x_6, x_7\}$. Then we can consider the resource constraints and get $P_C(S) = \{\{x_6, x_7\}, \{x_2, x_6, x_7\}, \{x_4, x_6, x_7\}\}$, which shows that each portfolio is contained, and the coefficients of x_6, x_7 can be adjusted.

Step 3: Use formula (26) to solve the three portfolios in the set one by one. The number is normalized and satisfies the additivity. Here, the authors only describe $\{x_4, x_6, x_7\}$ and list the results of others.

Taking $p_{false} = 0.4$, the loss of misjudgments is less than $l_0 = 0.03$, by using formula (26) and the normalized matrix, we get the following problem:

$$\begin{cases} \text{objective function：} \quad Max \quad V(x,v) = 0.386x_6 + 0.416x_7 + 0.410 \\ \text{constraint conditions：} \quad \begin{cases} 28 + x_6 + x_7 \le 50 \\ 29 + x_6 + 1.2x_7 \le 50 \\ x_6 \ge 0 \\ x_7 \ge 0 \\ 0.092x_6 + 0.104x_7 \le 1.872 \\ 0.112x_6 + 0.118x_7 \ge 0.422 \end{cases} \end{cases}$$

We can solve this problem and arrive at $\begin{cases} x_6 = 9.16 \\ x_7 = 9.89 \end{cases}$, and for the project x_4, the whole solution set is given as $\{x_4 = 1, x_6 = 9.16, x_7 = 9.89\}$.

Similarly, the authors can arrive at

$$\{x_2 = 1, x_6 = 9.74, x_7 = 10.50\}, \{x_6 = 21.81, x_7 = 23.50\}.$$

Step 4: According to Algorithm 3.3.1, and comparing three portfolios, then we have the optimal solution as $\{x_4 = x_6 = 9.158, x_7 = 9.89\}$.

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