A Non-Parametric Approach of Heteroskedasticity Robust Estimation of Vector-Autoregressive (VAR) Models

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Abstract

This contribution studies the application of heteroskedasticity robust estimation of Vector-Autoregressive (VAR) models. VAR models have become one of the most applied models for the analysis of multivariate time series. Econometric standard software usually provides parameter estimators that are not robust against unknown forms of heteroskedasticity. Different bootstrap methodologies are available which are able to generate heteroskedasticity robust parameter estimates. However, common literature is mostly focused on univariate time series models. This study applies a natural extension of the non-parametric pairs bootstrap methodology to different VAR models, taking into account empirical stock market data of the FTSE 100, DAX 30 and S&P 500. A comparison shows that the t-values of the bootstrap models' parameters are considerably lower than the

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ordinary ones and that the determinants of the covariance matrices are clearly smaller.

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1   Introduction and Literature Review

Since the pioneer work of [11], Vector-autoregressive (VAR) models have become one of the most applied models for the analysis of multivariate time series. These models have proven to be useful for describing and forecasting the dynamic behavior of economic and financial time series. Common standard econometric software like EViews and JMulti, for instance, allow for modeling VAR models. According to [9], the t-values of estimated VAR models' parameter matrices have their standard asymptotic distributions if the lag order is chosen to be equal or larger than two, even if the variables employed are $I(1)$, as shown by [12] and [1]. This property makes time series analysis in a VAR model framework to a flexible tool for researchers.

VAR model parameter estimates, provided by standard econometric software, typically rely on this asymptotic theory. Furthermore, these estimators do usually not account for the presence of heteroskedasticity which has an essential impact on the estimators' robustness. Volatility clustering in financial time series is according to [6] a well known as stylized fact. Even if the lag order of a VAR model is chosen to be large enough to generate independent errors, the data may still remain heteroskedastic distributed. Figure 1 shows the monthly log-returns of the FTSE 100, DAX 30 and the S&P 500 stock indices from December 2000-December 2010. A visual inspection of figure 1 depicts at least two

outcomes: Firstly, all three stock markets exhibit alternating periods of high and low volatility. The second is that these clusters apparently occur at the same time. From January 2001 until January 2003 the stock markets show a relatively high volatility, whereas the period from May 2004 until September 2007 was relatively low. Afterwards the volatility increased again. The latter increase of stock markets' volatilities was associated with the financial crisis.

Employing VAR models to analyze correlation, dependence and causality schemes requires robustness of the parameter estimates. [7] report that, despite the superconsistency of standard estimators for the cointegration parameters in a VAR model, the small sample properties are often poor. In their simulation study they show that their proposed Generalized Least Squares Estimator (GLS3) provides estimates which are robust against conditional heteroskedasticity, even in smaller samples. The GLS3 estimator as proposed by [7] requires a numerical optimization procedure to estimate the parameters of the underlying multivariate GARCH process and, thus, this approach can be considered as parametric. [4] discusses different bootstrapping methodologies that account for the presence of an unknown form of heteroskedasticity in the data. In the context of dynamic regression models [4] considers studies of [5] who analyze three approaches, namely a recursive-design wild bootstrap, a fixed design wild bootstrap procedure and a pairs bootstrap approach where the lagged dependent variables used as regressors are treated as if they were exogenous. [5] show, that all three approaches yield the correct asymptotic distribution of the OLS estimator, given suitable assumptions. However, in the multivariate VAR framework the contemporaneous correlation structure between the endogenous variables has to be taken into account. Wild bootstrapping is according to [4] more closely linked to the discussions of regression models than pairs bootstrap. Employing wild bootstrap in a univariate model framework makes use of a so called pick-distribution. The latter is uncorrelated with the error of the autoregressive model and generates heteroskedasticity in the repeated sampling procedure.

However, in the multivariate VAR model, applying different drawn vectors from the pick-distribution, such as proposed by [8] for instance, to each error vector of the endogenous variables employed in each bootstrap sample would lead to different correlation structures in the covariance matrix. In contrast, using the same vector of the pick-distribution for all endogenous variables' error vectors in each sample would generate a correlation structure that is higher than the empirical one. Consequently, both approaches would lead to incorrect correlation structures of the VAR model's contemporaneous covariance matrix. Figure 1 shows that from January 2003 – August 2003 the volatilities of the S&P 500 and the FTSE 100 are relatively low, whereas the volatility of the DAX 30 seems to be still in a high volatility state. Even if the volatility clustering apparently occurs simultaneously, the volatility clusters are not perfectly correlated.
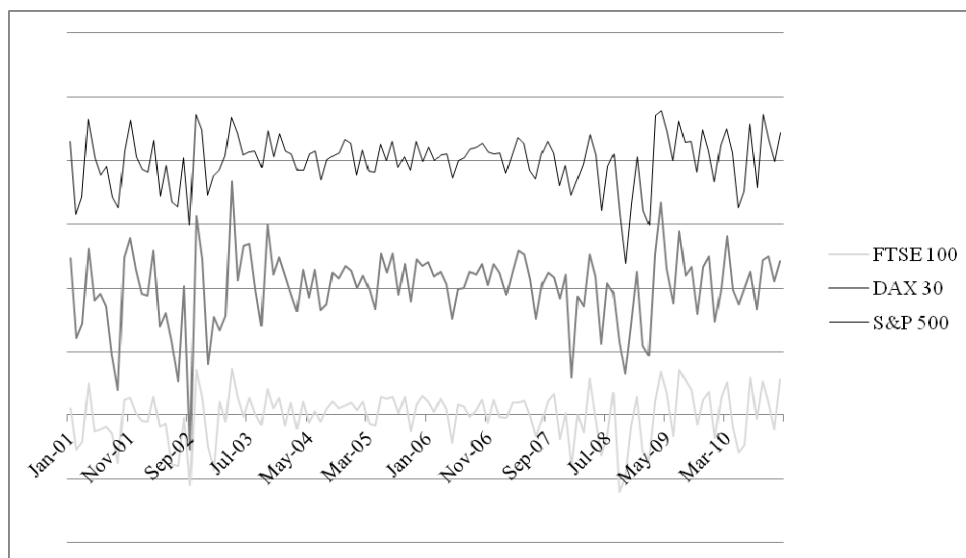

Figure 1:   Monthly log-returns of different stock indices over a 10-years period

Due to the potential problems associated with wild bootstrapping procedure, a natural extension of heteroskedasticity robust bootstrapping in the context of VAR models is the application of the pairs bootstrap approach where the lagged

dependent variables used as regressors are treated as if they were exogenous. To the best of my knowledge there are no other studies available that apply the latter bootstrap methodology to VAR models in order to estimate heteroskedasticity robust parameters. This paper is organized as follows. The next part presents the econometric methodology of an extension of the pairs bootstrap methodology and its application to VAR models. Thereafter this methodology is applied to three different bivariate VAR models describing the dynamic stochastic processes between some preselected stock markets. The last section concludes.

2   Econometric Methodology

In the academic literature different model selection criteria for VAR-models are discussed. The purpose of the latter is to choose a lag order ensuring the error distribution to exhibit no serial correlation. Let a standard VAR-model of lag order p be given by

$$Y_t = c + A_1 Y_{t-1} + ... + A_p Y_{t-p} + \varepsilon_t, \tag{1}$$

where Y_t is a $K \times 1$ -vector of endogenous variables, $A_1,...,A_p$ are $K \times K$ -parameter matrices of lagged vectors $Y_{t-1},...,Y_{t-p}$. Provided that the lag order p is chosen to be large enough to generate independent errors, the error distribution is assumed to be independent distributed and given by $\varepsilon_t \sim (0, \Sigma)$. According to [9], the VAR model of equation (1) can be simply estimated via OLS, so that the parameter vector then is given by

$$\begin{pmatrix} \hat{b}_1 \\ \vdots \\ \hat{b}_K \end{pmatrix} = \left(\begin{pmatrix} Z & \mathbf{0} & \cdots \\ \mathbf{0} & \ddots & \mathbf{0} \\ \cdots & \mathbf{0} & Z \end{pmatrix}^T \begin{pmatrix} Z & \mathbf{0} & \cdots \\ \mathbf{0} & \ddots & \mathbf{0} \\ \cdots & \mathbf{0} & Z \end{pmatrix} \right)^{-1} \begin{pmatrix} Z & \mathbf{0} & \cdots \\ \mathbf{0} & \ddots & \mathbf{0} \\ \cdots & \mathbf{0} & Z \end{pmatrix}^T \begin{pmatrix} Z_{01} \\ \vdots \\ Z_{0K} \end{pmatrix}, \tag{2}$$

where Z and $\mathbf{0}$ are $(T-p) \times (K+1)$ matrices with

$Z = \left(\mathbf{1}, y_{1,t-1},...., y_{K,t-1},...., y_{1,t-p},...., y_{K,t-p}\right)$ and $\mathbf{0}$ is a matrix with zeros and

$\left(Z_{01}...Z_{0K}\right)'$ is $\left(K\cdot(T-p)\right)\times 1$ vector with $\left(Z_{01}...Z_{0K}\right)' = \left(y_{1,t-p},...., y_{K,t-p}\right)'$.

The sample values $T-1,...,T-p$ are in line with [4] treated as pre-sample values.

The parameter vector $\left(\hat{b}_1...\hat{b}_K\right)'$ stacks the rows of the matrices $A_1,...,A_p$ into

columns, that is $\hat{b}_1'$ stacks the first rows of parameter matrices $A_1,...,A_p$ into a

column, $\hat{b}_K$ stacks the $K-$th rows of parameter matrices $A_1,...,A_p$ into a

column and so forth. The properties in line with standard asymptotic theory can in

line with [9] be summarized with $\sqrt{T}\left(\left(\hat{b}_1...\hat{b}_K\right)'-\left(b_1...\hat{b}_K\right)'\right) \xrightarrow{d} N\left(0,\Sigma_{\hat{b}}\right)$ where

$\Sigma_{\hat{b}} = \text{plim}\left(Z'Z/T\right)^{-1}\otimes\Sigma$.

[4] gives an overview of different non-parametric methodologies that can be
used to generate bootstrapping samples. In the presence of an unknown form of
heteroskedasticity, a non-parametric bootstrapping methodology may be employed
to generate artificial samples. If the lag order p is large enough, the pairs

$\left(\left(y_{1t},...., y_{Kt}\right);\left(y_{1t-1},...., y_{Kt-1}\right),....,\left(y_{1t-p},...., y_{Kt-p}\right)\right)$ can be considered as *iid*

drawings from some joint distribution. Therefore, the model of equation (1) can be
pairwise bootstrapped. Given a multivariate framework, pairwise bootstrapping
should ensure that the correlation structure of the non-diagonal elements of Σ is
maintained.

Let the endogenous variables be given by $\left(y_{1t},...., y_{Kt}\right)'$, then the pairs

bootstrap sample $S*$ of the corresponding multivariate data generating
autoregressive process of order p is then given by

$$S* = \left\{\left(\left(y_{i1t}^*,...., y_{iKt}^*\right);\left(y_{i1t-1}^*,...., y_{iKt-1}^*\right),....,\left(y_{i1t-p}^*,...., y_{iKt-p}^*\right)\right),\ i=1,...n\right\} \qquad (3)$$

where $i=1,...n$ denotes the number of bootstrapped samples. Each observation in

$S*$ is pairwise drawn from $\left(\left(y_{1t},...., y_{Kt}\right);\left(y_{1t-1},...., y_{Kt-1}\right),....,\left(y_{1t-p},...., y_{Kt-p}\right)\right)$

with replacement. Given equation (2), $n \cdot \left(p \cdot K^2 + K \right)$ parameter estimates corresponding to n artificial samples can be estimated easily via OLS with

$$
\begin{pmatrix} \hat{b}_1^* \\ \vdots \\ \hat{b}_K^* \end{pmatrix} = \left(\begin{pmatrix} Z^* & \mathbf{0} & \cdots \\ \mathbf{0} & \ddots & \mathbf{0} \\ \cdots & \mathbf{0} & Z^* \end{pmatrix}^T \begin{pmatrix} Z^* & \mathbf{0} & \cdots \\ \mathbf{0} & \ddots & \mathbf{0} \\ \cdots & \mathbf{0} & Z^* \end{pmatrix} \right)^{-1} \begin{pmatrix} Z^* & \mathbf{0} & \cdots \\ \mathbf{0} & \ddots & \mathbf{0} \\ \cdots & \mathbf{0} & Z^* \end{pmatrix}^T \begin{pmatrix} Z_{01}^* \\ \vdots \\ Z_{0K}^* \end{pmatrix}, \qquad (4)
$$

whereby the empirical density function E then, is given by

$$
\text{E}: \quad \text{probability } \frac{1}{T} \text{ on } \left(\left(y_{i1t}^*, ..., y_{iKt}^* \right); \left(y_{i1t-1}^*, ..., y_{iKt-1}^* \right), ..., \left(y_{i1t-p}^*, ..., y_{iKt-p}^* \right) \right), \quad (5)
$$

with $t = 1, ..., T$ and $i = 1, ..., n$. This methodology is a natural extension of the pairs bootstrap approach applied in [5] study and treats the lagged dependent variables in the regressor set of (2) as if they were fixed. [5] show that pairwise bootstrapping yield the correct asymptotic distribution for the OLS estimator so that $E \left(\hat{b}_1^* \ldots \hat{b}_K^* \right)' = \left(b_1 \ldots b_K \right)'$ as $n \rightarrow \infty$. However, [5] study is rather focused on univariate data generating processes in the context of dynamic regression models instead of simultaneous equation regression models as analyzed in this study.

3   Discussion of the Results

The pairs bootstrapping approach is applied to three VAR models. The first VAR model takes into account the log prices of the German leading stock index DAX 30 and the British leading stock index FTSE 100. The second VAR model involves the log prices of the FTSE 100 and the US-stock index S&P 500, whereas the third model takes into account the log prices of the DAX 30 and the S&P 500. Even though these stock markets are chosen for illustration reasons, they are of economically importance due to the industrial production. In particular, they are considered in the literature analyzing stochastic linkages, such as cointegration relationships, among stock markets (see [3], [2] & [10]).

The models involve 10 years of monthly stock market data running from December 2000 until December 2010 and, thus, corresponding to 120 observations. Even though the Schwarz Criterion suggests a lag order of $p = 1$, the VAR models are fitted with a lag order of $p = 2$ in order to maintain standard properties and to ensure independency concerning the error distribution, satisfying the condition for the pairs bootstrap methodology. The Portmanteau- and LM-tests are performed with 16, respectively 5 lags for all models.[2] The Portmanteau- test statistics suggest that there is no remaining autocorrelation on a common significance level of 5%. Even though the LM-test statistic rejects the null hypothesis of no autocorrelation concerning the first VAR-(2) model, the further analysis assumes that there is more evidence against autocorrelation as the Portmanteau-tests account for a higher lag-order.

Furthermore, the multivariate ARCH-LM tests accounting for 5 lags show clearly that the residuals are heteroskedastic distributed as all p-values are below the common significance level of 5%.[3] Equations (6)-(8) of Table 1 in the appendix show the EViews output of the parameter estimates of the VAR-(2) models, whereas equations (9)-(11) of table 2 presents the parameter estimates of the corresponding pairwise bootstrapped models. The parameter estimates of equations (9)-(11) are based upon $n = 1000$ bootstrap samples. Equations (6)-(11) suggest that the ordinary parameter estimates are quite close to the bootstrapped ones. The standard deviations which are given in parenthesis are, as expected, slightly higher for the bootstrapped parameter estimates. Furthermore, the determinants of covariance matrices of the bootstrapped models are smaller in comparison to the ordinary models of equations (6)-(8), even though the estimates of the covariance matrices are within 1.5 times the standard deviation of the

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[2] The p-values of the Portmanteau tests are 0.0949, 0.0764 and 0.1169, whereas the corresponding p-values of the LM-tests are 0.0260, 0.4740 and 0.2261.

[3] The corresponding p-values are 0.0003, 0.0013 and 0.0001.

corresponding bootstrapped estimates of equations (9)-(11).

In order to figure out if the covariances are significantly different from each other, the difference between the log-determinant corresponding to the ordinary model's covariance matrix and the log-determinant of the bootstrapped model's covariance matrix is multiplied with $T - p = 118$ which is assumed to result in a test statistic that is chi-square distributed with 10 degrees of freedom. The idea rests upon the Wald-test statistic as the latter compares the magnitude of covariance matrices between restricted and unrestricted model. As each model contains 10 parameters and the bootstrap model is considered as restricted model, the test statistic is under the null hypothesis assumed to be chi-square distributed with 10 degrees of freedom. The corresponding test statistics are estimated to be $\lambda_1 = 21.3763$ (equations (6) and (9)), $\lambda_2 = 93.2989$ (equations (7) and (10)) and $\lambda_3 = 22.2200$ (equations (8) and (11)) and thus, the null hypothesis can be rejected for all models.[4] It is worth noting, however, that the application of this test requires the assumption that the underlying stochastic processes generating the ordinary and bootstrapped model is the same. Given this assumption, the bootstrapped parameter estimates, which exhibit robustness against an unknown form of heteroskedasticity, fit the data better in comparison to the ordinary models of equations (6)-(9).

In contrast to the 3-step estimation set-up for estimating the cointegration parameters in the VECM, as suggested by [7], the approach as proposed in this study does not involve to estimation of an underlying multivariate GARCH-process. The simulation of $n = 1000$ samples allows for any unknown form of heteroskedasticity as the samples are drawn with replacement from the underlying data generating process while the parameter variances can directly be estimated over the simulated samples.

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[4] The corresponding p-values are 0.0186, 0.0000 and 0.0140.

4 Conclusion

Applying the pairs bootstrap methodology to empirical heteroskedastic distributed data shows that the parameter estimates are quite close to each other. Models, as the VAR models discussed here, can be used in applied work to analyze and parameterize stock markets causality schemes such as Granger-causality, for instance. The non-parametric bootstrapping procedure as discussed in this study can be also employed for heteroskedasticity robust hypothesis testing within the VAR model framework which is, however, left for future research. Future simulation studies may provide further evidence for robust hypothesis testing when applying this methodology to VAR models exhibiting an unknown form of heteroskedasticity. Furthermore, standard software like EViews and JMulti, for instance, provide Vector-Error-Correction models (VECM) as a natural extension of the VAR methodology of cointegrated time series data. The 3-step generalized least squares estimator, as proposed by [7], may be modified in future studies such that the estimation of a multivariate GARCH process may be substituted by an adequate bootstrapping procedure which is left for future studies, too.

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Appendix

Table 1: VAR-(2)-models parameter estimates from the standard software EViews

$$\begin{pmatrix} DAX_t \\ FTSE_t \end{pmatrix} = \begin{pmatrix} 0.0030 \\ (0.2168) \\ 0.1786 \\ (0.1399) \end{pmatrix} + \begin{pmatrix} 0.9004 & 0.3154 \\ (0.1818) & (0.2862) \\ 0.0637 & 0.9811 \\ (0.1166) & (0.1835) \end{pmatrix} \begin{pmatrix} DAX_{t-1} \\ FTSE_{t-1} \end{pmatrix}$$

$$+ \begin{pmatrix} 0.0322 & -0.2489 \\ (0.1806) & (0.2857) \\ -0.0669 & -0.0261 \\ (0.1158) & (0.1831) \end{pmatrix} \begin{pmatrix} DAX_{t-2} \\ FTSE_{t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_{DAX,t} \\ \varepsilon_{FTSE,t} \end{pmatrix}$$

with

$$\Sigma_{DAX,FTSE} = \begin{pmatrix} 9.14e-04 & 5.05e-04 \\ (----) & (----) \\ 5.05e-04 & 3.75e-04 \\ (----) & (----) \end{pmatrix}, \qquad (6)$$

$$\begin{pmatrix} FTSE_t \\ S\&P_t \end{pmatrix} = \begin{pmatrix} 0.1800 \\ (0.1004) \\ 0.1887 \\ (0.1064) \end{pmatrix} + \begin{pmatrix} 0.8116 & 0.2713 \\ (0.1922) & (0.1777) \\ 0.1148 & 1.0782 \\ (0.2038) & (0.1884) \end{pmatrix} \begin{pmatrix} FTSE_{t-1} \\ S\&P_{t-1} \end{pmatrix}$$

$$+ \begin{pmatrix} 0.1295 & -0.2588 \\ (0.1910) & (0.1777) \\ -0.1252 & -0.1273 \\ (0.2025) & (0.1883) \end{pmatrix} \begin{pmatrix} FTSE_{t-2} \\ S\&P_{t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_{FTSE,t} \\ \varepsilon_{S\&P,t} \end{pmatrix}$$

with

$$\Sigma_{FTSE,S\&P} = \begin{pmatrix} 3.69e-04 & 3.44e-04 \\ (----) & (----) \\ 3.44e-04 & 4.15e-04 \\ (----) & (----) \end{pmatrix}, \qquad (7)$$

$$\begin{pmatrix} DAX_t \\ S\&P_t \end{pmatrix} = \begin{pmatrix} 0.0670 \\ (0.1243) \\ 0.1629 \\ (0.0845) \end{pmatrix} + \begin{pmatrix} 0.7644 & 0.5275 \\ (0.1835) & (0.2694) \\ -0.0671 & 1.2503 \\ (0.1248) & (0.1832) \end{pmatrix} \begin{pmatrix} DAX_{t-1} \\ S\&P_{t-1} \end{pmatrix}$$

$$+ \begin{pmatrix} 0.1981 & -0.5038 \\ (0.1844) & (0.2673) \\ 0.0492 & -0.2821 \\ (0.1254) & (0.1817) \end{pmatrix} \begin{pmatrix} DAX_{t-2} \\ S\&P_{t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_{FTSE,t} \\ \varepsilon_{S\&P,t} \end{pmatrix}$$

with

$$\Sigma_{DAX,S\&P} = \begin{pmatrix} 8.94e-04 & 5.25e-04 \\ (----) & (----) \\ 5.25e-04 & 4.13e-04 \\ (----) & (----) \end{pmatrix}. \qquad (8)$$

Table 2:   VAR-(2)-models with pairwise bootstrapped parameter estimates

$$
\begin{pmatrix} DAX_t \\ FTSE_t \end{pmatrix} = \begin{pmatrix} -0.0054 \\ (0.2368) \\ 0.1782 \\ (0.1481) \end{pmatrix} + \begin{pmatrix} 0.9051 & 0.3073 \\ (0.2305) & (0.3330) \\ 0.0687 & 0.9732 \\ (0.1579) & (0.2506) \end{pmatrix} \begin{pmatrix} DAX_{t-1} \\ S\&P_{t-1} \end{pmatrix}
$$

$$
+ \begin{pmatrix} 0.0295 & -0.2404 \\ (0.2097) & (0.3029) \\ -0.0689 & -0.0211 \\ (0.1467) & (0.2379) \end{pmatrix} \begin{pmatrix} DAX_{t-2} \\ FTSE_{t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_{FTSE,t} \\ \varepsilon_{S\&P,t} \end{pmatrix}
$$

with

$$
\Sigma_{DAX,FTSE} = \begin{pmatrix} 8.35e-04 & 4.58e-04 \\ (1.71e-04) & (0.84e-04) \\ 4.58e-04 & 3.39e-04 \\ (0.84e-04) & (0.51e-04) \end{pmatrix}, \tag{9}
$$

$$
\begin{pmatrix} FTSE_t \\ S\&P_t \end{pmatrix} = \begin{pmatrix} 0.1760 \\ (0.0948) \\ 0.1851 \\ (0.0955) \end{pmatrix} + \begin{pmatrix} 0.8036 & 0.2764 \\ (0.2232) & (0.1808) \\ 0.1033 & 1.0794 \\ (0.2630) & (0.2202) \end{pmatrix} \begin{pmatrix} FTSE_{t-1} \\ S\&P_{t-1} \end{pmatrix}
$$

$$
+ \begin{pmatrix} 0.1366 & -0.2615 \\ (0.2178) & (0.1772) \\ -0.1162 & -0.1243 \\ (0.2479) & (0.2001) \end{pmatrix} \begin{pmatrix} FTSE_{t-2} \\ S\&P_{t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_{FTSE,t} \\ \varepsilon_{S\&P,t} \end{pmatrix}
$$

with

$$
\Sigma_{FTSE,S\&P} = \begin{pmatrix} 3.36e-04 & 3.12e-04 \\ (0.51e-04) & (0.51e-04) \\ 4.58e-04 & 3.37e-04 \\ (0.84e-04) & (0.60e-04) \end{pmatrix}, \tag{10}
$$

$$
\begin{pmatrix} DAX_t \\ S\&P_t \end{pmatrix} = \begin{pmatrix} 0.0484 \\ (0.1428) \\ 0.1515 \\ (0.0948) \end{pmatrix} + \begin{pmatrix} 0.7923 & 0.5093 \\ (0.2000) & (0.2656) \\ -0.0459 & 1.2308 \\ (0.1382) & (0.2129) \end{pmatrix} \begin{pmatrix} DAX_{t-1} \\ S\&P_{t-1} \end{pmatrix}
$$

$$
+ \begin{pmatrix} 0.1695 & -0.4788 \\ (0.1913) & (0.2617) \\ 0.0280 & -0.2587 \\ (0.1318) & (0.1973) \end{pmatrix} \begin{pmatrix} DAX_{t-2} \\ S\&P_{t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_{FTSE,t} \\ \varepsilon_{S\&P,t} \end{pmatrix}
$$

with

$$
\Sigma_{DAX,S\&P} = \begin{pmatrix} 8.16e-04 & 4.77e-04 \\ (1.67e-04) & (0.85e-04) \\ 4.77e-04 & 3.74e-04 \\ (0.85e-04) & (0.62e-04) \end{pmatrix}. \tag{11}
$$