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Numerical Approach of an Optimal Control Problem for Sand dune formation in Aquatic Environment

Yahaya Alassane Mahaman Nouri[1], Saley Bisso[2] and Abani Maidaoua Ali[3]

Abstract

The aim of this paper is to determine the optimal initial height of a sand dune that may favor its formation when it is completely immersed in an aquatic environment. We formulate an optimal control problem governed by the equations which model the formation dynamics of this dune through its height under the effect of the incompressible flows in space dimension 2, where the control plays the role of an uncertainty on the initial height. To solve this problem, we use a Chebyshev-Gauss-Lobatto spectral approach $\mathbb{P}_{N-2,M-2}$-type in space and the Second-order backward Euler scheme. The Chebyshev-Gauss-Lobatto quadrature and the Composite-Trapezoidal method are also used. Further numerical tests are given to illustrate our approch and compare the approach and optimal solutions.

[1] Département de Mathématiques et Informatique, Université d'Agadez, Niger. E-mail: alassanenouri@yahoo.fr

[2] Département de Mathématiques et Informatique, Université Abdou Moumouni, Niamey, Niger. E-mail: bsaley@yahoo.fr

[3] Département de Mathématiques et Informatique, Université Dan Dicko Dankoulodo, Maradi, Niger. E-mail: indice002@yahoo.fr

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1 Introduction

Sanding is an environmental phenomenon whose stake has been the subject of many contributions for an effective struggle [12, 13, 15, 16, 17, 18, 19]. Yet we can not influence this phenomenon until we have a good understanding of the process that governs its formation. It is in this perspective that we have developed and studied numerically a mathematical model [17, 18] that describes the sand dune formation dynamics across its height in an incompressible flows where the dune is supposed to be completely submerged and occupies a bounded open regulated domain $\Gamma_\mu =]\frac{-1}{\mu}, \frac{1}{\mu}[^2$, $(\mu > 1)$ of $\mathbb{R}^2$.

The results obtained allowed us to understand the sand dune formation dynamics in an aquatic environment over a time interval $[0, T]$, $T > 0$ [17, 18]. Thus, in order to implement these results, we were interested in this work to determine the optimal initial height which can favor the dune formation at a given instant t, with the same initial data [17, 18] that we consider as the observation data. And to better understand the control action on the approximate height, we use this optimal value as initial data to calculate the optimum height. To achieve this, we formulate an optimal control problem governed by the equations which model the dune formation dynamics, while acting on the initial height of this dune with a control that plays the uncertainty role on This one [1, 5, 20, 22].

Several approaches are used to solve a large class of optimal control problems [1, 4, 6, 7, 14, 20, 24, 26]. For our problem, we use the Second-order backward Euler scheme for time semi-discretization and the Chebyshev-Gauss-Lobatto spectral approach $\mathbb{P}_{N-2,M-2}$-type [2, 8, 18, 21] for spatial discretization. This approach is based on use Chebyshev polynomials of degree at most $N-2$ following x and at most $M-2$ following y to approximate the functions and their derivatives on the Chebyshev-Gauss-Lobatto usual grid of collocation points. Furthermore, we approximate the cost function using the Chebyshev-Gauss-Lobatto quadrature method for integral on Γ_μ domain and the Composite-Trapezoidal method for integral on time interval [3, 9].

The paper is organized as follows : Section 2 is devoted to the formulation of optimal control problem. In Section 3 we present the numerical schemes that we used. Numerical results are presented and discussion in Section 4. We concludes this paper in section 5.

2 Problem Formulation

Let $\Gamma_\mu =]\frac{-1}{\mu}, \frac{1}{\mu}[^2$, $(\mu > 1)$, a regular bounded domain occupied by a sand dune which is supposed to be completely immersed in an incompressible flows in a regular open domain $\Omega =]-1,1[^2$ of $\mathbb{R}^2$.
Let $T > 0$. Note : $Q =]0,T[\times\Gamma_\mu$ and the control space $\mathcal{U} = L^2(\Gamma_\mu)$.
The model problem under consideration is to find the optimal control v^{opt} and the optimal height h^{opt} which minimize the cost function :

$$J(v) = \frac{1}{2}\int_0^T \parallel h(t,x,y) - h^{obs} \parallel^2_{L^2_w(\Gamma_\mu)} dt + \frac{\alpha}{2} \parallel v \parallel^2_{L^2_w(\Gamma_\mu)}, \tag{1}$$

subject to :

$$\frac{\partial h}{\partial t} - \nabla.(m\nabla h) = \Phi(t,x,y) \quad \text{in } Q \tag{2}$$

$$\parallel \nabla h \parallel \leq 1, \; m(\parallel \nabla h \parallel -1) = 0 \quad \text{in } Q \tag{3}$$

$$h(0,x,y) = h^{obs} + v(x,y) \quad \text{on } \Gamma_\mu, \tag{4}$$

where
• $h(t,x,y)$ is the dune height;
• h^{obs} is an observation data;
• $\Phi(t,x,y)$ is a source term;
• $m(t,x,y)$ is the mass density of the sand grains transported by the flows;
• $v(x,y)$ denotes the control variable that plays the role of an uncertainty on the initial height of the dune;
• α denotes a real coefficient of regularization.
The norm $\parallel \parallel_{L^2_w(\Gamma_\mu)}$ is defined for a continuous function ϕ to a weight function w [8, 21], by the following relation :

$$\parallel \phi \parallel_{L^2_w(\Gamma_\mu)} = \left(\int_{-\frac{1}{\mu}}^{\frac{1}{\mu}} \int_{-\frac{1}{\mu}}^{\frac{1}{\mu}} \mid \phi(x,y) \mid^2 w(x)w(y)dxdy\right)^{\frac{1}{2}}. \tag{5}$$

Note $\mathcal{U}_{ad} = \{u \in \mathcal{U} \ : \| \nabla u \| \leq 1\}$, the admissibles controls set.
Choose Φ, m and h in $L^2(Q)$, and the observation data h^{obs} in $L^2(\Gamma_\mu)$.
We assume that problem (1)-(4) has a unique solution (h^{opt}, v^{opt}). We propose a reformulation as follows :

$$J(u) = \min_{v \in \mathcal{U}_{ad}} J(v), \tag{6}$$

subject to Eqs. (2)-(4).

3 Numerical Schemes

In this section, we give the numerical Schemes that we use to discrete problem (1)-(4). The approximate process of the considered problem includes the approximation as well as the discretization of the cost function and the constraints model.

3.1 Approximation of the Constraints model

For a given positif integer r, we consider a time step discretisation $\Delta t = \frac{T}{r}$, with $T \geq 1$. Then we define the knots of the interval $[0;T]$ given by $t_n = n\Delta t$, with $n \in \{0, \dots, r\}$.

For a given continues function $\varphi(t,x,y)$, we approximate φ at the knots t_n by $\varphi(t_n,x,y) \approx \varphi^n(x,y)$.

In order to approach in time the Eq. (2), we used second-order backward Euler scheme which is given by :

$$\partial_t \varphi(t_{n+1},x,y) \approx \frac{3\varphi^{n+1}(x,y) - 4\varphi^n(x,y) + \varphi^{n-1}(x,y)}{2\Delta t}, \text{ for } n = 1, \dots, r. \tag{7}$$

Let $\Lambda =]-1,1[$. For a given positive integers N and M we denote by $\mathbb{P}_{N-2}(\Lambda)$ and $\mathbb{P}_{M-2}(\Lambda)$ sets of orthogonal polynomials of degree less than or equal to $N-2$ and $M-2$, respectively.

Let denote $\mathbb{P}_{N-2,M-2}(\Lambda \times \Lambda) = \mathbb{P}_{N-2}(\Lambda) \otimes \mathbb{P}_{M-2}(\Lambda)$, the set of polynomials defined on $\Lambda \times \Lambda$ of degree $N-2$ according to the variable x and degree $M-2$ according to the variable y, where $\otimes$ denotes Kronecker product [11].

The Chebyshev-Gauss-Lobatto spectral approach $\mathbb{P}_{N-2,M-2}$-type consists in approaching functions and its derivatives using Chebyshev polynomials and the Chebyshev-Gauss-Lobatto mesh [8, 21]. For $\mu > 1$, interval $]\frac{-1}{\mu}, \frac{1}{\mu}[$ subset $]-1,1[$, $\mathbb{P}_{N-2,M-2}(\Gamma_\mu)$ subset $\mathbb{P}_{N-2,M-2}(\Lambda \times \Lambda)$.
Let (xx_i, yy_j) a grid of Γ_μ defined by : $xx_i = \frac{1}{\mu} cos(\frac{i\pi}{N}), \quad i = 1, \dots, N-1$ $yy_j = \frac{1}{\mu} cos(\frac{j\pi}{N}), \quad j = 1, \dots, M-1$. We write Eqs. (2)-(4) at the nodes (xx_i, yy_j) and at point t_{n+1} for $i = 1, \dots, N-1$, $j = 1, \dots, M-1$ and $n = 0, 1, \dots, r$.

Let us consider the following approximations :

$$\begin{aligned} h(t_{n+1}, xx_i, yy_i) &\approx h_{i,j}^{n+1}, \\ m(t_{n+1}, xx_i, yy_i) &\approx m_{i,j}^{n+1}, \\ \phi(t_{n+1}, xx_i, yy_i) &\approx \phi_{i,j}^{n+1}, \\ v(xx_i, yy_i) &\approx v_{i,j}. \end{aligned} \tag{8}$$

We approach the first and secondary operators of derivation of $\varphi = m, h$ in $\mathbb{P}_{N-2,M-2}(\Gamma_\mu)$

$$\begin{aligned} \frac{\partial \varphi^{n+1}(x_i; y_j)}{\partial x} &= \sum_{k=0}^{N} \widetilde{d}_{i,k}^{N,1} \varphi_{k,j}^{n+1}; \\ \frac{\partial \varphi^{n+1}(x_i; y_j)}{\partial y} &= \sum_{l=0}^{M} \widetilde{d}_{j,l}^{M,1} \varphi_{i,l}^{n+1} \\ \frac{\partial^2 \varphi^{n+1}(x_i; y_j)}{\partial x^2} &= \sum_{k=0}^{N} \widetilde{d}_{i,k}^{N,2} \varphi_{k,j}^{n+1}; \\ \frac{\partial^2 \varphi^{n+1}(x_i; y_j)}{\partial y^2} &= \sum_{l=0}^{M} \widetilde{d}_{j,l}^{M,2} \varphi_{i,l}^{n+1}, \end{aligned} \tag{9}$$

where $\widetilde{d}_{i,k}^{N,1}$ and $\widetilde{d}^{N,2}$, $1 \le i \le N-1;\ 1 \le k \le N-1$ are coefficients of the Chebyshev differentiation matrix of order 1 $\widetilde{D_N}$ and order 2 $(\widetilde{D_N})^2$ in $\mathbb{P}_{N-2}(\Lambda)$ [8, 21].
Using the approximations (8) and the schemes (7) and (9) we obtain the dis-

crete form of Eq. (2) as follows :

$$\frac{3h_{i,j}^{n+1}-4h_{i,j}^{n}+h_{i,j}^{n-1}}{2\Delta t}-\Big(\sum_{k=1}^{N-1}\widetilde{d}_{i,k}^{N,1}m_{k,j}^{n+1}\Big)\Big(\sum_{k=1}^{N-1}\widetilde{d}_{i,k}^{N,1}h_{k,j}^{n+1}\Big)-m_{i,j}^{n+1}\sum_{k=1}^{N-1}\widetilde{d}_{i,k}^{N,2}h_{k,j}^{n+1}$$
$$-\Big(\sum_{l=1}^{M-1}\widetilde{d}_{j,l}^{M,1}m_{i,l}^{n+1}\Big)\Big(\sum_{l=1}^{M-1}\widetilde{d}_{j,l}^{M,1}h_{i,l}^{n+1}\Big)-m_{i,j}^{n+1}\sum_{l=1}^{M-1}\widetilde{d}_{j,l}^{M,2}h_{i,l}^{n+1}=\Phi_{i,j}^{n+1}, \tag{10}$$

For $n=0,1,\dots,r$ Let

$$A_1^{n+1}=2\Delta t(diag[(\widetilde{D_N}\otimes I_{N-1})\mathbf{m}^{n+1}]\mathbb{I}).*(\widetilde{D_N}\otimes I_{M-1}), \tag{11}$$
$$A_2^{n+1}=2\Delta t\big[diag\big(\mathbf{m}^{n+1}\big)\mathbb{I}\big].*\big((\widetilde{D_N})^2\otimes I_{M-1}\big), \tag{12}$$
$$A_3^{n+1}=2\Delta t\big(diag\big[(I_{M-1}\otimes\widetilde{D_M})\mathbf{m}^{n+1}\big]\mathbb{I}\big).*\big(I_{N-1}\otimes\widetilde{D_M}\big), \tag{13}$$
$$A_4^{n+1}=2\Delta t\big[diag\big(\mathbf{m}^{n+1}\big)\mathbb{I}\big].*\big(I_{N-1}\otimes(\widetilde{D_M})^2\big), \tag{14}$$

where I_{N-1}, I_{M-1} are $(N-1)\times(N-1)$ and $(M-1)\times(M-1)$ dimensional identity matrices,
$\mathbb{I}$ is $(N-1)(M-1)\times(N-1)(M-1)$ dimensional matrix with entries equal to 1,
$\mathbf{m}^{n+1}$ is a vector of order $(N-1)(M-1)\times 1$ given by:

$$\mathbf{m}^{n+1}=(m_{1,1}^{n+1};...;m_{1,M-1}^{n+1};m_{2,1}^{n+1};...;m_{2,M-1}^{n+1};.....;m_{N-1,1}^{n+1};...;m_{N-1,M-1}^{n+1})^t,$$

$A_1^{n+1},\ A_2^{n+1},\ A_3^{n+1},\ A_4^{n+1}$ are $(N-1)(M-1)\times(N-1)(M-1)$ dimensional matrices given by the second, third, fourth and fifth terms, respectively, in the first member of Eq. (10).
$.*$ denotes multiplication element per element of the same dimensional matrices. Then using Eqs. (11)-(14), we can write the matrice formulation for Eq. (10) as follows :

$$C^{n+1}H^{n+1}=4H^{n}-H^{n-1}+R^{n+1},\quad n=1,\dots,r, \tag{15}$$

where H^{n+1}, R^{n+1} are vectors of order $(N-1)(M-1)\times 1$ given by :

$$H^{n+1}=(h_{1,1}^{n+1},\dots,h_{1,M-1}^{n+1},h_{2,1}^{n+1},\dots,h_{2,M-1}^{n+1},\dots,h_{N-1,1}^{n+1},\dots,h_{N-1,M-1}^{n+1})^t,$$
$$R^{n+1}=2\Delta t(\Phi_{1,1}^{n+1},\dots,\Phi_{1,M-1}^{n+1},\Phi_{2,1}^{n+1},\dots,\Phi_{2,M-1}^{n+1},\dots,\Phi_{N-1,1}^{n+1},\dots,\Phi_{N-1,M-1}^{n+1})^t.$$

C^{n+1} is $(N-1)(M-1)\times(N-1)(M-1)$ dimensional matrix given by :

$$C^{n+1}=3(I_{N-1}\otimes I_{M-1})-A_1^{n+1}-A_2^{n+1}-A_3^{n+1}-A_4^{n+1}. \tag{16}$$

Assuming C^{n+1} reversible, we can rewrite Eq. (15) as follows :

$$H^{n+1} = (C^{n+1})^{-1}(4H^n - H^{n-1} + R^{n+1}), \quad n = 1, \dots, r. \tag{17}$$

3.2 Approximation of the Cost Function

The basic principle of the Chebyshev-Gauss-Lobatto quadrature and the Composite-Trapezoidal method is describe in many references [3, 9, 10, 23, 25]. Using Eq. (5) and the Chebyshev-Gauss-Lobatto quadrature, we obtain the following approximations :

$$\begin{aligned} \parallel h(t,x,y) - h^{obs} \parallel^2_{L^2_w(\Gamma_\mu)} &= \int_{-\frac{1}{\mu}}^{\frac{1}{\mu}} \int_{-\frac{1}{\mu}}^{\frac{1}{\mu}} \mid h(t,x,y) - h^{obs} \mid^2 w(x)w(y)dxdy \\ &= \int_{-\frac{1}{\mu}}^{\frac{1}{\mu}} \Big(\int_{-\frac{1}{\mu}}^{\frac{1}{\mu}} \mid h(t,x,y) - h^{obs} \mid^2 w(x)dx \Big) w(y)dy \\ &\approx \int_{-\frac{1}{\mu}}^{\frac{1}{\mu}} \Big(\sum_{i=1}^{N-1} \mid h(t,xx_i,y) - h^{obs} \mid^2 w_i \Big) w(y)dy \\ &\approx \sum_{j=1}^{M-1} \Big(\sum_{i=1}^{N-1} \mid h(t,xx_i,yy_j) - h^{obs} \mid^2 w_i \Big) w_j \end{aligned} \tag{18}$$

and

$$\begin{aligned} \parallel v(x,y) \parallel^2_{L^2_w(\Gamma_\mu)} &= \int_{-\frac{1}{\mu}}^{\frac{1}{\mu}} \int_{-\frac{1}{\mu}}^{\frac{1}{\mu}} \mid v(x,y) \mid^2 w(x)w(y)dxdy \\ &= \int_{-\frac{1}{\mu}}^{\frac{1}{\mu}} \Big(\int_{-\frac{1}{\mu}}^{\frac{1}{\mu}} \mid v(x,y) \mid^2 w(x)dx \Big) w(y)dy \\ &\approx \int_{-\frac{1}{\mu}}^{\frac{1}{\mu}} \Big(\sum_{i=1}^{N-1} \mid v(xx_i,y) \mid^2 w_i \Big) w(y)dy \\ &\approx \sum_{j=1}^{M-1} \Big(\sum_{i=1}^{N-1} \mid v(xx_i,yy_j) \mid^2 w_i \Big) w_j, \end{aligned} \tag{19}$$

where w_i, is the Chebyshev-Gauss-Lobatto coefficient [8, 9, 21], define by :

$$w_i = \begin{cases} \frac{\pi}{2N}, & i = 0, N \\ \frac{\pi}{N}, & i = 1, \dots, N-1, \end{cases} \tag{20}$$

as well as w_j, $j = 0, 1, \ldots, M$.

By subtituting Eqs. (18)-(19) in Eq. (1) we get for $i = 1, \ldots, N-1$ and $j = 1, \ldots, M-1$:

$$J_{N,M}(v) \approx \frac{1}{2}\int_0^T \Big(\sum_{j=1}^{M-1}\sum_{i=1}^{N-1} \mid h(t, xx_i, yy_j) - h^{obs} \mid^2 w_i w_j\Big)dt$$
$$+\frac{\alpha}{2}\sum_{j=1}^{M-1}\sum_{i=1}^{N-1} \mid v(xx_i, yy_j) \mid^2 w_i w_j. \tag{21}$$

Using composite trapezoidal formula on interval $[0, T]$, we obtain from Eq. (21) :

$$J^n_{N,M}(v) \approx \Big(\frac{\Delta t}{4}\sum_{k=0}^{n}\Big(\sum_{j=1}^{M-1}\sum_{i=1}^{N-1}[(h^k_{i,j} - h^{obs})^2 + (h^{k+1}_{i,j} - h^{obs})^2]\Big)$$
$$+\frac{\alpha}{2}\sum_{j=1}^{M-1}\sum_{i=1}^{N-1} v^2_{i,j}\Big) w_i w_j. \tag{22}$$

We can rewrite Eq. (22) in the following form :

$$J^n_{N,M}(V) \approx \Big(\frac{\Delta t}{4}\sum_{k=0}^{n}\Big(\big[(diag(H^k - H^{obs}))(H^k - H^{obs})\big]^t$$
$$+\big[(diag(H^{k+1} - H^{obs}))(H^{k+1} - H^{obs})\big]^t\Big) + \frac{\alpha}{2}\big((diag(V))V\big)^t\Big)(W_N \otimes W_M), \tag{23}$$

where H^{obs}, V are vectors of order $(N-1)(M-1) \times 1$ given by :

$$H^{obs} = (h^{obs}, \ldots, h^{obs})^t, \tag{24}$$
$$V = (v_{1,1}, \ldots, v_{1,M-1}, v_{2,1}, \ldots, v_{2,M-1}, \ldots, v_{N-1,1}, \ldots, v_{N-1,M-1})^t, \tag{25}$$

W_N, W_M are vectors of order $(N-1) \times 1$ and $(M-1) \times 1$, respectively, given by :

$$W_N = (w_1, \ldots, w_{N-1})^t,$$
$$W_M = (w_1, \ldots, w_{M-1})^t,$$

diag(X) is $L \times L$ dimensional matrix define from $X = (X_1, X_2, \ldots, X_L)^t$ by :

$$diag(X) = \begin{pmatrix} X_1 & 0 & & & \\ 0 & X_2 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & X_{L-1} & 0 \\ & & & 0 & X_L \end{pmatrix}, \quad \text{with } L = (N-1)(M-1). \tag{26}$$

From the discrete form of Eq. (4), we obtain :

$$H^0_{opt} = H^{obs} + V_{opt}, \tag{27}$$

where H^0_{opt} and V_{opt} are vectors of order $(N-1)(M-1)\times 1$ denotes initial optimal height and optimal control, respectively, given by:

$$H^0_{opt} = (h^{0,opt}_{1,1}, \ldots, h^{0,opt}_{1,M-1}, h^{0,opt}_{2,1}, \ldots, h^{0,opt}_{2,M-1}, \ldots, h^{0,opt}_{N-1,1}, \ldots, h^{0,opt}_{N-1,M-1})^t, \tag{28}$$

$$V_{opt} = (v^{opt}_{1,1}, \ldots, v^{opt}_{1,M-1}, v^{opt}_{2,1}, \ldots, v^{opt}_{2,M-1}, \ldots, v^{opt}_{N-1,1}, \ldots, v^{opt}_{N-1,M-1})^t. \tag{29}$$

We deduce from Eq. (17) and (27) the optimal height vector H_{opt} given by :

$$H^{n+1}_{opt} = (C^{n+1})^{-1}(4H^n_{opt} - H^{n-1}_{opt} + R^{n+1}), \quad n = 1, \ldots, r. \tag{30}$$

4 Numerical Results

We choose $N = M = 20$, $T = 1$, $\alpha = 10^{-2}$, $\Delta t = 2.10^{-3}$ and consider observation data as follows :

$$h^{obs} = (1-x^2)(1-y^2).$$

Figures 1, 2, 3 and 4 describe the spatial profile of the optimal control for 501 time discretization points and 361 nodes of the Γ_μ domain for $\mu = 10, 150, 200, 300$. These graphics show that, as the domain is small, the optimal control is compact.

With the same parameters, the spatial profiles of the approximate height (Figures 5, 6, 9, 10) and the optimum height (Figures 7, 8, 11, 12) of the dune are shown at $t = 0.042$ for a time steps $\Delta t = 2.10^{-3}$. These graphics show that more the Γ_μ domain is small, more the control affects the approximate height, significantly. The control action generates a significant disturbance of this height, causing a sharpening of the dune. The spatial profile of optimum height in Figures. 7, 8, 11 and 12 confirm this.

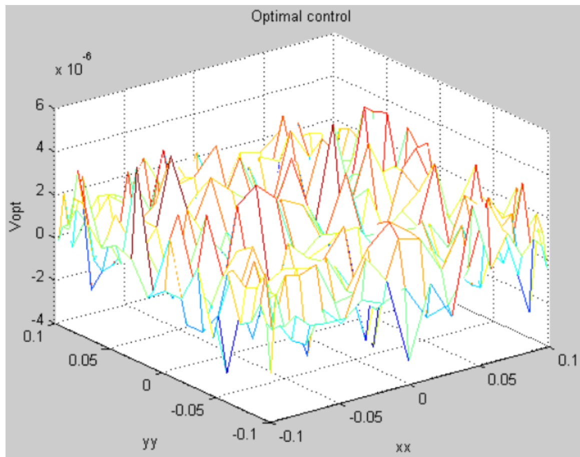

Figure 1: Spatial profile of optimal control for $\Delta t = 2.10^{-3}, N = M = 20$ and $\mu = 10$.

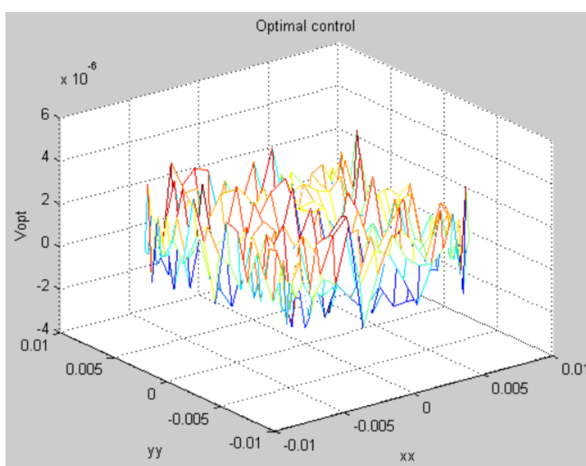

Figure 2: Spatial profile of optimal control for $\Delta t = 2.10^{-3}, N = M = 20$ and $\mu = 150$.

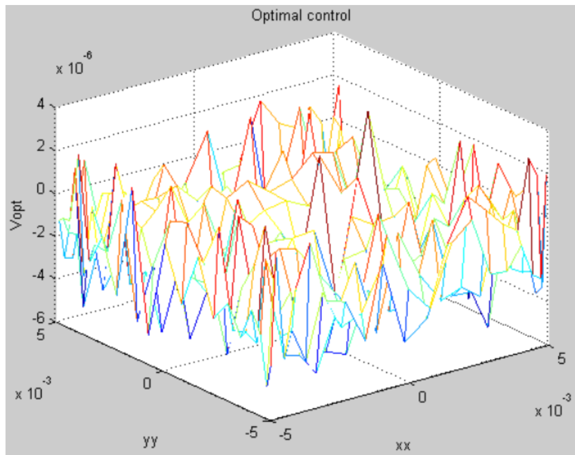

Figure 3: Spatial profile of optimal control for $\Delta t = 2.10^{-3}, N = M = 20$ and $\mu = 200$.

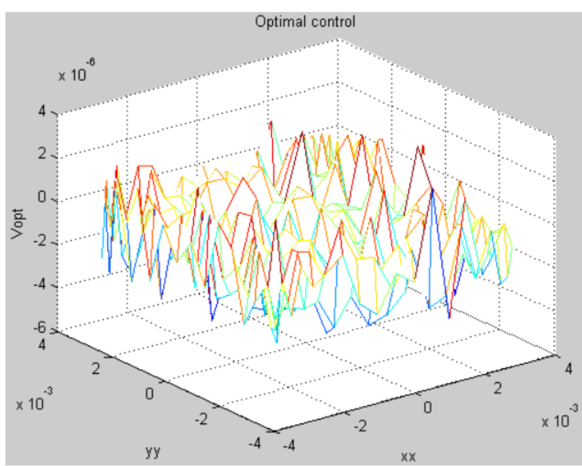

Figure 4: Spatial profile of optimal control for $\Delta t = 2.10^{-3}, N = M = 20$ and $\mu = 300$.

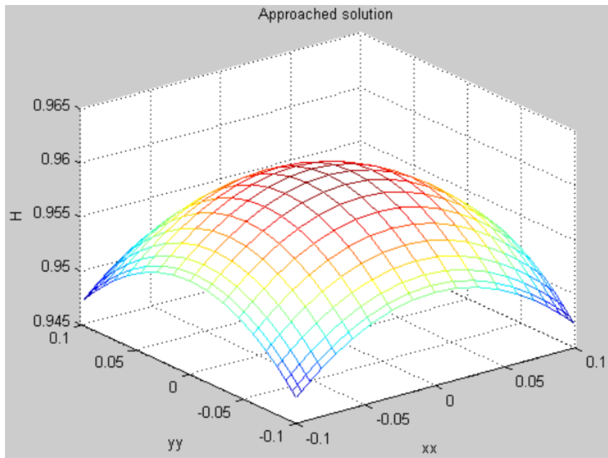

Figure 5: Spatial profile of approach dune height at $t = 0,042$, for $\Delta t = 2.10^{-3}, N = M = 20$ and $\mu = 10$.

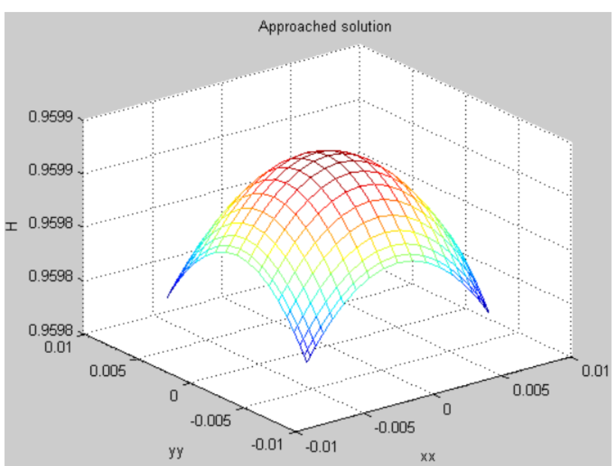

Figure 6: Spatial profile of approach dune height at $t = 0,042$, for $\Delta t = 2.10^{-3}, N = M = 20$ and $\mu = 150$.

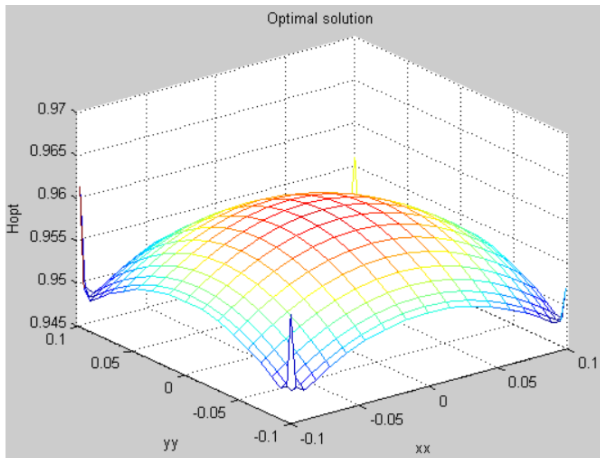

Figure 7: Spatial profile of optimum dune height at $t = 0,042$, for $\Delta t = 2.10^{-3}, N = M = 20$ and $\mu = 10$.

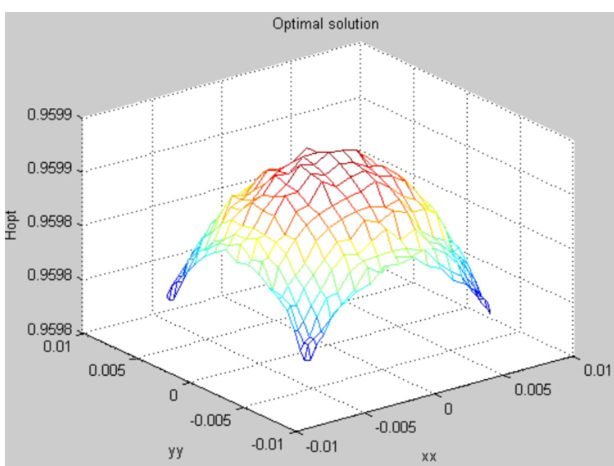

Figure 8: Spatial profile of optimum dune height at $t = 0,042$, for $\Delta t = 2.10^{-3}, N = M = 20$ and $\mu = 150$.

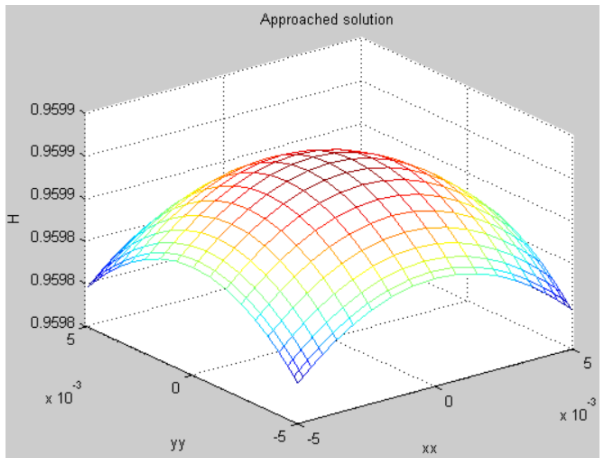

Figure 9: Spatial profile of approach dune height at $t = 0,042$, for $\Delta t = 2.10^{-3}, N = M = 20$ and $\mu = 200$.

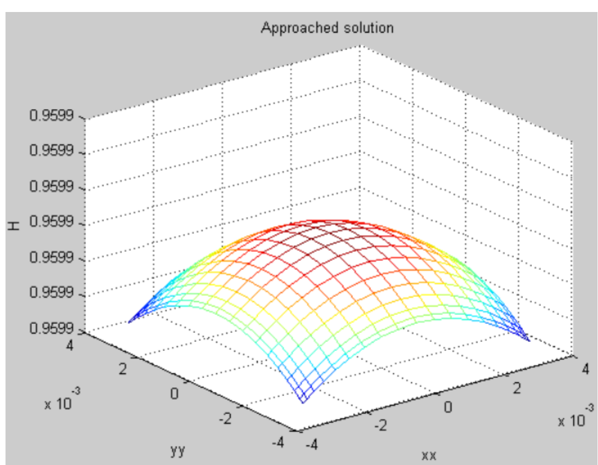

Figure 10: Spatial profile of approach dune height at $t = 0,042$, for $\Delta t = 2.10^{-3}, N = M = 20$ and $\mu = 300$.

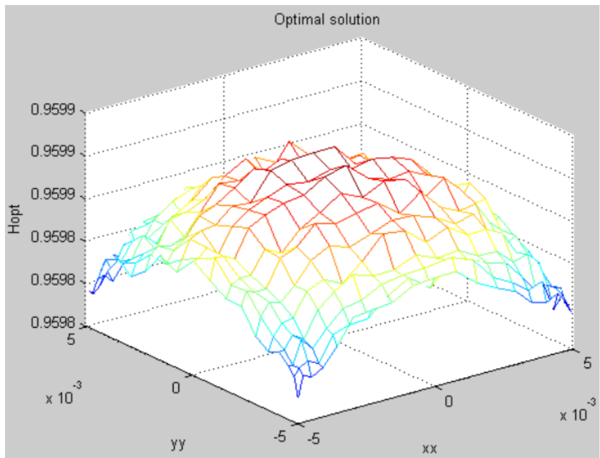

Figure 11: Spatial profile of optimum dune height at $t = 0,042$, for $\Delta t = 2.10^{-3}, N = M = 20$ and $\mu = 200$.

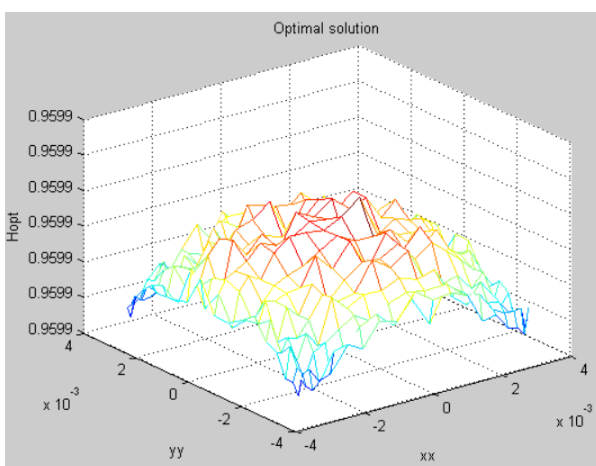

Figure 12: Spatial profile of optimum dune height at $t = 0,042$, for $\Delta t = 2.10^{-3}, N = M = 20$ and $\mu = 300$.

5 Conclusion

In this paper we have studied numerically an optimal control problem of sand dune formation dynamics in an aquatic environment. The aim is to determine the initial optimal height which favor dune formation in aquatic environment. we are formulate an optimal control problem governed by the equations which model the dune formation dynamics, while acting on the initial height of this dune with a control that plays the uncertainty role on This one. We are using the Chebyshev-Gauss-Lobatto spectral approach $\mathbb{P}_{N-2,M-2}$-type and the second-order backward Euler scheme to approach the constraints model. The Chebyshev-Gauss-Lobatto quadrature and the Composite-Trapezoidal method are used to approximate the cost function. Numerical results that we obtain show that the methodology that we used is effective and convenient to approach the optimal control problem considered. Our futur work will be devoted to study an optimal distributed control problem of the dune formation dynamics in an aquatic environment or under the wind effect.

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