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On Quadratic Abstract Measure Integro-Differential Equations

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Abstract

In this paper a nonlinear abstract measure quadratic integro – differential equation is studied in Banach Algebra. The existence of solution abstract measure integro – differential equations is proved for extremal solutions for Caratheodory as well as discontinuous cases of the non-linearity involved in the equations.

Mathematics Subject Classification: 34G99

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1 Introduction

For a given closed and bounded interval $J = [0,a]$ in R the set of real numbers,

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consider the integro differential equation in short (IGDE)

$$\left(\frac{x(t)}{f\left(t,x\left(\xi(t)\right)\right)}\right)^{1} = g\left(t,x\left(\delta(t)\right),\int_0^t k\left(s,x\left(\eta(s)ds\right)\right), \quad a.e. \quad t \in J \tag{1.1}$$

Where $f : J \times R \to R - \{0\}$ is continuous,

$$g : J \times R \times R \to R \ \& \ k : J \times R \to R \qquad and \qquad \xi, \delta, \eta : J \to J$$

The existence of solutions of IGDE (1.1) is proved in Dhage$\left[7\right]$ by using a new non-linear alternative of Leray – Schauder type developed in same paper. In this chapter we apply a nonlinear alternative of Leray – Schauder type involving the product of two operators in a Banach algebra under some weaker conditions than that given in Dhage and Regan$\left[8\right]$ to a quadratic measure differential equation related to IGDE (1.1) for proving the existence results. The existence of exterimal solutions is also proved using a fixed point theorem of in ordered Banach algebras.

In the first section introduction is given in section II we state the abstract measureintegro differentiate equation to be discussed in this paper. The section III the auxiliary results are given and the existence result is discussed in section IV.Finally the existence results for extermal solutions for the integro differential equations is discussed in section V.

2 Quadratic Integro differential equations

Let X be a real Banach algebra with a convenientnorm $\|\cdot\|$.

Let $x, y \in X$. Then the line segment $\overline{xy}$ in X is defined by

$$\overline{xy} = \left\{ z \in X \,\middle|\, z = x + r\left(y - x\right),\, 0 \le r \le 1 \right\} \tag{2.1}$$

Let $x_0 \in X$ be a fixed pointand $z \in X$.

Then for any $x \in \overline{x_0 z}$, we define the sets S_x and $\overline{S}_x$ in X by

$$\left.\begin{aligned} S_x &= \{rx \mid -\infty < r \le 1\} \\ \overline{S}_x &= \{rx \mid -\infty < r \le 1\} \end{aligned}\right\} \tag{2.2}$$

Let $x_1, x_2 \in \overline{xy}$ be arbitrary. We say $x_1 < x_2$ if $S_{x_1} \subset S_{x_2}$, or equivalently $\overline{x_0 x_1} \subset \overline{x_0 x_2}$. In this case we also write $x_2 > x_1$ let M denote the σ-algebra of all subsets of X such that (X, M) is a measureable space. Let $AC'(X, M)$ be the space of all vector measures (real signed measures) and define a norm $|\cdot|$ on $AC'(X, M)$ by

$$\|p\| = |p|(X) \tag{2.3}$$

where $|p|$ is a total variation measure of p and is given by

$$|p|(X) = sup \sum_{i=1}^{\infty} |p(Ei)|,\ Ei \subset X \tag{2.4}$$

where supremum is taken over all possible partition $\{Ei; i \in N\}$ of X. It is known that $AC(X, M)$ is a Banach space with respect to the norm $\|\cdot\|$, given by (2.3). For any nonempty subset S of X, let $L^1_\mu(S, R)$ denote the space of μ-integrable real valued functions on S which is equipped with the norm $\|\cdot\|_{L^1_\mu}$ given by

$$\|\phi\|_{L^1_\mu} = \int_S |\phi(x)| d\mu$$

For $\phi \in L^1_\mu(S, R)$. Let $p_1, p_2 \in AC(X, M)$ and define a multiplication composition in $AC(X, M)$ by

$$(p_1 * p_2)(E) = p_1(E)\, p_2(E) \text{ for all } E \in M. \text{ Then we have}$$

Lemma 2.1 $AC(X, M)$ *is a Banach algebra.*

Proof. Let $p_1, p_2 \in AC(X, M)$ be two elements. Let $\sigma = \{E_1, E_2 .....E_n .....\}$ be a disjoint partition of X. Then by (2.3) – (2.4),

$$\begin{aligned}\|p_1 \;\; p_2\| &= |p_1\, p_2|(X)\\ &= \underset{\sigma}{sup}\sum_{i=1}^{\infty}\left|(p_1 * p_2)(Ei)\right|\\ &= \underset{\sigma}{sup}\sum_{i=1}^{\infty}|p_1(E_1)||p_2(E_2)|\\ &\leq \underset{\sigma}{sup}\left(\sum_{i=1}^{\infty}|p_1(E_1)|\right)\left(\sum_{i=1}^{\infty}|p_2(E_2)|\right)\\ &= \left(\underset{\sigma}{sup}\sum_{i=1}^{\infty}|p_1(E_1)|\right)\left(\underset{\sigma}{sup}\sum_{i=1}^{\infty}|p_2(E_2)|\right)\\ &= \|p_1\|\,\|p_2\|\end{aligned}$$

Hence $AC(X,M)$ is a Banach algebra.

Let μ be a σ finite measure on X, and let $p \in AC(X,M)$. We say p is a absolutely continuous with respect to the measures μ if $\mu(E)=0$ implies $p(E)=0$ for some $E \in M$. In this case we also write $p << \mu$.

Let $x_0 \in X$ be fixed and let M_0 denote the σ-algebra on Sx_0 let $z \in X$ be such that $z > x_0$ and let M_z denote the σ-algebra of all sets containing M. and the sets of the form $\overline{S}x, x \in \overline{x_0 z}$. Given a $p \in AC(X,M)$ with $p << \mu$, consider the abstract measure integro - differential equation (AMIGDE) of the form

$$\frac{d}{d\mu}\left(\frac{p(\overline{S}_x)}{f\left(x, p(\overline{S}_x(\xi))\right)}\right) = g\left(x, p(\overline{S}_x(\delta)), \int_{\overline{S}x} k\left(t, p(\overline{S}_t(\eta))\right)d\mu\right), \; a.e.[\mu] \;\; on \;\; \overline{x_0 z} \tag{2.5}$$

and

$$p(E) = q(E), \qquad E \in M_0 \tag{2.6}$$

where q is a given known vector measure, $\lambda(\overline{S}_x) = \dfrac{p(\overline{S}_x)}{f\left(x, p(\overline{S}_x)\right)}$ is a singed measure suchthat $\lambda << \mu$, $\dfrac{d\lambda}{d\mu}$ is a Radon Nikodym derivative of λ with respect to μ,

$f : S_x \times R \rightarrow R - \{0\}$, $g : S_z \times R \times R \rightarrow R$ and the map

$$x \rightarrow g\left(x, p\left(\overline{S}_x(\delta)\right), \int_{\overline{S}x} k\left(k, p\left(\overline{S}_t(\eta)\right) d\mu\right)$$

is μ -integrable for each $p \in AC(X, M_z)$.

Definition 2.1 *Given an initial real number q on* M_0 *, a vector* $p \in AC(S_z, M_z), (z > x_0)$ *is said to be a solution of AMIGDE (2.5) – (2.6), if*

i) $p(E) = q(E)$, $E \in M_0$

ii) $p << \mu$ *on* $\overline{x_0 z}$, &

iii) *satisfies* (2.5) *a.e.*$[\mu]$ *on* $\overline{x_0 z}$

Remark 2.1 *The AMIGDE (2.5) (2.6) is equivalent to the abstract measure integral equation (in short AMIE).*

$$p(E) = \left[f\left(x, p\left(E_1(\xi)\right)\right)\right] \left(\int_E g\left(x, p\left(\overline{S}_x(\delta)\right), \int_{\overline{S}x} k\left(t, p\left(\overline{S}_x(\eta)\right) d\mu\right)\right),$$

$$\text{if } E \in M_z, \quad E \subset \overline{x_0 z} \tag{2.7}$$

$$p(E) = q(E) \qquad \text{if } E \in M_0 \tag{2.8}$$

A solution p of abstract measure AMIGDE (5.2.5) – (5.2.6) in $\overline{x_0 z}$ will be denoted by $p\left(\overline{S}x_0, q\right)$.

Note that the above equations includes the abstract measure differential equation considered in Dhage and Bellale $[6]$ as a special case. The see this, define $f(x, y) = 1$ for all $x \in \overline{x_0 z}$ & $y \in R$ then AMIGD (2.5) – (2.6) reduces to

$$\frac{dp}{d\mu} = g\left(x, p\left(\overline{S}_x(\delta)\right), \int_{\overline{S}x} k\left(t, p\left(\overline{S}_t(\eta)\right) d\mu\right) \left(a.e.[\mu] \text{ on } \overline{x_0 z} \quad (2.9)\right)$$

$$p(E) = q(E) \ , \ E \in M_0 \tag{2.10}$$

Thus our AMIGDE (2.5) – (2.6) is more general and we claim that it is a new to the literature on measure differential equations.

Now we shall prove the existence theorem.

3 Auxiliary Results

Let X be a Banach space andlet $T : X \to Y$. T is called compact if $\overline{T(x)}$ is a compact subset of X. T is called totally bounded if for any bounded subset S of X, T (S) is a totally bounded subset of X. T is called completely continuous if T is continuous and totally bounded on X. Every compact operator is totally bounded, but the converse may not be true, however two notions are equivalent on a bounded subset of X.

An operator $T : X \to Y$ is called D – Lipschitz if these exists a continuous and non-decreasing function $\psi : R^{+} \to R^{+}$ such that

$$\|Tx - Ty\| \leq \psi\left(\|x - y\|\right) \tag{3.1}$$

for all $x, y \in X$, where $\psi(0) = 0$. The function ψ is called a D – function of T on X. In particular if $\psi(r) = \propto r$, $\propto > 0$, T is called a Lipschitz with the Lipschitz constants $\propto$ further if $\propto < 1$, then T is called a contraction with contraction constant $\propto$. Again if $\psi(r) < r$ for $r > 0$, then T is called a non-linear contraction on X with D – function ψ.

Theorem 3.1 *Let U* and $\overline{U}$ *denote respectively the open and closed bounded subset of a Banach algebra X such that* $0 \in U$ *let* $A, B : \overline{U} \to X$ *be two operators such that*

i) *A is D – Lipschitz*

ii) *B is completely continuous and*

iii) $M\phi(r) < r, r > 0$ *where* $M = \left\|B\left(\overline{U}\right)\right\|$.

Then either

a) *The equation* $Ax\,Bx = x$ *has a solution in* $\overline{U}$ *, as*

b) *These is a point* $u \in d\,U$ *such that* $u = \lambda Au\,Bu$ *for some* $0 < \lambda < 1$*,*

where ∂U *is a boundary of U in X.*

Corollary 3.1 *Let* $B_r(0)$ & $\overline{B_r}(0)$ *denote respectively the open and closed balls in a Banach algebra centred at origin 0 of radius r for some real number* $r > 0$. *Let* $A,B: \overline{B_r}(0) \to X$ *be two operators such that*

i) *A is Lipschitz with Lipschitzcontent* α .

ii) *B is compact* and *continuous,* and

iii) $\alpha M < 1$, *where* $M = \left\| \left(B\,\overline{B_r}(0) \right) \right\|$ *then either*

a) *The operator equation* $Ax\,Bx = x$ *has a solution x in X with* $\|x\| \leq r$ *or*

b) *These is an* $u \in X$ *with* $\|u\| = r$ *such that* $\lambda AuBu = u$ *for some* $0 < \lambda < 1$.

We define an order relation $\leq$ in $AC(S_z, M_z)$ with the help of the cone K in $AC(S_z, M_z)$ given by

$$k = \left\{ p \in AC(S_z, M_z) \;\middle|\; p(E) \geq 0 \text{ for all } E \in M_z \right\} \tag{3.2}$$

Thus for any $p_1, p_2 \in AC(Sz, M_z)$ are have $p_1 \leq p_2$ if and only if

$$p_2 - p_1 \in k \tag{3.3}$$

or equivalently

$$p_1 \leq p_2 \Leftrightarrow p_1(E) \leq p_2(E) \text{ ....}(3.4) \text{ for all } E \in M_z .$$

Obviously the cone K is positive in $AC(S_z, M_z)$. To see this,let $p_1\, p_2 \in K$.

Then $p_1(E) \geq 0$ and $p_2(E) \geq 0$ for all $E \in M_z$. By multiplication composition $(p_1 * p_2)E = p_1(E)\; p_2(E) \geq 0$ for all $p_1 * p_2 \in k$, and so K is a positive come in $AC(S_z, M_z)$.

The following lemma follow immediately from the definition of the positive cone K in $AC(S_z, M_z)$

Lemma 3.1 *Dhage*[3] *if* $u_1, u_2, v_1, v_2 \in K$ *are such that* $u_1 < v_1 \,\&\, u_2 < v_2$, *then* $u_1 u_2 \leq v_1 v_2$.

Lemma 3.2 *The cone K is normalin* $AC(S_z, M_z)$.

Proof. To prove it is enough to show that norm $\|\cdot\|$ is semi monotone on K.Let $p_1, p_2 \in K$ be such that $p_1 \leq p_2$ on M_z. Then we have

$$0 \leq p_1(E) \leq p_2(E)$$

For all $E \in M_z$.

Now for a countable partition

$$\sigma = \{E_n : n \in N\} \text{ of } S_2 \text{,one has}$$

$$\begin{aligned} \|p_1\| &= |p_1|(S_z) \\ &= sup \sum_{i=1}^{\infty} |p_1(Ei)| \\ &\leq \sup_{\sigma} \sum_{i=1}^{\infty} |p_2(Ei)| \\ &= \|p_2\|(S_z) \\ &= \|p_2\| \end{aligned}$$

As a result $\|\cdot\|$ is semi monotone on k andconsequently the cone K is normal in $AC(S_z, M_z)$. The proof of the lemma is complete.

An operator $T : X \to X$ is called positive if the range $r(T)$ of T is contained in the cone K in X.

Theorem 3.2 Dhage$[4]$. *Let* $[u,v]$ *be an order interval in the real Banach algebra X and let* $A, B : [u,v] \to [u,v]$ *be positive and non-decreasing operators such that*

i) *A is Lipschitz with a Lipschitzconstant* α ,

ii) *B is compact & continuous, and*

iii) *The elements* $u,v \in X$ *with* $u \le v$.

Satisfy $u \le AuBu$ *and* $AvBv \le v$.

Further if the cone K is normal, then the operator $AxBx = x$ has a least and a greatest positive solution in $[u,v]$, whenever $\alpha M < 1$, where $M = \|B([u,v])\| = sup\{\|Bx\| : x \in [u,v]\}$.

Theorem 3.3 Dhage $[4]$ *let k be a positive cone in a real Banach algebra X and let* $A,B : K \to K$ *be non-decreasing operators such that*

i) *A is Lipschitz with the Lipschitz constant* $\propto$

ii) *B is bounded, and*

iii) *There exist elements* $u,v \in k$ *such that* $u \le v$ *satisfying* $u \le AuBu$ *and* $AvBv \le v$.

Further, if the cone k is normal then the operator equation $AxBx = x$ *has a least and greatest positive solution in* $[u,v]$*, whenever* $\propto M < 1$ *where*

$$M = \|B([u,v])\| = sup\{\|Bx\| : x \in [u,v]\}$$

4 Existence Result

We need the following definition.

Definition 4.1 *A functions* $\beta : S_z \times R \times R \to R$ *is called Caratheodory if*

i) $x \to \beta(x, y_1, y_2)$ *is* μ*-measurable for each* $y_1, y_2 \in R$ *and*

ii) *The function* $(y_1, y_2) \to \beta(x, y_1, y_2)$ *is continuous almost everywhere* $[\mu]$ *on* $\overline{x_0 z}$.

A Caratheodary function β *on* $S_z \times R \times R$ *is called* L_μ^1 *Caratheodary if*

iii) For each real number $r > 0$ *there exists a functions* $h_r \in L_\mu^1\left(S_z R_t\right)$ *such that*

$$\left|\beta\left(x, y_1, y_2\right)\right| \leq h_r\left(x\right) \; a.e.[\mu] \; on \; \overline{x_0 z}$$

For all $y_1, y_2 \in R$ *with* $\left|y_1\right| \leq r \,\&\, \left|y_2\right| \leq r$.

A function $\psi : R_+ \to R_+$ *is called sub multiplicative if* $\psi\left(\lambda r\right) \leq \lambda \psi\left(r\right)$ *for all real number* $\lambda > 0$. *Let* ψ *denote the class of function* $\psi : R_+ \to R_+$ *satisfying the following properties:*

i) ψ *is continuous*

ii) ψ *is non-decreasingand*

iii) ψ *issub multiplicative*

A member $\psi \in \Psi$ is called a D – function on R_+. These do exist D – function, in fact, the function $\psi : R_+ \to R_+$ defined by $\psi\left(\lambda\right) = \lambda r$, $\lambda > 0$ is a D – function on R_+ we consider the following set of assumptions:

$\left(A_0\right)$ for any $z > x_0$, the σ-algebra M_z is compact with respect to the topology generated by the Pseudometric d defined on M_z by

$$D\left(E_1, E_2\right) = \left|\mu\right|\left(E_1 \Delta E_2\right),\; E_1 E_2 \in M_z$$

$\left(A_1\right)$ The function $x \to \left|f\left(x, o\right)\right|$ is bounded with $Fo = sup_{x \in S_z} \left|f\left(x, o\right)\right|$

$\left(A_2\right)$ The function is continuous and these exists a bounded function $\alpha : S_z \to R^+$ with bound $\left\|\alpha\right\|$ such that

$$\left|f\left(x, y_1\right) - f\left(x, y_2\right)\right| \leq \alpha\left(x\right)\left|y_1 - y_2\right|,\; a.e.[\mu], x \in \overline{x_0 z}$$

For all $y_1, y_2 \in R$

$\left(H_0\right)$ q is continuous on M_z with respect to the pseudo-metric d defined in $\left(A_1\right)$.

$\left(H_1\right)$ The function $x \to k\left(x, p\left(\overline{S}_x\left(\eta\right)\right)\right\}$ is μ-integrable& satisfies

$$\left|k\left(t,y\right|\le\gamma\left(x\right)\left|y\right|,\ a.e.\left[\mu\right]on\ \overline{x_0z}\qquad \text{For all}\quad y\in R$$

$\left(H_2\right)$ The function $g\left(x,y_1,y_2\right)$ is Caratheodary

$\left(H_3\right)$ There exists a function $\phi\in L^1_\mu\left(S_z,R^+\right)$ such that $\phi\left(x\right)>0\ a.e.\left[\mu\right]$ on $\overline{x_0z}$ and D – function $\psi:\left[0,\infty\right]\to\left(0,\infty\right)$ such that

$$\left|g\left(x,y_1,y_2\right)\right|\le\phi\left(n\right)\psi\left(\left|y_1\right|+\left|y_2\right|\right)\ a.e.\left[\mu\right]\ on\ \overline{x_0z}$$

For all $y_1,y_2\in R$

We frequently use the following estimate of the function g in the subsequent part of the paper. For any $p\in AC\left(S_z,M_z\right)$, one has

$$\begin{aligned}
&\left|g\left(x,p\left(\overline{S}_x\left(\delta\right)\right),\int_{\overline{S}x}k\left(t,p\left(\overline{S}_t\left(\eta\right)\right)\right)d\mu\right)\right|\\
&\le\phi\left(x\right)\psi\left(\left|p\left(\overline{S}_x\left(\delta\right)\right|+\int_{\overline{S}x}\left|k\left(t,p\left(\overline{S}_t\left(\eta\right)\right)\right)\right|d\mu\right)\\
&\le\phi\left(x\right)\psi\left(\left|p\right|\left(S_z\left(\delta\right)\right)+\int_{\overline{S}z}\gamma\left(x\right)\left|p\left(\overline{S}_z\left(\eta\right)\right)\right|d\mu\right)\\
&\le\phi\left(x\right)\psi\left(\left\|p\right\|+\int_{\overline{S}z}\gamma\left(x\right)\left\|p\right\|d\mu\right)\\
&\le\phi\left(x\right)\psi\left(\left\|p\right\|+\left\|\gamma\right\|_{L^1_\mu}\left\|p\right\|\right)\\
&\le\phi\left(x\right)\left(1+\left\|\gamma\right\|_{L^1_\mu}\right)\psi\left(\left\|p\right\|\right)
\end{aligned}$$

Theorem 4.1 *Suppose that the assumptions* $\left(A_0\right)-\left(A_2\right)\&\left(H_1\right)-\left(H_3\right)$ *holds. Suppose that there exists a real number* $r>0$ *such that*

$$r>\frac{F_0\left[\left\|q\right\|+\left\|\phi\right\|_{L^1_\mu}\left(1+\left\|\gamma\right\|_{L^1_\mu}\right)\psi\left(r\right)\right]}{1-\left\|\alpha\right\|\left[\left\|q\right\|+\left\|\phi\right\|_{L^1_\mu}\left(1+\left\|\gamma\right\|_{L^1_\mu}\right)\psi\left(r\right)\right]}\tag{4.1}$$

where $\left\|\alpha\right\|\left[\left\|q\right\|+\left\|\phi\right\|_{L^1_\mu}\left(1+\left\|\gamma\right\|_{L^1_\mu}\right)\psi\left(r\right)\right]<1$

& $F_0=\sup\limits_{x\in S_z}\left|f\left(x,o\right)\right|$. Then the AMIGDE (2.5) – (2.6) has a solution on $\overline{x_0z}$.

Proof :- Consider on open ball $\overline{B}_r(0)$ in $AC(S_z, M_z)$ centered at the origin 0and of radius r. Where r satisfies the inequalities in (5.4.1). Define two operators.

$$A, B : \overline{B}r(0) \to AC(S_z, M_z)$$

by

$$Ap(E) = \begin{cases} 1 & if\ E \in M_0 \\ f\left(x, p\left(E(\xi)\right)\right) & if\ E \in M_z\ E \subset \overline{x_0 z} \end{cases} \tag{4.2}$$

&

$$Bp(E) = \begin{cases} q(E) & if\ E \in M_0 \\ \int\limits_E g\left(x, p\left(\overline{S}_x(\delta)\right), \int_{\overline{S}x} k\left(t, p\left(\overline{S}_t(\eta)\right) d\mu\right) d\mu\right. & \\ & if\ E \in M_z, E \subset \overline{x_0 z} \end{cases} \tag{4.3}$$

We show that the operators A and B satisfy all the condition of Corollary 3.1 on $\overline{B}r(0)$.

Step – I First, we show that A is a Lipschitz on $\overline{B}r(0)$. Let $p_1, p_2 \in \overline{B}r(0)$ be arbitrary, then by assumption (A_2),

$$\begin{aligned} \left|Ap_1(E) - Ap_2(E)\right| &= \left| f\left(x, p_1\left(E(\xi)\right) - f\left(x, p_2\left(E(\xi)\right)\right)\right|\right. \\ &\leq \alpha(x) \left| p_1\left(E(\xi)\right) - p_2\left(E(\xi)\right)\right| \\ &\leq \alpha \|x\| \, |p_1 - p_2|(E) \end{aligned}$$

for all $E \in M_z$. Hence by definition of the norm in $AC(S_z, M_z)$ one has

$$\|Ap_1 - Ap_2\| \leq \|\alpha\| \, \|p_1 - p_2\|$$

For all $p_1, p_2 \in AC(S_z, M_z)$. As a result. A is a Lipschitz operator $\overline{B}r(0)$ with the Lipschitzconstant $\|\alpha\|$.

Step – II We show that B is continuous on $\overline{B}r(0)$. Let $\{p_n\}$ be sequence of vector measure in $\overline{B}r(0)$converging to a vector measure. Then by dominated convergence theorem,

$$\lim_{h\to\infty} \overline{B}p_n(E) = \lim_{h\to\infty} \int_E g\left(x, p_n\left(\overline{S}_x(\delta)\right), \int_{\overline{S}x} k\left(t, p\left(\overline{S}_t(\eta)\right)\right) d\mu\right.$$
$$= \int_E g\left(x, p\left(\overline{S}_x(\delta)\right), \int_{\overline{S}x}\left(t, p\left(\overline{S}_t(\eta)\right) d\mu\right) d\mu\right.$$
$$= \overline{B}p(E)$$

for all $E \in M_z$, $E \subset \overline{x_0 z}$. Similarly if $E \in M_0$ then

$\lim_{n\to\infty} \overline{B}p_n(E) = q(E) = Bp(E)$ and so B is a continuous operator on $\overline{B}r(0)$.

Step – III Next, we show that B is a totally bounded operator on $\overline{B}r(0)$. Let $\{p_n\}$ be a sequence an $\overline{B}r(0)$. Then we have $\|p_n\| \le r$ for all $n \in N$. We show that the set $\{Bp_n : n \in N\}$ is uniformly bounded andequicontinuous set in $AC(S_z, M_z)$. In this step, we first show that $\{Bp_n\}$ is uniformly bounded.

Let $E \in M_z$. Then there exists two subsets $F \in M_0 \,\&\, G \in M_z$, $G \subset \overline{x_0 z}$ such that

$$E = F \cup G \;\; \& \; F \wedge G = \phi$$

Hence ,

$$Bp_n(E(\sigma)) \le |q(F)| + \int_G \left| g\left(x, p_n\left(\overline{S}_x(\delta)\right), \int_{\overline{S}_x} k\left(t, p_n\left(\overline{S}_t(\eta)\right) d\mu\right| d\mu\right.$$
$$\le \|q\| + \int_G \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p_n\|)\, d\mu$$
$$\le \|q\| + \int_E \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p_n\|)\, d\mu$$
$$= \|q\| + \|\phi\|_{L^1_\mu}\left(1 + \|\gamma\|_{L^1_\mu}\,\psi(\|p_n\|)\right.$$

For all $E \in M_z$.

from (3.3) it follows that

$$\|Bp_n\| = |Bp_n|(S_z) = \sup_\sigma \sum_{i=1}^{\infty} |Bp_n(Ei)| = \|q\| + \|\phi\|_{L^1_\mu}\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p\|)$$
$$= \|q\| + \|\phi\|_{L^1}\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(r)$$

For all $n \in N$. Hence the sequence $\{Bp_n\}$ is uniformly bonded in $B\left(\overline{B}r(0)\right)$

Step – IV Next we show that $\{Bp_n : n\in N\}$ is equicontinuous set in $AC(S_z, M_z)$. Let $E_1, E_2 \in M_z$, then there exist

$$F_1, F_2 \in M_0 \ \& \ G_1, G_2 \in M_{z,} \ G_1 \subset \overline{x_0 z}$$

and $G_2 \subset \overline{x_0 z}$ such that $E_1 = F_1 \cup G_1$ with $(F_1 \cap G_1 = \phi)$ and $E_2 = F_2 \cup G_2$ with $F_2 \cap G_2 = \phi$.

We know the identities

$$\left.\begin{aligned} G_1 &= (G_1 - G_2) \cup (G_1 \cap G_2) \\ G_2 &= (G_2 - G_1) \cup (G_1 \cap G_2) \end{aligned}\right\} \tag{4.4}$$

There fore, we have

$$\begin{aligned} &Bp_n(E_1) - Bp_n(E_2) \le q(E_1) - q(F_2) \\ &+\int_{G_1 - G_2} g\left(x, p_n\left(\overline{S}_x(\delta)\right), \int_{\overline{S}x} k\left(t, p_n\left(\overline{S}_t(\eta)\right)\right) d\mu\right) d\mu \\ &+\int_{G_2 - G_1} g\left(x, p_n\left(\overline{S}_x(\delta)\right), \int_{\overline{S}x} k\left(t, p_n\left(\overline{S}_t(\eta)\right)\right) d\mu\right) d\mu \end{aligned}$$

Since g is Caratheodory and satisfies (H_3)

We have that

$$\begin{aligned} &\left|Bp_n(E_1) - Bp_n(E_2)\right| \le \left|q(F_1) - q(F_2)\right| \\ &+\int_{G_1 \Delta G_2} \left| g\left(x, p_n\left(\overline{S}(\delta)\right), \int_{\overline{S}_x} k\left(t, p_n\left(\overline{S}_t(\eta)\right)\right) d\mu\right| d\mu \\ &\le \left|q(F_1) - q(F_2)\right| + \int_{G_1 \Delta G_2} \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right) \psi\left(\|p_n\|\right) d\mu \end{aligned}$$

Assume that $d(E_1, E_2) = |\mu|(E_1 \Delta E_2) \to 0$.

Then we have $E_1 \to E_2$. As a result $F_1 \to F_2$ and $|\mu|(G_1 \Delta G_2) \to 0$. As q is continuous on compact M_z, it is uniformly continuous and so

$$\begin{aligned} &\left|Bp_n(E_1) - Bp_2(E_2)\right| \le \left|q(F_1) - q(F_2)\right| + \int_{G_1 \Delta G_2} \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right) \psi\left(\|p_n\|\right) d\mu \to 0 \\ &as \quad E_1 \to E_2 \end{aligned}$$

This shows that $\{Bp_n : n \in N\}$ is a equicontinuous set in $AC(S_z M_z)$. Now an application of the Arzela – Ascolli theorem yields that B is a totally bounded operator on $\overline{B}r(0)$. Now B is continuous and totally bounded operator on $\overline{B}r(0)$, it is completely continuous operator on $\overline{B}r(0)$.

Step V Finally we show that hypothesis (iii) of Corollary 3.1. The Lipschitz constant of A is $\|\alpha\|$. Here the number M in the hypothesis (iii) is given by

$$\begin{aligned} M &= \left\|B\left(\overline{B}r(0)\right)\right\| \\ &= sup\left\{\|Bp\| : p \in \overline{B}r(0)\right\} \\ &= sup\left\{|Bp|(S_z) : p \in \overline{B}r(0)\right\} \end{aligned} \tag{4.5}$$

Now let $E \in M_z$. Then there are sets $F \in M_0$ and $G \in M_z$, $G \subset \overline{x_0 z}$ such that

$$E = F \cup G \ and \ F \cap G = \phi$$

From the definition of B is follows that

$$Bp(E) = q(F) + \int_G g\left(x, p\left(\overline{S}_x(\delta)\right), \int_{\overline{S}_x} k\left(t, p\left(\overline{S}_t(\eta)\right) d\mu\right) d\mu\right.$$

Therefore,

$$\begin{aligned} \left|Bp(E)\right| &\le \left|q(F)\right| + \int_G \left|g\left(x, p\left(\overline{S}_x(\delta)\right), \int_{\overline{S}_x} k\left(t, p\left(\overline{S}_t(\eta)\right) d\mu\right)\right| d\mu \\ &\le \|q\| + \int_G \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi\left(\|p\|\right) d\mu \\ &\le \|q\| + \int_{\overline{x_0 z}} \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi\left(\|p\|\right) d\mu \\ &= \|q\| + \|\phi\|_{L^1_\mu}\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi\left(\|p\|\right) \end{aligned}$$

Hence from (4.6) it follows that

$$\|Bp\| \le \|q\| + \|\phi\|_{L^1_\mu}\left(1 + \|\gamma\|_{L^1_{gm}}\right)\psi\left(\|p\|\right)$$

For all $p \in \overline{B}r(0)$. As a result are have

$$M = \left\| B\left(\overline{B}r(0)\right)\right\| \le \|q\| + \|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|p\|\right)$$

$$\text{Now } \alpha M \le \|\alpha\| \cdot \left[\|q\| + \|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(r)\right]$$

$$<1$$

and so, hypothesis (iii) of corollary 3.1 is satisfied.

Now an application of corollary 3.1 yields that either the operator $Ax\,Bx = x$ has a solution, or there exist $u \in AC\left(S_z, M_z\right)$ such that $\|u\| = r$ satisfying $u = \lambda AxBx$ for some $0 < \lambda < 1$. We show that this letter assertion does not hold. Assume the contrary. Then we have

$$u(E) = \begin{cases} \lambda\left[f\left(x,u(G)\right)\right]\left(\int g\left(x,u\left(\overline{S}_x(\delta)\right),\int_{\overline{S}_x} k\left(t,u\left(\overline{S}_t(\eta)\right)d\mu\right)d\mu\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad if \;\; E \in M_z,\; E \subset \overline{x_0 z}. \\ \lambda q(E)\;,\; if \;\; E \in M_0 \end{cases}$$

For some $0 < \lambda < 1$.

If $E \in M_z$, then these sets $F \in M_0$ and $G \in \overline{Mz}, G \subset \overline{x_0 z}$ such that $E = F \cup G \;\; and \; F \cap G = \phi$. Then we have

$$u(E) = \lambda\, Au(E),\quad Bu(E)$$

$$= \lambda\left[f\left(x,u(G)\right)\right]\left(q(F) + \int_G g\left(x,u\left(\overline{S}_x(\delta)\right),\int_{\overline{S}x} k\left(t,u\left(\overline{S}_t(\eta)\right)d\mu\right)\right.$$

$$= \lambda\left[f\left(x,u(G)\right) - f(x,0)\right]\left(q(F) + \int_G g\left(x,u\left(\overline{S}_x(\delta)\right)d\mu,\int_{\overline{S}x} k\left(t,u\left(\overline{S}_t(\eta)\right)d\mu\right)\right)\right.$$

$$+\lambda\, f(x,0)\left(q(F) + \int_G g\left(x,u\left(\overline{S}_x(\delta),\int_{\overline{S}x} k\left(t,u\left(\overline{S}_t(\eta)\right)d\mu\right)d\mu\right)\right.\right.$$

Hence

$$
\begin{aligned}
&\left|u(E)\right| \le \left(\left|1+\left(x,u(G)-f(x,0)\right|\right)\right. \\
&.\left(\left|q(F)\right|+\int_G \left|g\left(x,u\left(\overline{S}_x(\delta)\right)d\mu,\int_{\overline{S}_x} k\left(t,u\left(\overline{S}_t(\eta)\right)d\mu\right)\right|d\mu\right) \\
&+\left|f(x,0)\right|\left(\left|q(F)\right|+\int_G \left|g\left(x,u\left(\overline{S}_x(\delta)\right),\int_{\overline{S}_x} k\left(t,u\left(\overline{S}_t(\eta)\right)d\mu\right)\right|d\mu\right) \\
&\le \lambda\left(\alpha(x)\left|u(G)\right|+F_0\right)\left(\|q\|+\int_G \phi(x)\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|u\|\right)d\mu\right) \\
&\le \left[\|\alpha\|\left|u\right|(E)\right|+F_0\left]\left(\|q\|+\int_{\overline{x_0 z}} \phi(x)\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|u\|\right)d\mu\right)\right. \\
&\le \left[\|\alpha\|\,\|u\|+F_0\right]\left[\|q\|+\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|u\|\right)\right]
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
&\|u\| \le \left(\|\alpha\|\,\|u\|\left[\|q\|+\|\phi\|_{L^1_{gm}}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|u\|\right)\right]\right)+F_0\left[\|q\|+\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|u\|\right)\right] \\
&\le \frac{F_0\left[\|q\|+\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\left(\|u\|\right)\right)}{1-\|\alpha\|\left[\|q\|+\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi\,\|u\|\right]}
\end{aligned}
$$

Substituting $\|u\|=r$ in the above inequality yields

$$
r \le \frac{F_0\left[\|q\|+\|\phi\|_{L^1_\mu}\left(1+\|r\|_{L^1_\mu}\right)\psi\left(\|\gamma\|\right)\right)}{1-\|\alpha\|\left[\|q\|+\|\phi\|_{L^1_\mu}\left(1+\|r\|_{L^1_\mu}\right)\psi\,\|\gamma\|\right]} \tag{4.6}
$$

Which is a contradiction to the first inequality in (4.4). In consequence, the operator equation $p(E)=Ap(E)Bp(E)$ has a solution $u\left(\overline{S}x_0,q\right)$ in $AC\left(S_z,M_z\right)$ with $\|u\|\le r$. This further implies that the AMIGDE (2.5) – (2.6) has a solution on $\overline{x_0 z}$. This completes the proof.

5 Existence of Extremal Solutions

In this section we shall prove the existence of a minimal and a maximal solutions for the AMIGDE (2.5) – (2.6) on $\overline{x_0 z}$ under Carathedory as well as discontinuous case of non-lineality g involved in it.

5.1 Carathedory Case

We need following definitions

Definition 5.1 *A vector measure* $u \in AC(S_z, M_z)$ *is called a lower solution of the AMIGDE (2.5) – (2.6) if*

$$\frac{d}{d\mu}\left(\frac{u(\overline{S}_x)}{f\left(x, u\left(\overline{S}_x(\xi)\right)\right)}\right) \leq g\left(x, u\left(\overline{S}_x(\delta)\right), \int_{\overline{S}_x} k\left(t, u\left(\overline{S}_t(\eta)\right)\right) d\mu\right) \ a.e.\ [\mu] \text{ on } \overline{x_0 z}$$

$$\& \quad u(E) \leq q(E)\ , \ E \in M_0$$

Similarly a vector measure $v \in AC(S_z, M_z)$ is called an upper solution to AMIGDE (2.5) – (2.6) if

$$\frac{d}{d\mu}\left(\frac{v(\overline{S}_x)}{f\left(x, v\left(\overline{S}_x(\xi)\right)\right)}\right) \geq g\left(x, v\left(\overline{S}_x(\delta)\right), \int_{\overline{S}_x} k\left(t, v\left(\overline{S}_t(\eta)\right)\right) d\mu\right) \ a.e.\ [\mu] \text{ on } \overline{x_0 z}$$

$$\text{and} \quad v(E) \geq q(E)\ , \ E \in M_0$$

A vector measure $p \in A(S_z, M_z)$ is a solution to AMIGDE (2.5) – (2.6) it is upper as well as lower solution to AMIGDE (2.5) – (2.6) on $\overline{x_0 z}$.

Definition 5.2 *A solution* P_M *is called as maximal solution to AMIGDE (2.5) – (2.6) if for any other solution* $p(\overline{S}x_0, q)$ *for the AMIGDE (2.5) – (2.6) we have that*

$$p(E) \leq p_M(E), \forall E \in M_z$$

Similarly a minimal solution $p_m\left(\overline{S}x_0,q\right)$ *of AIGDE (2.5) – (2.6) is defined on* $\overline{x_0 z}$.

We consider the following assumptions :

(C_0) $f \,\&\, g$ define the functions

$$F : \overline{x_0 z}\times R \to R^+ - \{0\} \text{ and } g : \overline{x_0 z} + R\times R \to R^+$$

(C_1) The functions $f\left(x,y_1\right), k\left(x,y_1\right)$ *and* $g\left(x,y_1,y_2\right)$ are non-decreasing in y_1, y_2 for each $x \in \overline{x_0 z}$.

(C_2) The AMIGDE (2.6) – (2.6) has a lower solution u and an upper solution v such that $u \le v$ on M_z.

(C_3) The function $g\left(x,y_1,y_2\right)$ is L^1_μ Caratheodary.

Theorem 5.1 *Suppose that the assumptions*

$\left(A_0\right)-\left(A_2\right)$, $\left(B_0\right)-\left(B_2\right)$, & $\left(C_0\right)-\left(C_3\right)$ *holds. Further suppose that*

$$\|\alpha\|\left(\|q\|+\|h_r\|_{L^1_\mu}\right) < 1 \tag{5.1}$$

where $r = \|u\|+\|v\|$. *Then the AMIGDE (2.5) - (2.6) has a minimal and maximal solution defined on* $\overline{x_0 z}$.

Proof. AMIGDE (2.5) - (2.6) is equivalent to the abstract measures integral equation $(5.7)\&(2.8)$. Define the operators $A,B : AC\left(S_z,M_z\right)\to AC\left(S_z,M_z\right)$ by (4.2)and(4.3) respectively. Then the AMIGDE (2.5) – (2.6) is equivalent to the operator equation

$$p(E) = Ap(E)\; Bp(E)\,,\;\; E \in M_z \tag{5.2}$$

We shall show that the operator A and B satisfy all the conditions of theorem 3.2 on $AC\left(S_z,M_z\right)$ since μ is a positive measure, from assumption $\left(C_0\right)$ if follows that A and B on positive operators on $AC\left(S_z,M_z\right)$. To show this let

$p_1, p_2 \in AC(S_z, M_z)$ be such that $p_1 \le p_2$ on M_z. From (C_2) it follows that

$$Ap_1(E) = f(x, p_1(E)) \le f(x, p_2(E)) = Ap_2(E)$$

For all $E \in M_z$, $E \subset \overline{x_0 z}$ and $Ap_1(E) = 1 = Ap_2(E)$

For $E \subset M_0$. Hence A is non-decreasing on $AC(S_z, M_z)$

Similarly, we have

$$\begin{aligned} Bp(E) &= \int_E g\left(x, p_1\left(\overline{S}_x(\delta)\right), \int_{\overline{S}x} k\left(t, p_1\left(\overline{S}_t(\eta)\right)\right) d\mu\right) d\mu \\ &= \int_E g\left(x, p_2\left(\overline{S}_x(\delta)\right), \int_{\overline{S}x} k\left(t, p_2\left(\overline{S}_t(\eta)\right)\right) d\mu\right) d\mu \\ &= Bp_2(E) \end{aligned}$$

For all $E \in M_z$, $E \subset \overline{x_0 z}$

Again if $E \in M_0$, then

$$Bp_1(E) = q(E) = Bp_2(E)$$

Therefore the operator B is also non-decreasing on $AC(S_z, M_z)$. Now it can be shown that as the proof of theorem 3.1 that A is Lipschitz operator on $[u, v]$ with the Lipschitz constant $\|\alpha\|$. Since the cone K is normal in X, the order interval $[u, v]$ is norm-bounded.

Hence there is a real number $r > 0$ such that $\|x\| \le \|u\| + \|v\| = r$ for all $x \in [u, v]$. As g is L_μ^1 Caratheodary, there is a function $hr : L_\mu^1(S_z, R^+)$ such that $|g(x, y_1, y_2)| \le hr(x)$ on $\overline{x_0 z}$ for all $y_1, y_2 \in R$. Now proceeding with the arguments as in the proof of theorem 4.1 with $\overline{B}r(0) = [u, v]$, $\gamma(u) = hr(x)$ and $\psi(r) = 1$, it can be proved that B is compact and continuous operator on $[u, v]$. Since u is lower solution of AMIGDE (2.5) – (2.6) we have

$$u(E)\le\left|f\left(x,u(E)\right)\right|\left(\left(\int_E g\left(x,u\left(\overline{S}_x(\delta)\right),\int_{\overline{S}_x} k\left(t,u\left(\overline{S}_t(\eta)\right)\right)d\mu\right)d\mu\right)\right)$$

$$E\in M_z,\ E\subset\overline{x_0 z}$$

and

$$u(E)\le q(E),\quad if\quad E\in M_0$$

From the above inequality in the gives

$$u(E)\le Au(E)\ Bu(E)\ ,if\ \ E\in M_z$$

and so $u\le AuBu$. Similarly, since $v\in AC\left(S_z,M_z\right)$ is an upper solution of AMIGDE (2.5) – (2.6) it can be proved that $Av(E)Bv(E)\le v(E)$ for all $E\in M_z$ and consequently $AvBv\le v$ on M_z . Thus hypothesis (iii) of theorem 3.2 is satisfied. Now definition of norm, it follows that

$$\begin{aligned}M&=\left\|B([u,v])\right\|\\&=sup\left\{\|Bp\|:p\in[u,v]\right\}\\&=sup\left\{|Bp|\left(S_z\right):p\in[u,v]\right\}\\&=\underset{p\in[u,v]}{sup}\left\{\underset{\sigma}{sup}\sum_{i=1}^{\infty}|B|_p(Ei)\right\}\end{aligned}$$

For any partition $\sigma=\{Ei:i\in N\}$ of S_z such that $S_z=\cup_{i=1}^{\infty}Ei,\ Ei\cap Ej=\phi,\forall i,j\in N$

Let $E\in M_z,\ E\cap\overline{x_0 z}$. Then, for any $p\in[u,v]$ one has

$$\begin{aligned}|Bp|(E)&\le\underset{\sigma}{sup}\sum_{i=1}^{\infty}\int_{Ei}\left|g\left(x,v\left(\overline{S}_x(\delta)\right),\int_{\overline{S}_x}\left(t,v\left(\overline{S}_t(\eta)\right)\right)d\mu\right)\right|d\mu\\&\le\underset{\sigma}{sup}\sum_{i=1}^{\infty}\int_{Ei}hr(x)\,d\mu\\&=\int_E hr(u)\,d\mu=\|hr\|_{L_\mu^1}\end{aligned}$$

Therefore for any $E\in M_z$, there are sets $E\in M_0$ and $G\in\overline{x_0 z}$ such that $E=F\cup G,\ F\cap G=\phi$.

Hence, we obtain

$$M = \left\| B\left(\left[u,v\right]\right)\right\| \le \|q\| + \|hr\|_{L^1_{gm}}$$

As $\alpha M \le \|\alpha\| \left(\|q\| + \|hr\|_{L^1_\mu} \right) < 1$. Thus the operator A and B satisfy all the conditions of theorem 3.2 and so an application of it yields that the operator equation $ApBp = p$ has a maximal and a minimal solution in $[u,v]$. Thus further implies that AMIGDE (2.5) – (2.6) has a maximal & a minimal solution on $\overline{x_0 z}$. This completes the proof.

5.2 Discontinuous Case

In the following we obtain an existence result for external solution for the AMIGDE (2.5) – (2.6) when the nonlinearity$g$ is a discontinuous function in all its three variables.

We consider the following assumptions :

(C_4) The function $h : \overline{x_0 z} \to R^+$ defined by

$$h(x) = g\left(x, v\left(\overline{S}_x(\delta)\right), \int_{\overline{S}_x} k\left(t, v\left(\overline{S}_t(\eta)\right)\right) d\mu\right)$$

is μ-integrableon $\overline{x_0 z}$.

Remark 5.1 *Assume that the hypothesis* (C_2) and (C_3) *hold. Then*

$$\left| g\left(x, v\left(\overline{S}_x(\sigma)\right), \int_{\overline{S}_x} k\left(t, v\left(\overline{S}_x(\eta)\right)\right) d\mu\right) \right| \le h(x)$$

All $p \in [u,v]$.

Theorem 5.2 *Suppose that the assumptions* $(A_0)-(A_2),(B_0)-(B_2)$ *and* $(C_0)-(C_2),(C_4)$ *hold. Further suppose that*

$$\|\alpha\|\left(\|q\|+\|h\|_{L^1_\mu}\right)<1 \tag{5.3}$$

Then the AMIGDE (2.5) – (2.6) has a minimal and a maximal solution defined on $\overline{x_0 z}$

Proof. The proof is similar to Theorem 5.2 with appropriate modifications. Here, the function h plays the role of $h_r\left[u,v\right]$. Now the desired conclusion follows by an application of theorem 5.3.Notice that we do not need any type of continuity of the nonlinear function g in above theorem 5.2 for guaranteeing the existence of extermal solutions for the AMIGDE (2.5) - (2.6) on $\overline{x_0 z}$ instead we assumed the monotonicity condition on it.

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