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Kinematics of one dimensional (1D) carrier wave propagating in a visco-elastic medium

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Abstract

Every material contains particles. When a wave travels through a material, the oscillating field in the wave will set some of these particles into forced vibration, and the vibrating particles will generate new waves of their own. The initial energy of the propagating wave is attenuated due to absorption and scattering by the medium as it passes. The aim of the present study was to characterize the mechanism of Fourier transform technique in determining the energy attenuation profile of a carrier wave equation (CWE) as it propagates in a pipe containing a visco-elastic fluid. This study also provides a deductive method for determining the independent characteristics of two superposed waves whose initial characteristics were not known. We showed in this work that the interference of a

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‘parasitic wave’ with a ‘host wave’ will results to a drastic reduction in the energy propagation time of the CWE and a narrow frequency bandwidth.

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1 Introduction

The term wave is often intuitively understood as referring to a transport of spatial disturbances that are generally not accompanied by a motion of the medium occupying this space as a whole. In a wave, the energy of a vibration is moving away from the source in the form of a disturbance within the surrounding medium [1]. However, this notion is problematic for a standing wave (for example, a wave on a string), where energy is moving in both directions equally, or for electromagnetic (e.g., light) waves in a vacuum, where the concept of medium does not apply and interaction with a target is the key to wave detection and practical applications. There are two velocities that are associated with waves, the phase velocity and the group velocity and to understand them, one must consider several types of waveform [2], [3].

Although arbitrary wave shapes will propagate unchanged in lossless linear time-invariant systems, in the presence of dispersion the sine wave is the unique shape that will propagate unchanged but for phase and amplitude, making it easy to analyze [4]. Due to the Kramers-Kronig relations a linear medium with dispersion also exhibits loss, so the sine wave propagating in a dispersive medium is attenuated in certain frequency ranges that depend upon the medium [5]. The sine function is periodic, so the sine wave or sinusoid has a wavelength in space and a period in time [6].

The sinusoid is defined for all times and distances, whereas in physical situations we usually deal with waves that exist for a limited span in space and duration in time. Fortunately, an arbitrary wave shape can be decomposed into an infinite set of sinusoidal waves by the use of Fourier analysis. As a result the simple case of a single sinusoidal wave can be applied to more general cases [7].

In particular, many media are linear, or nearly so, so the calculation of arbitrary wave behaviour can be found by adding up responses to individual sinusoidal waves using the superposition principle to find the solution for a general waveform. When a medium is nonlinear, the response to complex waves cannot be determined from a sine-wave decomposition [8].

The superposition principle applies to any linear system, including algebraic equations, linear differential equations and systems of equations of those forms. The stimuli and response could be numbers, functions, vectors, vector fields, time-varying signals, or any other object which satisfies certain axioms. Note that when vectors or vector fields are involved, a superposition is interpreted as a vector sum. For example, in Fourier analysis, the stimulus is written as the superposition of infinitely many sinusoids [9].

Due to the superposition principle, each of these sinusoids can be analyzed separately, and its individual response can be computed. The response is itself a sinusoid, with the same frequency as the stimulus, but generally a different amplitude and phase. According to the superposition principle, the response to the original stimulus is the sum (or integral) of all the individual sinusoidal responses [10], [11].

The phenomenon of interference between waves is based on the idea of superposition of waves. When two or more waves traverse the same space, the net amplitude at each point is the sum of the amplitudes of the individual waves. In some cases, the summed variation has smaller amplitude than the component variations; this is called destructive interference. In other cases, the summed variation will have bigger amplitude than any of the components individually; this is called constructive interference [12].

If a wave is to travel through a medium such as water, air, steel, or a stretched string, it must cause the particles of that medium to oscillate as it passes. For that to happen, the medium must possess both mass (so that there can be kinetic energy) and elasticity (so that there can be potential energy). Thus the medium's mass and elasticity determines how fast the wave can travel in the medium [13].

Every material contains particles. When a wave travels through a material, the oscillating field in the wave will set some of these particles into forced vibration, and the vibrating particles will generate new waves of their own. If the participating particles are sufficiently close together, they will be driven coherently, with quite different results. In this case, the scattered waves can be superposed with the direct wave, giving rise to a new disturbance which will be the wave in the material.

Any actively defined physical system carries along with it an inbuilt attenuating factor such that even in the absence of any external influence the system will eventually come to rest after a specified time. This accounts for the non-permanent nature of any physical system. A 'parasitic wave' as the name implies, has the ability of destroying or transforming the intrinsic constituents of the 'host wave' to its form after a sufficiently long time. It contains an inbuilt multiplier λ which is capable of raising the intrinsic parameters of the 'parasitic wave' to become equal to those of the 'host wave'. Consequently, once this equality is achieved, then all the active components of the host wave would have been completely eroded and it ceases to exist.

This paper is outlined as follows. Section 1, illustrates the basic concept of the work under study. The mathematical theory is presented in Section 2. The results obtained are shown in Section 3. While in section 4, we present the analytical discussion of the results obtained. The conclusion of this work is shown in section

5. This is immediately followed by appendix of some useful identities and a list of references.

2 Mathematical Theory

2.1 Dynamical Theory of Superposition of Two Incoherent Waves

Let us consider two incoherent waves defined by the non-stationary displacement vectors

$$y_1(\vec{r},t) = a\beta\cos\left(\vec{k}\beta.\vec{r} - n\beta t - \varepsilon\beta\right) \tag{2.1}$$

$$y_2(\vec{r},t) = b\lambda\cos\left(\vec{k}'\lambda.\vec{r} - n'\lambda t - \varepsilon'\lambda\right) \tag{2.2}$$

$$\begin{aligned} y(\vec{r},t) &= y_1(\vec{r},t) + y_2(\vec{r},t) \\ &= a\beta\cos\left(\vec{k}\beta.\vec{r} - n\beta t - \varepsilon\beta\right) + b\lambda\cos\left(\vec{k}'\lambda.\vec{r} - n'\lambda t - \varepsilon'\lambda\right) \end{aligned} \tag{2.3}$$

where all the symbols have their usual wave related meaning. In this study, (2.1) is regarded as the 'host wave' whose propagation depends on the inbuilt raising multiplier β (= 0, 1, 2, . . ., $\beta_{\max}$). While (2.2) represents a 'parasitic wave' with an inbuilt raising multiplier λ (= 0, 1, 2, . . ., $\lambda_{\max}$). The inbuilt multipliers are both dimensionless and as the name implies, they have the ability of gradually raising the basic intrinsic parameters of both waves respectively with time. We have established in a previous paper [14] that when (2.1) is superposed on (2.1) according to (2.3) we get after some algebra that

$$y(\vec{r},t) = \sqrt{(a^2 - b^2\lambda^2) - 2\left(a - b\lambda\right)^2\cos\left((n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda\right)}\cos\left(\vec{k}_c.\vec{r} - (n - n'\lambda)t - E(t)\right) \tag{2.4}$$

Equation (2.4) is regarded as the carrier wave equation (CWE).

On interpretation, $E(t)$ represents total phase angle of the CWE and $\vec{k}_c.\vec{r} = (k - k'\lambda)\vec{r}$, is the coordinate of two dimensional (2D) position vector and

$\beta = 1$, that means it is assumed as a constant in this work and leaves its variation for future study. By definition: $(n - n'\lambda)$ the modulation angular frequency, the modulation propagation constant $(k - k'\lambda)$, the phase difference δ between the two interfering waves is $(\varepsilon - \varepsilon'\lambda)$ the interference term is $2\left(a - b\lambda\right)^2 \cos\left((n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda\right)$, while waves out of phase interfere destructively according to $\left(a - b\lambda\right)^2$ and waves in-phase interfere constructively according to $\left(a + b\lambda\right)^2$. In the regions where the amplitude of the wave is greater than either of the amplitude of the individual wave, we have constructive interference that means the path difference is $(\varepsilon + \varepsilon'\lambda)$, otherwise, it is destructive in which case the path difference is $(\varepsilon - \varepsilon'\lambda)$. If $n = n'$, then the average angular frequency say $(n + n'\lambda)/2$ will be much more greater than the modulation angular frequency say $(n - n'\lambda)/2$ and once this is achieved then we will have a slowly varying carrier wave with a rapidly oscillating phase.

Driving forces in antiphase $(\varepsilon - \varepsilon' = \pm\pi)$ provide full destructive superposition and the minimum possible amplitude; driving forces in phase $(\varepsilon = \varepsilon')$ provide full constructive superposition and the maximum possible amplitude. However, in one dimensional (1D) representation we can recast (2.4) as

$$y(x,t) = \sqrt{(a^2 - b^2\lambda^2) - 2\left(a - b\lambda\right)^2 \cos\left((n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda\right)} \cos\left(\vec{k}_C\, x - (n - n'\lambda)t - E(t)\right) \qquad (2.5)$$

$$E(t) = \tan^{-1}\left(\frac{a\sin\varepsilon + b\lambda\sin\left(\varepsilon'\lambda - (n - n'\lambda)t\right)}{a\cos\varepsilon + b\lambda\cos\left(\varepsilon'\lambda - (n - n'\lambda)t\right)}\right) \qquad (2.6)$$

The variation of the total phase angle with respect to time gives the characteristic angular velocity $Z(t)$. That is

$$\frac{dE(t)}{dt} = -Z(t) = -(n - n'\lambda)\left(\frac{b^2\lambda^2 + ab\lambda\cos\left((\varepsilon - \varepsilon'\lambda) + (n - n'\lambda)t\right)}{a^2 + b^2\lambda^2 + 2ab\lambda\cos\left((\varepsilon - \varepsilon'\lambda) + (n - n'\lambda)t\right)}\right) \qquad (2.7)$$

Note that $\vec{k}_c . x = (k - k'\lambda)\, x$. We can decompose the carrier wave equation CWE into two functions; function of the oscillating amplitude $f(A)$ and the function of the spatial oscillating phase $f(\theta)$, where

$$f(A) = \sqrt{(a^2 - b^2\lambda^2) - 2\left(a - b\lambda\right)^2 \cos\left((n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda\right)} \tag{2.8}$$

$$f(\theta) = \cos\left(\vec{k}_c\, x - (n - n'\lambda)\,t - E(t)\right) \tag{2.9}$$

2.2 Differentio-Binomial Expansion of the Carrier Wave Equation (CWE)

It will not be very easy to expand (2.8) using Fourier series technique. As a result, there is need for us to obtain a comprehensively valid approximate solution to it before applying Fourier series expansion. Hence to make it valid for the application of Fourier series expansion, we first minimize it using Binomial expansion and thereafter the resulting equation is differentiated with respect to the variable time t. Consequently, if we differentiate the result of the Binomial expansion with respect to time, the resulting oscillating amplitude will be converted from the usual dimension of length which is meters (m) to angular velocity whose unit is radian per second (rad./s) or velocity which is m/s. We can further rearrange (2.8) for the purpose of the approximation as follows.

$$f(A) = \sqrt{a^2 - b^2\lambda^2}\sqrt{1 - \frac{2\left(a - b\lambda\right)^2}{(a^2 - b^2\lambda^2)}\cos\left((n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda)\right)} \tag{2.10}$$

$$f(A) = \sqrt{(a^2 - b^2\lambda^2)}\,\frac{d}{dt}\left\{1 - \frac{\left(a - b\lambda\right)^2}{(a^2 - b^2\lambda^2)}\cos\left((n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda)\right) + \ldots\right\} \tag{2.11}$$

$$f(A) = \sqrt{(a^2 - b^2\lambda^2)}\left\{\frac{\left(a - b\lambda\right)^2 (n - n'\lambda)}{(a^2 - b^2\lambda^2)}\sin\left((n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda)\right) + \ldots\right\} \tag{2.12}$$

$$f(A) = \left\{\, D(n - n'\lambda)\sin\left((n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda)\right)\right\} \tag{2.13}$$

where for clarity of purpose we have set $D=(a-b\lambda)^2/\sqrt{(a^2-b^2\lambda^2)}$.

2.3 Fourier Series Expansion of the Oscillating Amplitude *f*(*A*) of the Carrier Wave Equation (CWE)

Now, by expanding the oscillating term of (2.13) in terms of Fourier series we get

$$\begin{aligned}F\left[f(A)\right]=C_0+C_1\left\{\sin\left((n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)\right\}+C_2\left\{\sin\left(2(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)\right\}\\ +C_3\left\{\sin\left(3(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)\right\}\\ +\ldots+C_\beta\left\{\sin\left(\beta(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)\right\}\end{aligned} \quad (2.14)$$

$$F\left[f(A)\right]_2=C_0+\sum_{\beta=1}^{\infty}C_\beta\left\{\sin\left(\beta(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)\right\} \quad (2.15)$$

Thus (2.15) represents the Fourier series expansion of the oscillating amplitude for only one phase described by the sine (odd) function. It is however not always convenient to specify amplitude and phase [15] we can express each term in the form

$$\begin{aligned}C_\beta\left\{\sin\left(\beta(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)\right\}\\ =A_\beta\cos\beta\left((n-n'\lambda)t\right)+B_\beta\sin\beta\left((n-n'\lambda)t\right)\end{aligned} \quad (2.16)$$

where

$$\left.\begin{aligned}A_\beta&=C_\beta\cos(\varepsilon-\varepsilon'\lambda)\\ B_\beta&=-C_\beta\cos(\varepsilon-\varepsilon'\lambda)\end{aligned}\right\}\quad\Rightarrow\quad C_\beta=\sqrt{A_\beta^2+B_\beta^2} \quad (2.17)$$

The negative sign indicates complex conjugate of the real part and the inclusions will make the dynamic components of the phase angle real. Thus (2.17) represents the amplitude of the nth harmonic. Where β is the Fourier index. From (2.16) if $\beta=0$ then;

$$C_0\left\{\sin\left(-(\varepsilon-\varepsilon'\lambda)\right)\right\}=A_0\quad\Rightarrow\quad C_0=-\frac{A_0}{\sin(\varepsilon-\varepsilon'\lambda)} \quad (2.18)$$

Thus the series expansion given by (2.15) can be rewritten using (2.16) and (2.18) as

$$F\left[f(A)\right]_2 = C_0 + \sum_{\beta=1}^{\infty} A_\beta \cos\beta\left((n-n'\lambda)t\right) + B_\beta \sin\beta\left((n-n'\lambda)t\right) \qquad (2.19)$$

where A_0, A_β and B_β are the Fourier coefficients of the series expansion of the CWE. Thus (2.19) represents simultaneously the Fourier series expansion for both the cosine (even) and sine (odd) functions. However, (2.15) and (2.19) is applicable to the study of wave interference, but in this study we shall utilize (2.19) for assumed convergence of both the cosine (even) and sine (odd) functions for maximum Fourier index β.

2.4 Determination of the Fourier Coefficients of the Fourier Series Expansion of the CWE

The Fourier components of $F\left[f(A)\right]_2$ in (2.15) and (2.19) are given by the Euler formulas

$$A_0 = \frac{1}{\tau}\int_0^\tau f(A)dt = \frac{1}{\tau}\int_0^\tau \left[D(n-n'\lambda)\sin\left((n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)\right]dt \qquad (2.20)$$

$$\begin{aligned} A_\beta &= \frac{1}{\tau}\int_0^\tau f(A)\cos\beta\left((n-n'\lambda)t\right)dt \\ &= \frac{1}{\tau}\int_0^\tau \left\{D\sin\left((n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)\right\}\cos\beta\left((n-n'\lambda)t\right)dt \end{aligned} \qquad (2.21)$$

$$\begin{aligned} B_\beta &= \frac{1}{\tau}\int_0^\tau f(A)\sin\beta\left((n-n'\lambda)t\right)dt \\ &= \frac{1}{\tau}\int_0^\tau \left\{D(n-n'\lambda)\sin\left((n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)\right\}\sin\beta\left((n-n'\lambda)t\right)dt \end{aligned} \qquad (2.22)$$

$$A_0 = -\frac{D}{\tau}\left\{\cos\left((n-n'\lambda)\tau-(\varepsilon-\varepsilon'\lambda)\right)-\cos\left(-(\varepsilon-\varepsilon'\lambda)\right)\right\} \qquad (2.23)$$

$$A_0 = \frac{(a-b\lambda)^2(n-n'\lambda)}{2\pi\sqrt{(a^2-b^2\lambda^2)}}\left\{ \cos\left(2\pi-(\varepsilon-\varepsilon'\lambda)\right)-\cos\left((\varepsilon-\varepsilon'\lambda)\right)\right\} \tag{2.24}$$

$$A_0 = -\frac{(a-b\lambda)^2(n-n'\lambda)\left(\sin(\pi)\sin\left(\pi-(\varepsilon-\varepsilon'\lambda)\right)\right)}{\pi\sqrt{(a^2-b^2\lambda^2)}} \tag{2.25}$$

$$C_0 = \frac{(a-b\lambda)^2(n-n'\lambda)\left(\sin(\pi)\sin\left(\pi-(\varepsilon-\varepsilon'\lambda)\right)\right)}{\pi\sqrt{(a^2-b^2\lambda^2)}\ \sin(\varepsilon-\varepsilon'\lambda)} \tag{2.26}$$

In which case $\tau(n-n'\lambda)=2\pi$ is the period of the CWE. Thus (2.26) gives the dimension of C_0 as radian per second (rad./s) or m/s. However, we are going to adopt m/s in this study. Please see the appendix for the identities we have used to get these results.

$$\begin{aligned} A_\beta = \frac{D(n-n'\lambda)}{2\tau}\{&\int_0^\tau \sin\left((1+\beta)(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)dt \\ &+\int_0^\tau \sin\left((1-\beta)(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)dt\} \end{aligned} \tag{2.27}$$

$$\begin{aligned} A_\beta = &-\frac{D(n-n'\lambda)}{2\tau}\{\frac{\cos\left((1+\beta)(n-n'\lambda)\tau-(\varepsilon-\varepsilon'\lambda)\right)-\cos\left(-(\varepsilon-\varepsilon'\lambda)\right)}{(1+\beta)(n-n'\lambda)}\} \\ &-\frac{D(n-n'\lambda)}{2\tau}\{\frac{\cos\left((1-\beta)(n-n'\lambda)\tau-(\varepsilon-\varepsilon'\lambda)\right)-\cos\left(-(\varepsilon-\varepsilon'\lambda)\right)}{(1-\beta)(n-n'\lambda)}\} \end{aligned} \tag{2.28}$$

The second term on the right side of (2.28) is ignored since if $\beta=1$, then according to the summation rule the expression in the parenthesis is infinite and it will not produce intended result. Hence

$$A_\beta = -\frac{(a-b\lambda)^2(n-n'\lambda)}{4\pi\sqrt{(a^2-b^2\lambda^2)}}\{\frac{\cos\left(2(1+\beta)\pi-(\varepsilon-\varepsilon'\lambda)\right)-\cos\left((\varepsilon-\varepsilon'\lambda)\right)}{(1+\beta)}\} \tag{2.29}$$

$$A_\beta = \frac{(a-b\lambda)^2(n-n'\lambda)}{2\pi\sqrt{(a^2-b^2\lambda^2)}}\{\frac{\sin\left((1+\beta)\pi\right)\sin\left((1+\beta)\pi-(\varepsilon-\varepsilon'\lambda)\right)}{(1+\beta)}\} \tag{2.30}$$

By following the same arithmetic procedure that led to (2.30) we then have for B_β that

$$B_\beta = \frac{D(n-n'\lambda)}{2\tau}\{\int_0^\tau \cos\big((1-\beta)(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\big)dt - \int_0^\tau \cos\big((1+\beta)(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\big)dt\} \tag{2.31}$$

$$B_\beta = \frac{D(n-n'\lambda)}{2\tau}\{\frac{\sin\big((1-\beta)(n-n'\lambda)\tau-(\varepsilon-\varepsilon'\lambda)\big)-\sin\big(-(\varepsilon-\varepsilon'\lambda)\big)}{(1-\beta)(n-n'\lambda)}\} - \frac{D(n-n'\lambda)}{2\tau}\{\frac{\sin\big((1+\beta)(n-n'\lambda)\tau-(\varepsilon-\varepsilon'\lambda)\big)-\sin\big(-(\varepsilon-\varepsilon'\lambda)\big)}{(1+\beta)(n-n'\lambda)}\} \tag{2.32}$$

$$B_\beta = -\frac{(a-b\lambda)^2(n-n'\lambda)}{4\pi\sqrt{(a^2-b^2\lambda^2)}}\{\frac{\sin\big(2(1+\beta)\pi-(\varepsilon-\varepsilon'\lambda)\big)+\sin\big((\varepsilon-\varepsilon'\lambda)\big)}{(1+\beta)}\} \tag{2.33}$$

$$B_\beta = -\frac{(a-b\lambda)^2(n-n'\lambda)}{2\pi\sqrt{(a^2-b^2\lambda^2)}}\{\frac{\sin\big((1+\beta)\pi\big)\cos\big((1+\beta)\pi-(\varepsilon-\varepsilon'\lambda)\big)}{(1+\beta)}\} \tag{2.34}$$

Upon the substitution of (2.30) and (2.34) into (2.19) we realize

$$F\left[f(A)\right]_2 = \frac{(a-b\lambda)^2(n-n'\lambda)\big(\sin(\pi)\sin\big(\pi-(\varepsilon-\varepsilon'\lambda)\big)\big)}{\pi\sqrt{(a^2-b^2\lambda^2)}\ \sin(\varepsilon-\varepsilon'\lambda)} + + \sum_{\beta=1}^{\infty}\frac{(a-b\lambda)^2(n-n'\lambda)}{2\pi\sqrt{(a^2-b^2\lambda^2)}}\frac{\sin\big((1+\beta)\pi\big)\sin\big((1+\beta)\pi-(\varepsilon-\varepsilon'\lambda)\big)}{(1+\beta)}\cos\beta\big((n-n'\lambda)t\big) - \sum_{\beta=1}^{\infty}\frac{(a-b\lambda)^2(n-n'\lambda)}{2\pi\sqrt{(a^2-b^2\lambda^2)}}\frac{\sin\big((1+\beta)\pi\big)\sin\big((1+\beta)\pi-(\varepsilon-\varepsilon'\lambda)\big)}{(1+\beta)}\sin\beta\big((n-n'\lambda)t\big)$$

$$F\left[f(A)\right]_2 = \frac{(a-b\lambda)^2(n-n'\lambda)\big(\sin(\pi)\sin\big(\pi-(\varepsilon-\varepsilon'\lambda)\big)\big)}{\pi\sqrt{(a^2-b^2\lambda^2)}\ \sin(\varepsilon-\varepsilon'\lambda)} + + \frac{(a-b\lambda)^2(n-n'\lambda)}{2\pi\sqrt{(a^2-b^2\lambda^2)}}\sum_{\beta=1}^{\infty}\frac{\sin(1+\beta)\pi}{(1+\beta)}\times \times\{\sin\big((1+\beta)\pi-(\varepsilon-\varepsilon'\lambda)\big)\cos\beta\big((n-n'\lambda)t\big) - -\cos\big((1+\beta)\pi-(\varepsilon-\varepsilon'\lambda)\big)\sin\beta\big((n-n'\lambda)t\big)\} \tag{2.35}$$

Since (2.35) represents the velocity of the oscillating amplitude of the CWE when the oscillating phase is assumed equal to one, it is also the equation of the maximum velocity v_m of the oscillating amplitude. That is,

$$v_m = F\left[f(A)\right]_2$$
$$= \frac{(a-b\lambda)^2 (n-n'\lambda)\sin(\pi)\sin\left(\pi-(\varepsilon-\varepsilon'\lambda)\right)}{\pi\sqrt{(a^2-b^2\lambda^2)}\,\sin(\varepsilon-\varepsilon'\lambda)} + \tag{2.36}$$
$$+\frac{(a-b\lambda)^2(n-n'\lambda)}{2\pi\sqrt{(a^2-b^2\lambda^2)}}\sum_{\beta=1}^{\infty}\frac{\sin(1+\beta)\pi}{(1+\beta)}\sin\left((1+\beta)\pi-\beta(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)$$

Thus (2.36) represents the Fourier transform of the maximum velocity of the oscillating amplitude of the CWE.

However, in the absence of the .parasitic wave' in which case $\lambda = 0$, the maximum velocity v_m becomes

$$v_m = F\left[f(A)\right]_3$$
$$= \frac{a\,n\sin(\pi)\sin(\pi-\varepsilon)}{\pi\,\sin(\varepsilon)} + \frac{an}{2\pi}\sum_{\beta=1}^{\infty}\frac{\sin(1+\beta)\pi}{(1+\beta)}\sin\left((1+\beta)\pi-\beta nt-\varepsilon\right) \tag{2.37}$$

2.5 Minimization of the Oscillating Amplitude *f*(*A*) of the CWE with respect to the Phase Angle

The motivation for the minimization of the oscillating amplitude $f(A)$ of the CWE with respect to the phase angle is that we want to make the oscillating amplitude invariant with respect to dimension, that is, the oscillating amplitude of the displacement vector y shall still retain its initial dimension as length. Now

$$y = f(A) = \sqrt{(a^2-b^2\lambda^2)}\sqrt{1-\frac{2\left(a-b\lambda\right)^2}{(a^2-b^2\lambda^2)}\cos\left((n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)} \tag{2.38}$$

$$y = f(A)$$
$$= \sqrt{(a^2-b^2\lambda^2)}\,\frac{d}{d\varphi}\{1-\frac{\left(a-b\lambda\right)^2}{(a^2-b^2\lambda^2)}\cos\left((n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)+\ldots\} \tag{2.39}$$

$$y = f(A) = \sqrt{(a^2-b^2\lambda^2)}\{-\frac{\left(a-b\lambda\right)^2}{(a^2-b^2\lambda^2)}\sin\left((n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)+\ldots\} \tag{2.40}$$

$$y = f(A)_4 = \left\{ D_2 \sin\left((n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda)\right) \right\} \tag{2.41}$$

where for clarity of purpose we have set; $D_2 = -(a - b\lambda)^2 / \sqrt{(a^2 - b^2\lambda^2)}$.

2.6 Fourier Series Expansion of the Oscillating Amplitude *y* = *f*(*A*)₄ of the CWE

We can now expand (2.41) in terms of Fourier series.

$$\begin{aligned} F\left[f(A)\right]_4 = C_0 &+ C_1\{\sin\left((n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda)\right)\} \\ &+ C_2\{\sin\left(2(n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda)\right)\} \\ &+ C_3\{\sin\left(3(n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda)\right)\} \\ &+ \ldots + C_\beta\{\sin\left(\beta(n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda)\right)\} \end{aligned} \tag{2.42}$$

$$F\left[f(A)\right]_5 = C_0 + \sum_{\beta=1}^{\infty} C_\beta \left\{\sin\left(\beta(n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda)\right)\right\} \tag{2.43}$$

Accordingly, it is not always convenient to specify amplitude and phase, we can express each term in the form

$$\begin{aligned} &C_\beta \left\{\sin\left(\beta(n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda)\right)\right\} \\ &\quad = A_\beta \cos\beta\left((n - n'\lambda)t\right) + B_\beta \sin\beta\left((n - n'\lambda)t\right) \end{aligned} \tag{2.44}$$

$$\left.\begin{aligned} A_\beta &= C_\beta \cos(\varepsilon - \varepsilon'\lambda) \\ B_\beta &= -C_\beta \sin(\varepsilon - \varepsilon'\lambda) \end{aligned}\right\} \Rightarrow C_\beta = \sqrt{A_\beta^2 + B_\beta^2} \tag{2.45}$$

From (2.44); if $\beta = 0$, then

$$C_0 \left\{\sin\left(-(\varepsilon - \varepsilon'\lambda)\right)\right\} = A_0 \Rightarrow C_0 = -\frac{A_0}{\sin(\varepsilon - \varepsilon'\lambda)} \tag{2.46}$$

Thus the series expansion given by (2.43) can be rewritten using (2.44) and (2.46) as

$$F\left[f(A)\right]_6 = C_0 + \sum_{\beta=1}^{\infty} \{A_\beta \cos\beta\left((n - n'\lambda)t\right) + B_\beta \sin\beta\left((n - n'\lambda)t\right)\} \tag{2.47}$$

The Fourier components of $F\left[f(A)\right]_5$ in (2.47) are given by the Euler formulas

$$A_0 = \frac{1}{\tau}\int_0^{\tau} f(A)_3 dt = \frac{1}{\tau}\int_0^{\tau}\{D\sin\left((n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)\}dt \tag{2.48}$$

$$\begin{aligned} A_\beta &= \frac{1}{\tau}\int_0^{\tau} f(A)_3 \cos\beta\left((n-n'\lambda)t\right)dt \\ &= \frac{1}{\tau}\int_0^{\tau}\{D\sin\left((n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)\}\cos\beta\left((n-n'\lambda)t\right)dt \end{aligned} \tag{2.49}$$

$$\begin{aligned} B_\beta &= \frac{1}{\tau}\int_0^{\tau} f(A)_3 \sin\beta\left((n-n'\lambda)t\right)dt \\ &= \frac{1}{\tau}\int_0^{\tau}\{D\sin\left((n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)\}\sin\beta\left((n-n'\lambda)t\right)dt \end{aligned} \tag{2.50}$$

$$A_0 = -\frac{D}{\tau(n-n'\lambda)}\left\{\cos\left((n-n'\lambda)\tau-(\varepsilon-\varepsilon'\lambda)\right)-\cos\left(-(\varepsilon-\varepsilon'\lambda)\right)\right\} \tag{2.51}$$

$$A_0 = \frac{(a-b\lambda)^2}{2\pi\sqrt{(a^2-b^2\lambda^2)}}\left\{\cos\left(2\pi-(\varepsilon-\varepsilon'\lambda)\right)-\cos\left((\varepsilon-\varepsilon'\lambda)\right)\right\} \tag{2.52}$$

$$A_0 = -\frac{(a-b\lambda)^2\left(\sin(\pi)\sin\left(\pi-(\varepsilon-\varepsilon'\lambda)\right)\right)}{\pi\sqrt{(a^2-b^2\lambda^2)}} \tag{2.53}$$

$$C_0 = \frac{(a-b\lambda)^2\left(\sin(\pi)\sin\left(\pi-(\varepsilon-\varepsilon'\lambda)\right)\right)}{\pi\sqrt{(a^2-b^2\lambda^2)}\,\sin(\varepsilon-\varepsilon'\lambda)} \tag{2.54}$$

Hence, C_0 has the dimension of meters (m). Also when we substitute (2.41) into (2.49), we have

$$\begin{aligned} A_\beta &= \frac{D}{2\tau}\int_0^{\tau}\sin\left((1+\beta)(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)dt \\ &+ \frac{D}{2\tau}\int_0^{\tau}\sin\left((1-\beta)(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)dt \end{aligned} \tag{2.55}$$

$$\begin{aligned} A_\beta &= -\frac{D}{2\tau}\left\{\frac{\cos\left((1+\beta)(n-n'\lambda)\tau-(\varepsilon-\varepsilon'\lambda)\right)-\cos\left(-(\varepsilon-\varepsilon'\lambda)\right)}{(1+\beta)(n-n'\lambda)}\right\} - \\ &-\frac{D}{2\tau}\left\{\frac{\cos\left((1-\beta)(n-n'\lambda)\tau-(\varepsilon-\varepsilon'\lambda)\right)-\cos\left(-(\varepsilon-\varepsilon'\lambda)\right)}{(1-\beta)(n-n'\lambda)}\right\} \end{aligned} \tag{2.56}$$

The second term on the right side of (2.56) is ignored since if $\beta = 1$ according to the summation rule the expression in the parenthesis is infinite and will not be useful. Hence

$$A_\beta = \frac{(a-b\lambda)^2}{4\pi\sqrt{(a^2-b^2\lambda^2)}}\left\{\frac{\cos\left(2(1+\beta)\pi-(\varepsilon-\varepsilon'\lambda)\right)-\cos\left((\varepsilon-\varepsilon'\lambda)\right)}{(1+\beta)}\right\} \quad (2.57)$$

$$A_\beta = -\frac{(a-b\lambda)^2}{2\pi\sqrt{(a^2-b^2\lambda^2)}}\left\{\frac{\sin\left((1+\beta)\pi\right)\sin\left((1+\beta)\pi-(\varepsilon-\varepsilon'\lambda)\right)}{(1+\beta)}\right\} \quad (2.58)$$

Finally, after careful substitution and simplification as the steps taken to arrive at (2.58) we have for B_β that

$$\begin{aligned} B_\beta &= \frac{D}{2\tau}\int_0^\tau \cos\left((1-\beta)(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)dt \\ &\quad -\frac{D}{2\tau}\int_0^\tau \cos\left((1+\beta)(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)dt \end{aligned} \quad (2.59)$$

$$B_\beta = -\frac{(a-b\lambda)^2}{4\pi\sqrt{(a^2-b^2\lambda^2)}}\left\{\frac{\sin\left(2(1+\beta)\pi-(\varepsilon-\varepsilon'\lambda)\right)+\sin\left((\varepsilon-\varepsilon'\lambda)\right)}{(1+\beta)}\right\} \quad (2.60)$$

$$B_\beta = -\frac{(a-b\lambda)^2}{2\pi\sqrt{(a^2-b^2\lambda^2)}}\left\{\frac{\sin\left((1+\beta)\pi\right)\cos\left((1+\beta)\pi-(\varepsilon-\varepsilon'\lambda)\right)}{(1+\beta)}\right\} \quad (2.61)$$

Also the substitution of (2.54), (2.58) and (2.61) into (2.47) we realize

$$\begin{aligned} y_m = F\left[f(A)\right]_6 &= \frac{(a-b\lambda)^2\sin(\pi)\sin\left(\pi-(\varepsilon-\varepsilon'\lambda)\right)}{\pi\sqrt{(a^2-b^2\lambda^2)}\ \sin(\varepsilon-\varepsilon'\lambda)} - \\ &\quad -\frac{(a-b\lambda)^2}{2\pi\sqrt{(a^2-b^2\lambda^2)}}\sum_{\beta=1}^{\infty}\frac{\sin\left((1+\beta)\pi\right)\sin\left((1+\beta)\pi-(\varepsilon-\varepsilon'\lambda)\right)}{(1+\beta)}\times \\ &\quad \times\cos\left(\beta(n-n'\lambda)t\right) \\ &\quad -\frac{(a-b\lambda)^2}{2\pi\sqrt{(a^2-b^2\lambda^2)}}\sum_{\beta=1}^{\infty}\frac{\sin\left((1+\beta)\pi\right)\cos\left((1+\beta)\pi-(\varepsilon-\varepsilon'\lambda)\right)}{(1+\beta)}\times \\ &\quad \times\sin\left(\beta(n-n'\lambda)t\right) \end{aligned} \quad (2.62)$$

However, since (2.62) represents the stationary oscillating amplitude when the oscillating phase is assumed equal to one, then it is also the equation of the maximum displacement y_m of the oscillating amplitude of the CWE. Hence, we can write after simplification that

$$y_m = F\left[f(A)\right]_6$$
$$= \frac{(a-b\lambda)^2 \sin(\pi)\sin\left(\pi-(\varepsilon-\varepsilon'\lambda)\right)}{\pi\sqrt{(a^2-b^2\lambda^2)}\ \sin(\varepsilon-\varepsilon'\lambda)} \tag{2.63}$$
$$-\frac{(a-b\lambda)^2}{2\pi\sqrt{(a^2-b^2\lambda^2)}}\sum_{\beta=1}^{\infty}\frac{\sin(1+\beta)\pi}{(1+\beta)}\sin\left(\ (1+\beta)\pi-\beta(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right).$$

Generally, $y_m = F\left[f(A)\right]_6$ has the space dimension which is meter m and it is slightly different from (2.36). However, in the absence of the .parasitic wave' in which case $\lambda = 0$ the maximum displacement y_m becomes

$$y_m = F\left[f(A)\right]_7$$
$$= \frac{a\sin(\pi)\sin(\pi-\varepsilon)}{\pi\ \sin(\varepsilon)} - \frac{a}{2\pi}\sum_{\beta=1}^{\infty}\frac{\sin(1+\beta)\pi}{(1+\beta)}\sin\left(\ (1+\beta)\pi-\beta\, n\, t-\varepsilon\right) \tag{2.64}$$

2.7 Fourier Series Expansion of the Spatial Oscillating Phase *f*(*θ*) of the CWE

Now the Fourier series expansion of the spatial oscillating phase given by (2.9) as a function of time is given by

$$F\left[f(\theta)\right] = C_0 + C_1\cos\left(\vec{k}.x-\left((n-n'\lambda)t-E\right)\right)+$$
$$+C_2\cos\left(\ \vec{k}.x-2\left((n-n'\lambda)t-E\right)\right)+C_3\cos\left(\ \vec{k}.x-3\left((n-n'\lambda)t-E\right)\right)$$
$$+\ldots+C_\beta\cos\left(\ \vec{k}.x-\beta\left((n-n'\lambda)t-E\right)\right) \tag{2.65}$$

$$F\left[f(\theta)\right]_1 = C_0 + \sum_{\beta=1}^{\infty} C_\beta\cos\left(\ \vec{k}.x-\beta\left((n-n'\lambda)t-E\right)\right) \tag{2.66}$$

However, there is need to separate the function in the summation sign into two components, space and time.

$$C_\beta \cos\left(\vec{k}.x - \beta\left((n-n'\lambda)t - E\right)\right) = A_\beta \cos\left(\beta\left((n-n'\lambda)t-E\right)\right) + B_\beta \sin\left(\beta\left((n-n'\lambda)t-E\right)\right) \tag{2.67}$$

With assumption that

$$\left.\begin{aligned} A_\beta &= C_\beta \cos\left(\vec{k}\, x\right) \\ B_\beta &= -C_\beta \sin\left(\vec{k}\, x\right) \end{aligned}\right\} \Rightarrow C_\beta = \sqrt{A_\beta^2 + B_\beta^2} \tag{2.68}$$

From (2.67); if $\beta = 0$, then

$$C_0 = \frac{1}{\cos\left(\vec{k}_c\, x\right)} A_0 \tag{2.69}$$

Consequently the series given by (2.66) can be rewritten more completely because of the additional modification, that is, after the substitution of (2.67) and (2.69) into (2.66) as

$$F\left[f(\theta)\right]_2 = \frac{1}{\cos\left(\vec{k}_c\, x\right)} A_0 + \sum_{\beta=1}^{\infty}\left\{A_\beta \cos\left(\beta\left((n-n'\lambda)t-E\right)\right) + B_\beta \sin\left(\beta\left((n-n'\lambda)t-E\right)\right)\right\} \tag{2.70}$$

Usually, the Fourier components of $F\left[f(\theta)\right]_2$ in (2.70) are given by the Euler formulas

$$A_0 = \frac{1}{\tau}\int_0^\tau f(\theta)dt = \frac{1}{\tau}\int_0^\tau \cos\left(\vec{k}_c\, x - (n-n'\lambda)t - E\right)dt \tag{2.71}$$

$$\begin{aligned} A_\beta &= \frac{1}{\tau}\int_0^\tau f(\theta)\cos\left(\beta\left((n-n'\lambda)t - E(t)\right)\right)dt \\ &= \frac{1}{\tau}\int_0^\tau \cos\left(\vec{k}_c x - (n-n'\lambda)t - E(t)\right)\cos\left(\beta\left((n-n'\lambda)t - E(t)\right)\right)dt \end{aligned} \tag{2.72}$$

$$\begin{aligned} B_\beta &= \frac{1}{\tau}\int_0^\tau f(\theta)\sin\left(\beta\left((n-n'\lambda)t - E(t)\right)\right)dt \\ &= \frac{1}{\tau}\int_0^\tau \sin\left(\vec{k}_c x - (n-n'\lambda)t - E(t)\right)\sin\left(\beta\left((n-n'\lambda)t - E(t)\right)\right)dt \end{aligned} \tag{2.73}$$

$$A_0 = -\frac{1}{\tau}\{\frac{1}{\left((n-n'\lambda)-Z(\tau)\right)}\sin(\vec{k}_c\, x-(n-n'\lambda)\,\tau-E(\tau))- \\ -\frac{1}{\left((n-n'\lambda)-Z(0)\right)}\sin(\vec{k}_c x-E(0))\} \tag{2.74}$$

$$A_0 = \frac{(n-n'\lambda)}{2\pi}\{\frac{\sin(\vec{k}_c x-E(0))}{\left((n-n'\lambda)-Z(0)\right)}-\frac{\sin(\vec{k}_c\, x-(n-n'\lambda)\,\tau-E(\tau))}{\left((n-n'\lambda)-Z(\tau)\right)}\} \tag{2.75}$$

$$C_0 = \frac{(n-n'\lambda)}{2\pi\cos(\vec{k}_c x)}\{\frac{\sin(\vec{k}_c x-E(0))}{\left((n-n'\lambda)-Z(0)\right)}-\frac{\sin(\vec{k}_c x-(n-n'\lambda)\,\tau-E(\tau))}{\left((n-n'\lambda)-Z(\tau)\right)}\} \tag{2.76}$$

$$A_\beta = \frac{1}{2\tau}\{\int_0^\tau \cos(\vec{k}_c x-(1-\beta)((n-n'\lambda)\,t-E))dt \\ +\int_0^\tau \cos(\vec{k}_c x-(1+\beta)((n-n'\lambda)\,t-E))dt\} \tag{2.77}$$

$$A_\beta = \frac{1}{2\tau}\{-\frac{\sin(\vec{k}_c x-(1-\beta)\left((n-n'\lambda)\,\tau-E(\tau)\right))}{(1-\beta)\left((n-n'\lambda)-Z(\tau)\right)}-\frac{\sin(\vec{k}_c x-E(0))}{(1-\beta)\left((n-n'\lambda)-Z(0)\right)}\} \\ +\frac{1}{2\tau}\{-\frac{\sin(\vec{k}_c x-(1+\beta)\left((n-n'\lambda)\,\tau-E(\tau)\right))}{(1+\beta)\left((n-n'\lambda)-Z(\tau)\right)}-\frac{\sin(\vec{k}_c x-E(0))}{(1+\beta)\left((n-n'\lambda)-Z(0)\right)}\} \; (2.78)$$

(2.78)

The first term on the right side of (2.78) is ignored since it becomes infinite if $\beta = 1$. As a result,

$$A_\beta = \frac{(n-n'\lambda)}{4\pi}\{\frac{\sin(\vec{k}_c x-E(0))}{(1+\beta)\left((n-n'\lambda)-Z(0)\right)}-\frac{\sin(\vec{k}_c x-(1+\beta)\left(2\pi-E(\tau)\right))}{(1+\beta)\left((n-n'\lambda)-Z(\tau)\right)}\} \tag{2.79}$$

By following the same procedure that led to (2.79) we obtain the equation for B_β from (2.73) as

$$B_\beta = \frac{1}{2\tau}\{\int_0^\tau \sin(\vec{k}_c x-(1-\beta)(n-n'\lambda)\,t-E)dt \\ -\int_0^\tau \sin(\vec{k}_c x-(1+\beta)(n-n'\lambda)\,t-E)dt\} \tag{2.80}$$

$$B_\beta = \frac{(n-n'\lambda)}{4\pi}\{\frac{\cos(\vec{k}_c x-E(0))}{(1+\beta)\left((n-n'\lambda)-Z(0)\right)}-\frac{\cos(\vec{k}_c x-(1+\beta)\left(2\pi-E(\tau)\right))}{(1+\beta)\left((n-n'\lambda)-Z(\tau)\right)}\} \tag{2.81}$$

Now upon the substitution of (2.76), (2.79) and (2.81) into (2.70) we get

$$
\begin{aligned}
F\left[f(\theta)\right]_2 &= \frac{(n-n'\lambda)}{2\pi\cos((k-k'\lambda)x)}\times \\
&\times\{\frac{\sin((k-k'\lambda)x-E(0))}{\left((n-n'\lambda)-Z(0)\right)}-\frac{\sin((k-k'\lambda)x-2\pi-E(\tau))}{\left((n-n'\lambda)-Z(\tau)\right)}\}+ \\
&+\sum_{\beta=1}^{\infty}\frac{(n-n'\lambda)}{4\pi}\{\frac{\sin((k-k'\lambda)x-E(0))}{(1+\beta)\left((n-n'\lambda)-Z(0)\right)}- \\
&-\frac{\sin((k-k'\lambda)x-(1+\beta)\left(2\pi-E(\tau)\right))}{(1+\beta)\left((n-n'\lambda)-Z(\tau)\right)}\}\times \\
&\times\cos(\beta((n-n'\lambda)t-E(t))) \\
&+\sum_{\beta=1}^{\infty}\frac{(n-n'\lambda)}{4\pi}\{\frac{\cos((k-k'\lambda)x-E(0))}{(1+\beta)\left((n-n'\lambda)-Z(0)\right)}- \\
&-\frac{\cos((k-k'\lambda)x-(1+\beta)\left(2\pi-E(\tau)\right))}{(1+\beta)\left((n-n'\lambda)-Z(\tau)\right)}\}\times \\
&\times\sin(\beta((n-n'\lambda)t-E(t)))
\end{aligned}
\tag{2.82}
$$

$$
\begin{aligned}
F\left[f(\theta)\right]_2 &= \frac{(n-n'\lambda)}{2\pi\cos((k-k'\lambda)x)}\times \\
&\times\{\frac{\sin((k-k'\lambda)x-E(0))}{\left((n-n'\lambda)-Z(0)\right)}-\frac{\sin((k-k'\lambda)x-2\pi-E(\tau))}{\left((n-n'\lambda)-Z(\tau)\right)}\}+ \\
&+\frac{(n-n'\lambda)}{4\pi\left((n-n'\lambda)-Z(0)\right)\left((n-n'\lambda)-Z(\tau)\right)}\sum_{\beta=1}^{\infty}\frac{1}{1+\beta}\times \\
&\times\{\left((n-n'\lambda)-Z(\tau)\right)\sin((k-k'\lambda)x+\beta\left((n-n'\lambda)t-E(t)\right)-E(0))- \\
&-\left((n-n'\lambda)-Z(0)\right)\times \\
&\times\sin((k-k'\lambda)x+\beta\left((n-n'\lambda)t-E(t)\right)-(1+\beta)\left(2\pi-E(\tau)\right))\}
\end{aligned}
\tag{2.83}
$$

Thus (2.83) represents the Fourier transform of the spatial oscillating phase and it is dimensionless. The first term represents the fundamental spatial oscillating phase, while the rest terms are the overtones. It should be noted that we converged the results of both the cosine (even) and the sine (odd) functions of (2.82) instead of applying them separately.

2.8 Convolution Theory of the Fourier Transform of the Velocity of the Oscillating Amplitude *F*[*f*(*A*)]$_2$ and the Spatial Oscillating Phase *F*[*f*(*θ*)]$_2$ of the CWE

Now that we have separately determined the Fourier transform of the oscillating amplitude $F\left[f(A)\right]_2$ and the spatial oscillating phase $F\left[f(\theta)\right]_2$, the necessary requirement now is to convolute them in order to obtain a concise equation of the velocity of the CWE. Convolution here means multiplying (2.36) by (2.83) term by term. Let us represent the result of the convolution of these functions by H and then with the same velocity vector v which represents the velocity of the CWE.

$$v = H\left\{F\left[f(A)\right]_2 ; F\left[f(\theta)\right]_2\right\} \equiv F\left[f(A)\right]_2 \otimes F\left[f(\theta)\right]_2 \tag{2.84}$$

$$\begin{aligned}
v = {} & \frac{(a-b\lambda)^2(n-n'\lambda)^2 \sin(\pi)\sin\left(\pi-(\varepsilon-\varepsilon'\lambda)\right)}{2\pi^2\sqrt{(a^2-b^2\lambda^2)}\ \cos\left(k-k'\lambda)x\right)\sin(\varepsilon-\varepsilon'\lambda)}\times \\
& \times\{\frac{\sin\left((k-k'\lambda)x-E(0)\right)}{\left((n-n'\lambda)-Z(0)\right)}-\frac{\sin\left((k-k'\lambda)x-2\pi-E(\tau)\right)}{\left((n-n'\lambda)-Z(\tau)\right)}\}+ \\
& +\frac{(a-b\lambda)^2(n-n'\lambda)^2 \sin(\pi)\sin\left(\pi-(\varepsilon-\varepsilon'\lambda)\right)}{4\pi^2\sqrt{(a^2-b^2\lambda^2)}\ \sin(\varepsilon-\varepsilon'\lambda)\left((n-n'\lambda)-Z(0)\right)\left((n-n'\lambda)-Z(\tau)\right)}\times \\
& \times\sum_{\beta=1}^{\infty}\frac{1}{1+\beta}\times \\
& \times\Big\{\left((n-n'\lambda)-Z(\tau)\right)\sin\left((k-k'\lambda)x+\beta\left((n-n'\lambda)t-E(t)\right)-E(0)\right)- \\
& -\left((n-n'\lambda)-Z(0)\right)\times \\
& \times\sin\left((k-k'\lambda)x+\beta\left((n-n'\lambda)t-E(t)\right)-(1+\beta)\left(2\pi-E(\tau)\right)\right)\Big\}+ \\
& +\frac{(a-b\lambda)^2(n-n'\lambda)^2}{4\pi^2\sqrt{(a^2-b^2\lambda^2)}\cos\left((k-k'\lambda)x\right)}\sum_{\beta=1}^{\infty}\frac{\sin(1+\beta)\pi}{(1+\beta)}\times \\
& \times\{\frac{\sin\left((k-k'\lambda)x-E(0)\right)}{\left((n-n'\lambda)-Z(0)\right)}-\frac{\sin\left((k-k'\lambda)x-2\pi-E(\tau)\right)}{\left((n-n'\lambda)-Z(\tau)\right)}\} \\
& \times\sin\left((1+\beta)\pi-\beta(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)+
\end{aligned}$$

$$+\frac{(a-b\lambda)^2(n-n'\lambda)^2}{8\pi^2\sqrt{(a^2-b^2\lambda^2)}\big((n-n'\lambda)-Z(0)\big)\big((n-n'\lambda)-Z(\tau)\big)}\times$$
$$\times\sum_{\beta=1}^{\infty}\frac{\sin(1+\beta)\pi}{(1+\beta)^2}\sin\big(\,(1+\beta)\pi-\beta(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\big)\times$$
$$\times\Big\{\big((n-n'\lambda)-Z(\tau)\big)\sin\Big((k-k'\lambda)x+\beta\big((n-n'\lambda)t-E(t)\big)-E(0)\Big)-$$
$$-\big((n-n'\lambda)-Z(0)\big)\times$$
$$\times\sin\Big((k-k'\lambda)x+\beta\big((n-n'\lambda)t-E(t)\big)-(1+\beta)\big(2\pi-E(\tau)\big)\Big)\Big\} \qquad (2.85)$$

Thus from (2.6) and (2.7) we realize

$$E(\tau)=\tan^{-1}\left(\frac{a\sin\varepsilon+b\lambda\sin(\varepsilon'\lambda-2\pi)}{a\cos\varepsilon+b\lambda\cos(\varepsilon'\lambda-2\pi)}\right);$$
$$E(0)=\tan^{-1}\left(\frac{a\sin\varepsilon+b\lambda\sin(\varepsilon'\lambda)}{a\cos\varepsilon+b\lambda\cos(\varepsilon'\lambda)}\right) \qquad (2.86)$$

$$Z(0)=(n-n'\lambda)\left(\frac{b^2\lambda^2+ab\lambda\cos(\varepsilon-\varepsilon'\lambda)}{a^2+b^2\lambda^2+2ab\lambda\cos(\varepsilon-\varepsilon'\lambda)}\right);$$
$$Z(\tau)=(n-n'\lambda)\left(\frac{b^2\lambda^2+ab\lambda\cos\big((\varepsilon-\varepsilon'\lambda)+2\pi\big)}{a^2+b^2\lambda^2+2ab\lambda\cos\big((\varepsilon-\varepsilon'\lambda)+2\pi\big)}\right) \qquad (2.87)$$

Thus (2.85) represents the Fourier transform of the velocity attained by the CWE since it is the result of the multiplication of the maximum velocity of the oscillating amplitude and the spatial oscillating phase. The first term on the right hand side represents the fundamental velocity while the rest term represents the available overtones of the velocity. In the absence of 'parasitic wave', that is when $\lambda=0$, we realize the below equation.

$$v=\frac{an^2\sin(\pi)\sin(\pi-\varepsilon)}{2\pi^2\,\cos(kx)\sin(\varepsilon)}\times\Big\{\frac{\sin(kx-\varepsilon)}{n}-\frac{\sin(kx-2\pi-\varepsilon)}{n}\Big\}+$$
$$+\frac{an\sin(\pi)\sin(\pi-\varepsilon)}{4\pi^2\,\sin(\varepsilon)}\times$$
$$\times\sum_{\beta=1}^{\infty}\frac{1}{1+\beta}\Big\{\sin\big(kx+\beta(nt-\varepsilon)-\varepsilon\big)-\sin\big(kx+\beta(nt-\varepsilon)-(1+\beta)(2\pi-\varepsilon)\big)\Big\}$$

$$+\frac{a\,n^2}{4\pi^2\cos\left(kx\right)}\times$$
$$\times\sum_{\beta=1}^{\infty}\frac{\sin(1+\beta)\pi}{(1+\beta)}\{\frac{\sin\left(k\,x-\varepsilon\right)}{n}-\frac{\sin\left(k\,x-2\pi-\varepsilon\right)}{n}\}\sin\left(\,(1+\beta)\pi-\beta\,(n\,t)-\varepsilon\right)+$$
$$+\frac{a\,n}{8\pi^2}\sum_{\beta=1}^{\infty}\frac{\sin(1+\beta)\pi}{(1+\beta)^2}\sin\left(\,(1+\beta)\pi-\beta\,(n\,t)-\varepsilon\right)\times$$
$$\times\{\sin\left(k\,x+\beta\,(n\,t-\varepsilon)-\varepsilon\right)-\sin\left(k\,x+\beta(n\,t-\varepsilon)-(1+\beta)(2\pi-\varepsilon)\right)\}\qquad(2.88)$$

Consequently, in the absence of ‘parasitic wave’, in which case $\lambda=0$, we realize that the values of the total phase angle and the characteristic angular velocity are; $E(t)=E(\tau)=E(0)=\varepsilon$ and $Z(t)=Z(\tau)=Z(0)=0$.

Note that we have assumed the same constraint for the product of the two convoluting functions. The assumption is possible because the two functions we are convoluting are of the same source and they are not incoherent. We should also observe that the dimension of the velocity v of the CWE after the application of the Fourier transform is m/s and so we are not using rad/s.

2.9 Evaluation of the Energy Attenuation Equation of the Carrier Wave Equation CWE

In natural systems, we can rarely find pure wave which propagates free from energy-loss mechanisms. But if these losses are not too serious we can describe the total propagation in time by a given force law $f(t)$. The propagating CWE in the pipe containing fluid is affected by three major factors: (i) the damping effect of the mass m of the surrounding fluid (ii) the damping effect of the dynamic viscosity of the fluid (η) and (iii) the damping effect of the fluid elastic property (μ). Then the force law equation governing the dissipation of the CWE in the pipe if the fluid is considered to be Newtonian is given by

$$f(t) = m\frac{\partial^2 y}{\partial t^2} + \eta\frac{\partial y^2}{dt} + \mu\, y \tag{2.89}$$

$$f(t) = \rho V\ \frac{\partial^2 y}{\partial t^2} + 2\eta\, y\frac{\partial y}{dt} + \mu\, y \tag{2.90}$$

where $V = \pi r^2 x$ is the volume of fluid in the cylindrical pipe. The notation r and x are the radius and the length of the pipe respectively. In this work, we arbitrarily considered the radius r = 0.04m, length x = 5000m, dynamic viscosity of the fluid $\eta =$ 0.003kg/ms and the fluid elastic property μ = 6 x 10^{-7}kg/s^2. The influence of gravity on the propagation of the carrier wave equation CWE in the pipe is however neglected.

$$f(t) = \rho V\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) + 2\eta y\frac{\partial y}{\partial t} + \mu\, y \tag{2.91}$$

$$\int f(t)dt = \rho V\int\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)dt + 2\eta\, y\int\left(\frac{\partial y}{\partial t}\right)dt + \mu\int y dt \tag{2.92}$$

$$\text{Im}pulse = \rho V\ \left(\frac{\partial y}{\partial t}\right) + 2\eta\, y^2 + \mu\int ydt \tag{2.93}$$

$$\text{Im}pulse = \rho V\ \ v + 2\eta\, y^2 + \mu\int ydt \tag{2.94}$$

We can now multiply through (2.94) by the velocity v in order to convert the unit of impulse to energy which is Joules (J) or (kgm^2/s^2) or (Nm).

$$\text{Im}pulse \times velocity\ \ v = \ \rho V\ \ v^2 + 2\eta\, v\, y^2 + \mu\int v\, ydt \tag{2.95}$$

$$Energy\ \ (E) = \rho V\ \ v^2 + 2\eta\, v\, y^2 + \mu\int\frac{\partial y}{\partial t}\, y\, dt \tag{2.96}$$

$$Energy\ \ (E) = \rho V\, v^2 + 2\eta v y^2 + \mu y^2 \tag{2.97}$$

For the energy to be a maximum then the spatial oscillating part of the CWE must be equal to one and the CWE would only have the oscillating amplitude which will now be maximum also. The velocity of the CWE is also a maximum if the oscillating amplitude is a maximum. Hence

$$Energy\ \ (E) = \rho V\ \ v_m^2 + 2\eta\, v_m\, y_m^2 + \mu\, y_m^2 \tag{2.98}$$

Although, if we are interested in the application of (2.97), which is the equation for the minimum energy of the CWE, then we must first convolute (2.63) and (2.83) before using the result of the convolution with (2.85) in (2.97). We should also note that since the maximum velocity v_m and the maximum displacement y_m of the CWE comprises of both the fundamental and the overtones or the nth harmonics, then the maximum energy E_m must also comprise of two parts; the fundamental, the overtones or the nth harmonics. In this study, we are going to implement (2.98). We shall also investigate the energy spectrum of the propagating CWE for both the presence and in the absence of the 'parasitic wave' for maximum value of the Fourier index $\beta = 505$. However, in the avoidance of error and for clarity of purpose we shall state v_m^2 and y_m^2 clearly from (2.36) and (2.63) respectively. Thus we have

$$\begin{aligned} v_m^2 = F^2\left[f(A)\right]_2 &= \left(\frac{(a-b\lambda)^2(n-n'\lambda)\sin(\pi)\sin\left(\pi-(\varepsilon-\varepsilon'\lambda)\right)}{\pi\sqrt{(a^2-b^2\lambda^2)}\,\sin(\varepsilon-\varepsilon'\lambda)}\right)^2 \\ &+\frac{(a-b\lambda)^4(n-n'\lambda)^2\sin(\pi)\sin\left(\pi-(\varepsilon-\varepsilon'\lambda)\right)}{\pi^2(a^2-b^2\lambda^2)\,\sin(\varepsilon-\varepsilon'\lambda)} \\ &\times\sum_{\beta=1}^{\infty}\frac{\sin(1+\beta)\pi}{(1+\beta)}\sin\left((1+\beta)\pi-\beta(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right) \\ &+\frac{(a-b\lambda)^4(n-n'\lambda)^2}{4\pi^2(a^2-b^2\lambda^2)}\sum_{\beta=1}^{\infty}\left(\frac{\sin(1+\beta)\pi}{(1+\beta)}\right)^2\sin^2\left((1+\beta)\pi-\beta(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right). \end{aligned} \tag{2.99}$$

$$\begin{aligned} y_m^2 = F^2\left[f(A)\right]_6 &= \left(\frac{(a-b\lambda)^2\sin(\pi)\sin\left(\pi-(\varepsilon-\varepsilon'\lambda)\right)}{\pi\sqrt{(a^2-b^2\lambda^2)}\,\sin(\varepsilon-\varepsilon'\lambda)}\right)^2 \\ &-\frac{(a-b\lambda)^4\sin(\pi)\sin\left(\pi-(\varepsilon-\varepsilon'\lambda)\right)}{\pi^2(a^2-b^2\lambda^2)\,\sin(\varepsilon-\varepsilon'\lambda)} \\ &\times\sum_{\beta=1}^{\infty}\frac{\sin(1+\beta)\pi}{(1+\beta)}\sin\left((1+\beta)\pi-\beta(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right) \\ &+\frac{(a-b\lambda)^4}{4\pi^2(a^2-b^2\lambda^2)}\sum_{\beta=1}^{\infty}\left(\frac{\sin(1+\beta)\pi}{(1+\beta)}\right)^2\sin^2\left((1+\beta)\pi-\beta(n-n'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right). \end{aligned} \tag{2.100}$$

Hence upon the application of (2.99) and (2.100) in (2.98), then the resulting energy attenuation equation will have two terms as we have noted before. Also in the absence of the 'parasitic wave' in which case $\lambda = 0$, we have from (2.37) and (2.64) that

$$v_m^2 = F^2\left[f(A)\right]_3 = \left(\frac{a\,n\sin(\pi)\sin(\pi-\varepsilon)}{\pi\,\sin(\varepsilon)}\right)^2 + \frac{a^2n^2\sin(\pi)\sin(\pi-\varepsilon)}{\pi^2\,\sin(\varepsilon)} + \sum_{\beta=1}^{\infty}\frac{\sin(1+\beta)\pi}{(1+\beta)}\times\sin\left((1+\beta)\pi-\beta nt-\varepsilon\right) + \left(\frac{an}{2\pi}\right)^2\sum_{\beta=1}^{\infty}\left(\frac{\sin(1+\beta)\pi}{(1+\beta)}\right)^2\sin^2\left((1+\beta)\pi-\beta nt-\varepsilon\right) \quad (2.101)$$

$$y_m^2 = F^2\left[f(A)\right]_7 = \left(\frac{a\sin(\pi)\sin(\pi-\varepsilon)}{\pi\,\sin(\varepsilon)}\right)^2 - \frac{a^2\sin(\pi)\sin(\pi-\varepsilon)}{\pi^2\sin(\varepsilon)}\sum_{\beta=1}^{\infty}\frac{\sin(1+\beta)\pi}{(1+\beta)}\times\sin\left((1+\beta)\pi-\beta nt-\varepsilon\right) + \left(\frac{a}{2\pi}\right)^2\sum_{\beta=1}^{\infty}\left(\frac{\sin(1+\beta)\pi}{(1+\beta)}\right)^2\sin^2\left((1+\beta)\pi-\beta nt-\varepsilon\right) \quad (2.102)$$

2.10 Determination of the 'host wave' parameters (α, n, ε and k)

Let us now discuss the possibility of obtaining the parameters of the 'host wave' which were initially not known from the carrier wave equation CWE. This is a very crucial stage of the study since there was no previous knowledge of the values. Now from (2.8), by using the boundary conditions that at time $t = 0$, $\lambda = 0$ and $A = a$, then

$$A = \sqrt{a^2 - 2a^2\cos(-\varepsilon)} = a\sqrt{1-2\cos(\varepsilon)} \quad (2.103)$$

$$\sqrt{1-2\cos(\varepsilon)} = 1 \Rightarrow \varepsilon = \cos^{-1}(0) = 90^0 = (1.5708\,rad.) \quad (2.104)$$

Any slight variation in the combined amplitude A to $A+\delta A$ of the CWE due to displacement with time t to $t+\delta t$ would invariably produce a negligible effect in the host amplitude a also under this situation $\lambda \approx 0$. Hence we can write

$$\lim_{\delta t\to 0}(A+\frac{\delta A}{\delta t})=a \tag{2.105}$$

$$\lim_{\delta t\to 0}(\sqrt{a^2-2a^2\cos\left(n(t+\delta t)-\varepsilon\right)}+\frac{n a^2\sin\left(n(t+\delta t)-\varepsilon\right)}{\sqrt{a^2-2a^2\cos\left(n(t+\delta t)-\varepsilon\right)}})=a \tag{2.106}$$

$$\sqrt{a^2-2a^2\cos(nt-\varepsilon)}+\frac{n a^2\sin(nt-\varepsilon)}{\sqrt{a^2-2a^2\cos(nt-\varepsilon)}}=a \tag{2.107}$$

$$\left(a^2-2a^2\cos(nt-\varepsilon)\right)+n a^2\sin(nt-\varepsilon)=a\sqrt{a^2-2a^2\cos(nt-\varepsilon)} \tag{2.108}$$

$$1-2\cos(nt-\varepsilon)+n\sin(nt-\varepsilon)=\sqrt{1-2\cos(nt-\varepsilon)} \tag{2.109}$$

At this point of our work, it may not be easy to produce a solution to the problem based on (2.109), this is due to the mixed sinusoidal wave functions. However, to get out of this complication we have implemented the below approximation technique to minimize the right hand side of (2.109). This approximation states that

$$\begin{aligned}&\left(1+\xi\, f(\varphi)\right)^{\pm n}\\ &=\frac{d}{d\varphi}\left(1+n\xi\, f(\varphi)+\frac{n(n-1)}{2!}\left(\xi\, f(\varphi)\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\xi\, f(\varphi)\right)^3+\dots\right)\end{aligned} \tag{2.110}$$

The general background of this approximation is the differentiation of the resulting binomial expansion of a given variable function. This approximation has the advantage of converging functions easily and also it produces intended applicable value of result. Hence (2.109) becomes

$$1-2\cos(nt-\varepsilon)+n\sin(nt-\varepsilon)=n\sin(nt-\varepsilon) \tag{2.111}$$

$$\begin{aligned}&nt-\varepsilon=\cos^{-1}(0.5)=60^0=1.0472\,rad.\Rightarrow nt=2.6182rad.\\ &\qquad\qquad\qquad\qquad\qquad\qquad\Rightarrow\ n=2.6182rad./s\end{aligned} \tag{2.112}$$

From (2.9), using the boundary conditions that for stationary state, that is, when

$\delta t = 0,\ \lambda \approx 0,\ \theta = \pi - (\varepsilon - \varepsilon'\lambda) = \pi - \varepsilon = 3.142 - 1.5708 = 1.5712\, rad,$
$E = \varepsilon = 1.5708\, rad$, then

$$\lim_{\delta t \to 0} \cos\left\{ (k - k'\lambda)\, r \cos\theta + (k - k'\lambda)\, r \sin\theta - (n - n'\lambda)(t + t\,\delta t) - E \right\} = 1 \qquad (2.113)$$

$$\left(k\, r\, (\cos\theta + r \sin\theta) - nt - \varepsilon \right) = 0 \quad \because \quad (\cos^{-1}(1) = 0) \qquad (2.114)$$

$$\left(k\, r\, (0.9996) - 2.6182 - 1.5708 \right) = 0$$

$$\Rightarrow\ k\, r = 4.1907 rad \ \Rightarrow\ k = 4.1907 rad\, /\, m \qquad (2.115)$$

The change in the resultant amplitude of the carrier wave is proportional to the frequency of oscillation of the spatial oscillating phase ϕ multiplied by the product of the variation with time t of the inverse of the oscillating phase with respect to the radial distance x and the variation with time t wave number $(k - k'\lambda)$. This condition would make us to write by using (2.8) and (2.9) that

$$\frac{dA}{dt} = \frac{(n - n'\lambda)(a - b\lambda)^2 \sin\left((n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda) \right)}{\sqrt{(a^2 - b^2\lambda^2) - 2(a - b\lambda)^2 \cos\left((n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda) \right)}} \qquad (2.116)$$

$$\frac{d\varphi}{dr} = -(k - k'\lambda)(\cos\theta + \sin\theta) \sin\left((k - k'\lambda) r\, (\cos\theta + \sin\theta) - (n - n'\lambda)t - E \right) \qquad (2.117)$$

$$\frac{d\varphi}{dt} = \left((n - n'\lambda) + Z \right) \sin\left((k - k'\lambda) r\, (\cos\theta + \sin\theta) - (n - n'\lambda)t - E \right) \qquad (2.118)$$

$$\frac{d\varphi}{d(k - k'\lambda)} = \left(-r(\cos\theta + \sin\theta) - E \right) \sin\left((k - k'\lambda) r\, (\cos\theta + \sin\theta) - (n - n'\lambda)t - E \right) \qquad (2.119)$$

$$\frac{dA}{dt} = \left(\frac{1}{2\pi} \frac{\partial\varphi}{\partial t} \right) \left(\frac{1}{r} \frac{\partial r}{\partial\varphi} \right) \left(\frac{\partial\varphi}{\partial(k - k'\lambda)} \right) = f\, l \qquad (2.120)$$

$$A = f\, l\, t \qquad (2.121)$$

That is the time rate of change of the resultant amplitude is equal to the frequency f of the spatial oscillating phase multiplied by the length l of the arc covered by the oscillating phase. Under this circumstance, we refer to A as the instantaneous amplitude of oscillation.

The first term in the parenthesis of (2.120) is the frequency dependent term, while the combination of the rest two terms in the parenthesis represents the angular length or simply the length of an arc covered by the spatial oscillating phase. Note that the second term in the right hand side of (2.120) is the inverse of (2.117).

With the usual implementation of the boundary conditions that at $t = 0$, $\lambda = 0$, $\theta = \pi - (\varepsilon - \varepsilon'\lambda) = \pi - \varepsilon = 3.142 - 1.5708 = 1.5712\,rad$, $E = \varepsilon = 1.5708\,rad$, $dA/dt = a$, we obtain the expression for the amplitude as

$$a = -\left(\frac{1}{2\pi}\right)\left(\frac{\left((\cos\theta + \sin\theta) - \varepsilon\right)}{k\sin\varepsilon(\cos\theta + \sin\theta)}\right) = 0.0217m \tag{2.122}$$

Note that $\cos(-\varepsilon) = \cos\varepsilon$ (even and symmetric function) and $\sin(-\varepsilon) = -\sin\varepsilon$ (odd and screw symmetric function). Thus generally we have established that the basic constituents parameters of the 'host wave are

$a = 0.0217m$, $n = 2.6182 rad/s$, $\varepsilon = 1.5708 rad$, and $k = 4.1907 rad/m$. (2.123)

2.11 Determination of the 'Parasitic Wave' Parameters (*b*, *n'*, *ε'* and *k'*)

Also we can now determine the basic parameters of the 'parasitic wave' which were initially not known before the interference from the derived values of the 'host wave' using the below method. We know that gradual depletion in the physical parameters of the system under study would mean that after a sufficiently long period of time all the active constituents of the 'host wave' would have been completely attenuated by the destructive influence of the 'parasitic wave'. On the basis of these arguments, we can now write as follows.

$$\left.\begin{aligned} a - b\lambda = 0 &\Rightarrow 0.0217 = b\lambda \\ n - n'\lambda = 0 &\Rightarrow 2.6182 = n'\lambda \\ \varepsilon - \varepsilon'\lambda = 0 &\Rightarrow 1.5708 = \varepsilon'\lambda \\ k - k'\lambda = 0 &\Rightarrow 4.1907 = k'\lambda \end{aligned}\right\} \tag{2.124}$$

Upon dividing the sets of relations in (2.124) with one another with the view to eliminate λ we get

$$\left.\begin{array}{l} 0.008288n' = b \\ 0.013820\varepsilon' = b \\ 0.005178k' = b \\ 1.6668\varepsilon' = n' \\ 0.6248k' = n' \\ 0.3748k' = \varepsilon' \end{array}\right\} \qquad (2.125)$$

However, there are several possible values that each parameter would take according to (2.125). But for a gradual decay process, that is for a slow depletion in the constituents of the host parameters we choose the least values of the 'parasitic wave' parameters. Thus a more realistic and applicable relation is when: $0.008288n' = 0.005178k'$. Based on simple ratio

$$\begin{array}{ll} n' = 0.00518\,rad/s & k' = 0.00829\,rad/m \\ \varepsilon' = 0.00311\,rad & b = 0.0000429\,m \end{array} \qquad (2.126)$$

Any of these values of the constituents of the 'parasitic wave' shall produce a corresponding approximate value of $\lambda = 505$ upon substituting it into (2.124). Hence we get the interval of the multiplier as $0 \le \lambda \le 505$.

2.12 Determination of the Attenuation Constant (η) of the CWE

Attenuation is a decay process caused by absorption and scattering of the medium where a wave is propagating. It brings about a gradual reduction and weakening in the initial strength of the basic parameters of a given physical system which the wave describes. In this study, the parameters are the amplitude (a), phase angle (ε), angular frequency (n) and the spatial frequency (k). The dimension of the attenuation constant (η) is determined by the system under study. However, in this work, attenuation constant is the relative rate of fractional change (FC) in the basic parameters of the carrier wave. There are 4 (four) attenuating parameters present in the CWE. Now, if a,η,ε,k represent the initial basic parameters of the 'host wave' that is present in the carrier wave and $a-b\lambda$, $n-n'\lambda$, $\varepsilon-\varepsilon'\lambda$, $k-k'\lambda$ represent the basic parameters of the 'host wave' that survives after a given time. Then, the FC is

$$\sigma=\frac{1}{4}\times\left(\left(\frac{a-b\lambda}{a}\right)+\left(\frac{\varepsilon-\varepsilon'\lambda}{\varepsilon}\right)+\left(\frac{n-n'\lambda}{n}\right)+\left(\frac{k-k'\lambda}{k}\right)\right) \tag{2.127}$$

$$\eta=\frac{FC|_{\lambda=i}-FC|_{\lambda=i+1}}{unit\,time(s)}=\frac{\sigma_i-\sigma_{i+1}}{unit\;time\,(s)} \tag{2.128}$$

The dimension is *per second* (s^{-1}). Thus (2.128) gives $\eta=0.001978\,\mathrm{s}^{-1}$ for all values of $\lambda\,(i=0,1,2,\ldots,505\,)$.

2.13 Determination of the time (*t*)

We used the information provided in section 2.9, to compute the various times taken for the carrier wave to attenuate to zero. The maximum time the carrier wave lasted as a function of the raising multiplier λ is also calculated from the attenuation equation shown by (2.128). The reader should note that we have adopted a slowly varying regular interval for the raising multiplier since this would help to delineate clearly the physical parameter space accessible to our model. However, it is clear from the calculation that the different attenuating fractional changes contained in the carrier wave are approximately equal to one another. We can now apply the attenuation time equation given below.

$$\sigma=e^{-(2^{\gamma}\,\eta\,t)\,/\,\lambda} \tag{2.129}$$

$$t=-\left(\frac{\lambda}{2^{\gamma}\,\eta}\right)\ln\sigma \tag{2.130}$$

where γ is the functional index of any physical system under study and here we assume $\gamma=1$. The equation is statistical and not a deterministic law. It gives the expected basic intrinsic parameters of the 'host wave' that survives after a given time t . Clearly, we used (2.130) to calculate the exact value of the decay time as a function of the raising multiplier. In this work, we used table scientific calculator and Microsoft excel to compute our results. Also the GNUPLOT 3.7 version was used to plot the corresponding graphs.

3 Presentation of Results

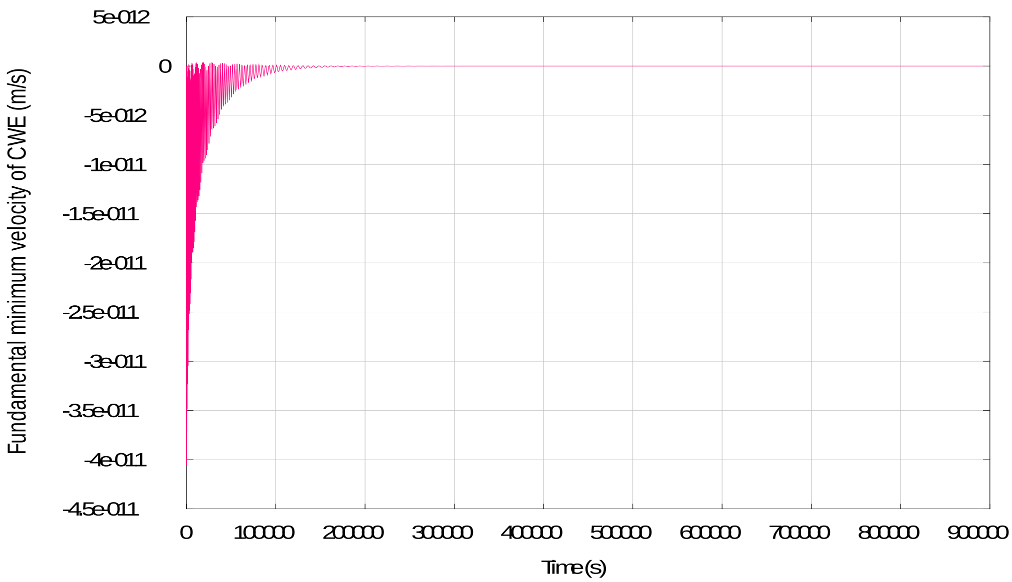

Figure 3.1: Represents λ [0, 505] and time [0, 892180s] or time = 892180s or 248 hrs (10 days), Fourier index $\beta = 0$. The fundamental velocity goes to zero after about 150000s or 42 hrs (1.8 days)

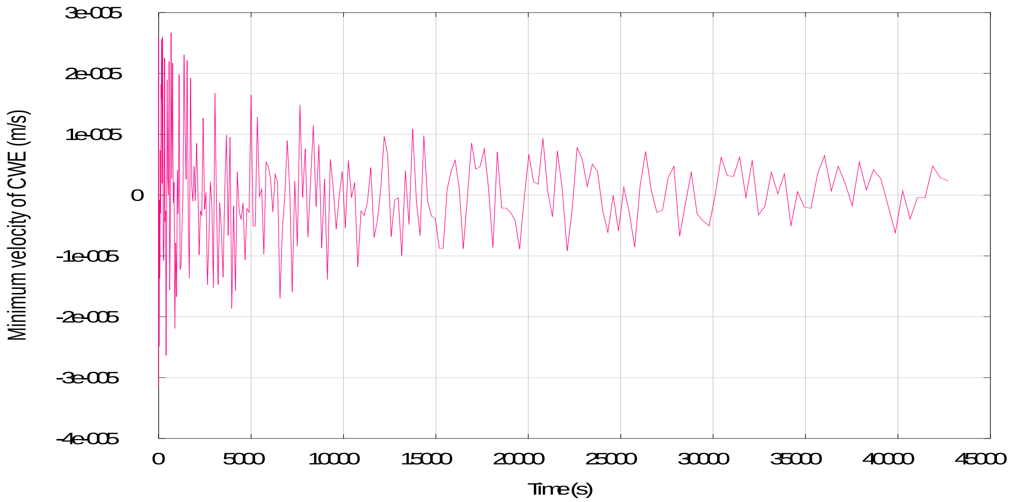

Figure 3.2: Represents the interval of the multiplier λ [0, 250] and time [0, 42705s] or time = 42705s (12hrs), Fourier index $\beta = 505$

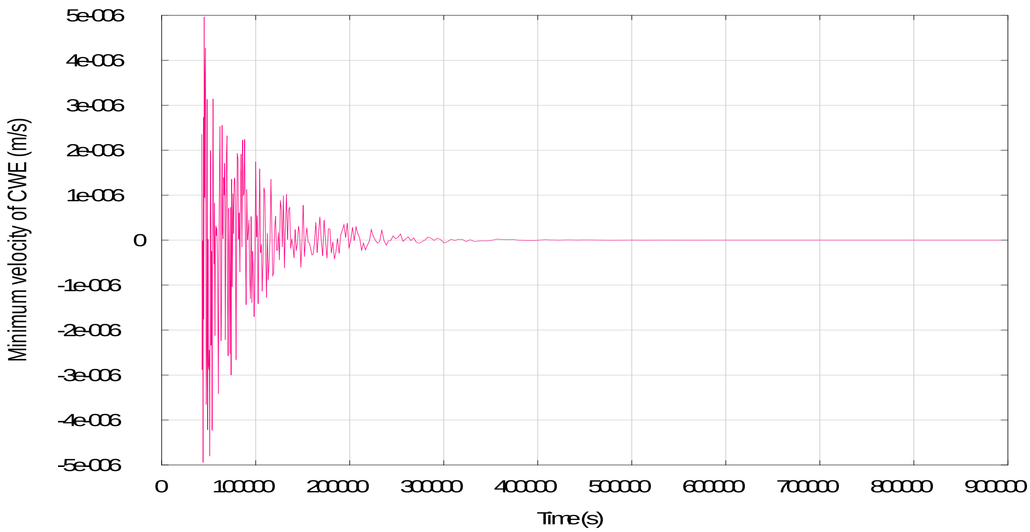

Figure 3.3: Represents the interval of the multiplier λ [250, 505] and time [42705s, 892180s] or time = 849475s or 236 hrs (10 days), Fourier index $\beta = 505$. The CWE attenuates to zero after about 300000s or 83hrs (3.4 days).

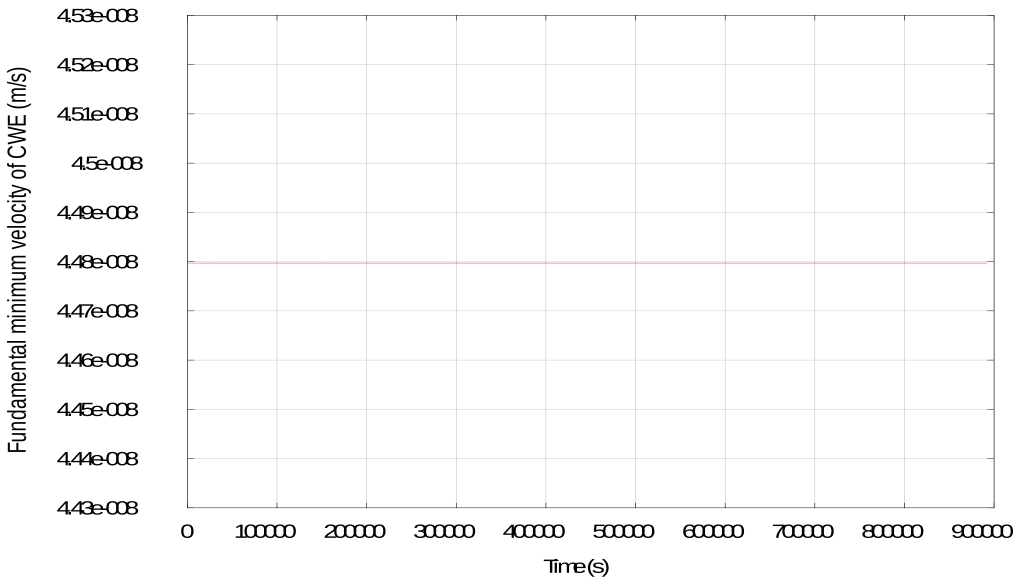

Figure 3.4: Represents the multiplier $\lambda = 0$ and the interval in time [0, 892180s] or time 892180s = 248 hrs (10 days), Fourier index $\beta = 0$

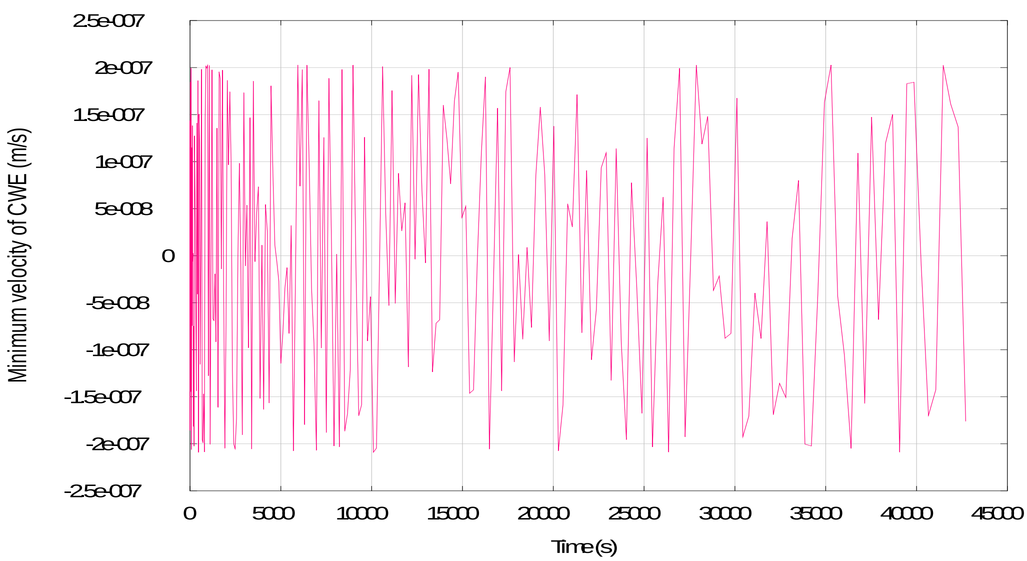

Figure 3.5: Represents the interval of the multiplier $\lambda = 0$ and time [0, 42705] or time = 42705s (11.8 hrs), Fourier index $\beta = 505$

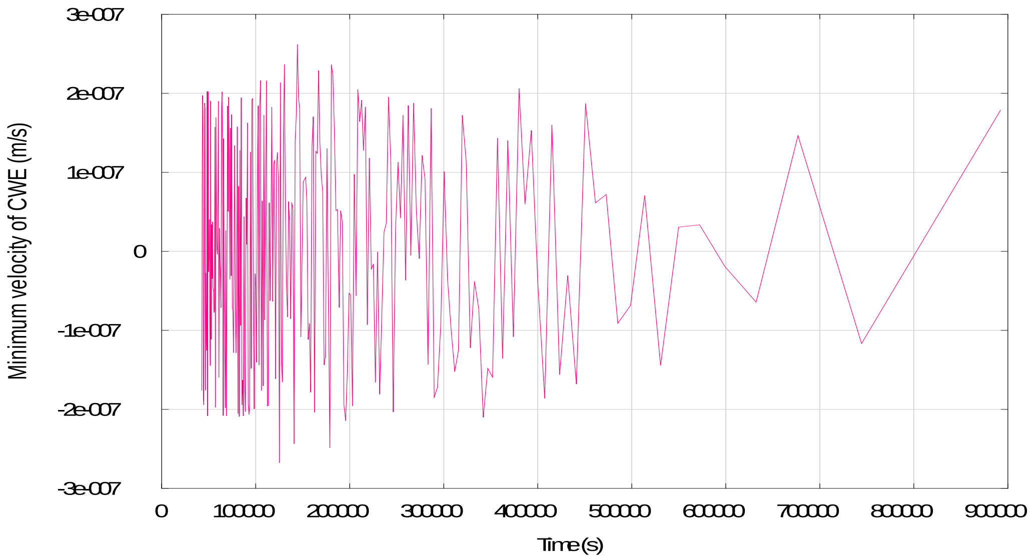

Figure 3.6: Represents the interval of the multiplier $\lambda = 0$ and time [42705s, 892180s] or time = 849475s or 236 hrs (10 days), Fourier index $\beta = 505$

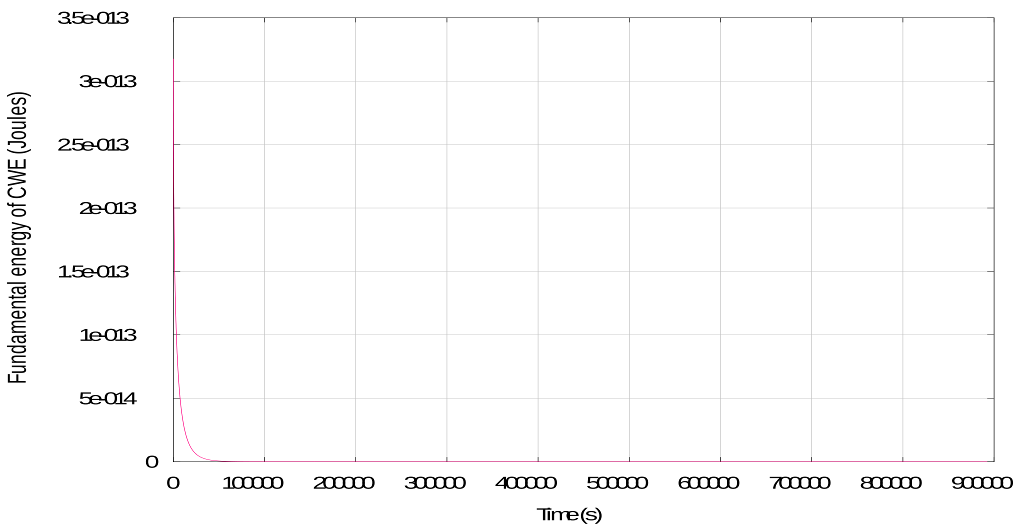

Figure 3.7: Represents the interval of the multiplier $\lambda =$ [0, 505] and time [0, 892180s] or time = 892180s or 248 hrs (10 days), Fourier index $\beta = 0$

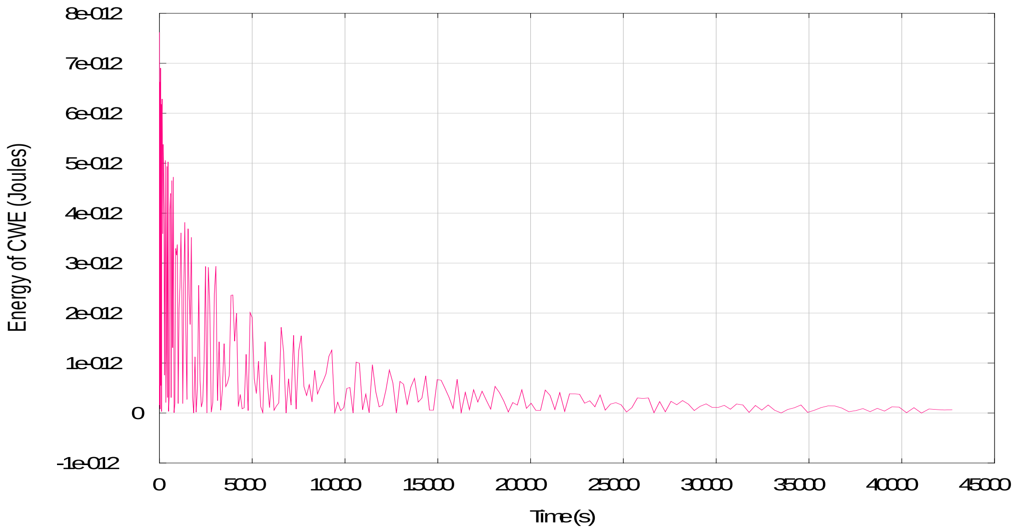

Figure 3.8: Represents the interval of the multiplier $\lambda =$ [0, 250] and time [0, 42705s] or time = 42705s or (12 hrs), Fourier index $\beta = 505$

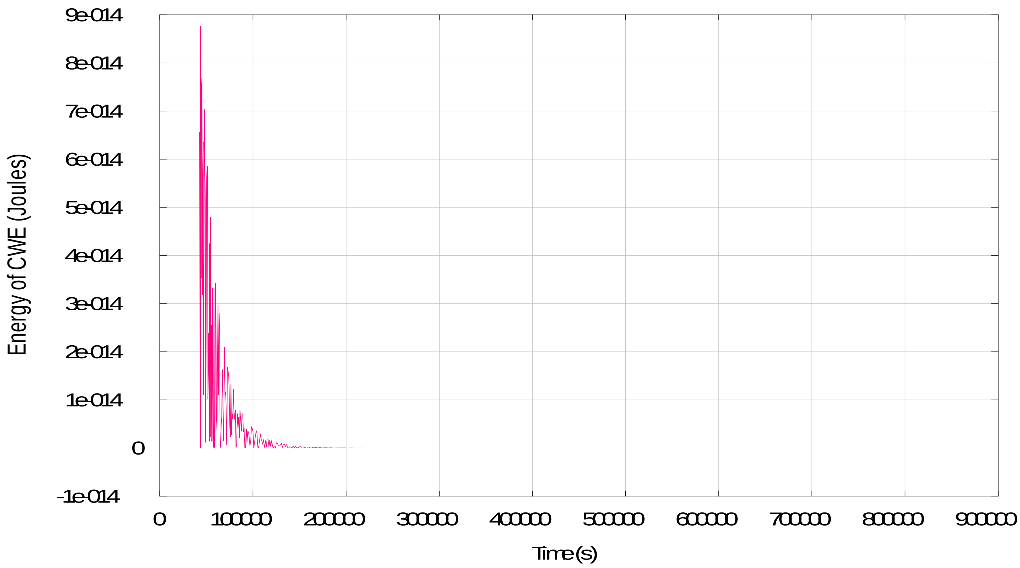

Figure 3.9: Represents the interval of the multiplier $\lambda =$ [250, 505] and time [42705s, 892180s] or time = 849475s or 236 hrs (10 days), Fourier index $\beta = 505$. The energy of the CWE goes to zero after 180000s or (50 hrs).

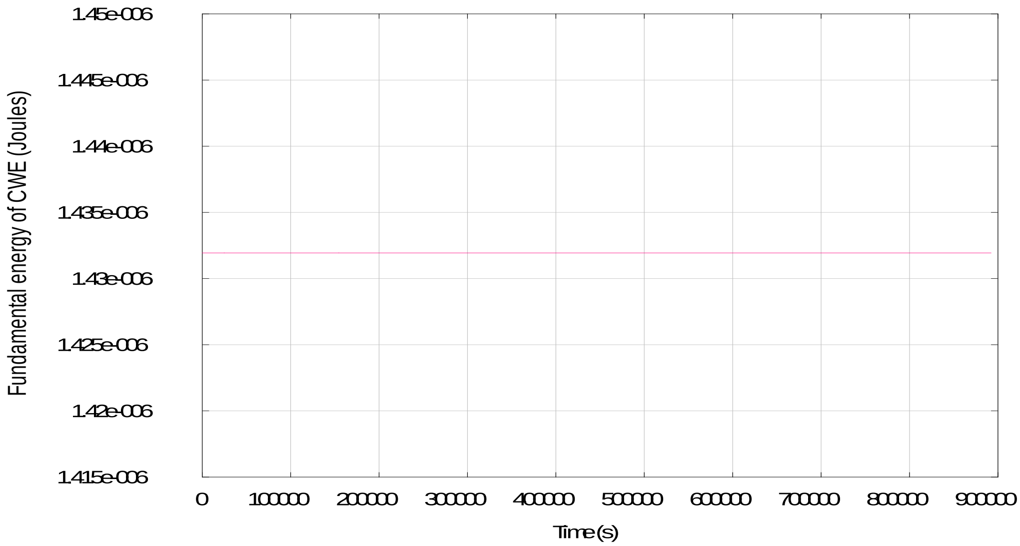

Figure 3.10: Represents the raising multiplier $\lambda = 0$ and the interval in time [0, 892180s] or time = 892180s or 248 hrs (10.3 days), Fourier index $\beta = 0$

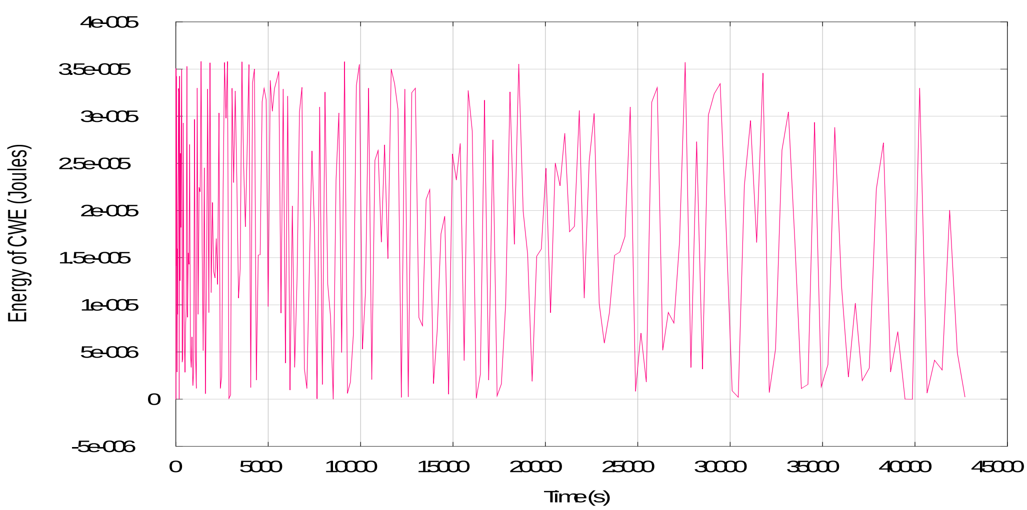

Figure 3.11: Represents the raising multiplier $\lambda = 0$ and the interval in time [0, 42705s] or precise time = 42705s (11.8 hrs), Fourier index $\beta = 505$

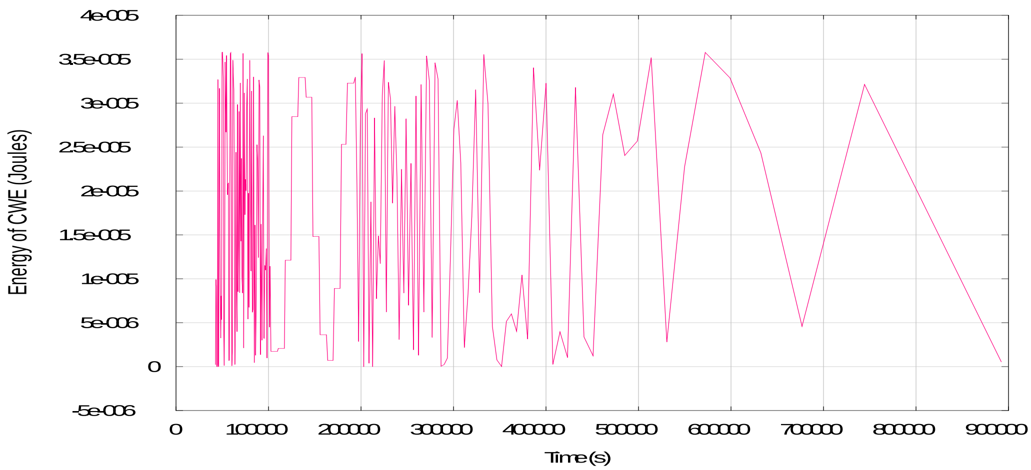

Figure 3.12: Represents the multiplier $\lambda = 0$ and the interval in time [42705s, 892180s] or precise time = 849475s or 236 hrs (10 days), Fourier index $\beta = 505$

4 Discussion of Results

Figure 3.1 – 3.3 shows the propagation of the CWE in a viscous fluid when the influence of the 'parasitic wave' is considered. It is revealed in Figure 3.1 that the fundamental minimum velocity of the CWE fluctuates between 0 and -4.5 x 10^{-11}m/s. The negative velocity means repulsion and hence destructive interference between the two interfering waves. The wave form of the fundamental minimum velocity is sinusoidal between the time intervals of 0 – 150000s or precisely 42 hrs (1.8 days) and thereafter it attenuate to zero.

It should be noted that because of the numerous waveforms involved when the Fourier index β=505 for every value of the multiplier λ, these figures could not really reflect all the possible waveforms available to the period of time that the CWE lasted, as a result, the figures almost displayed a straight line. Consequently, we classified our work based on the interval of the multiplier [0 – 250] and [250 – 505]. Although, our work was also confined to only when the

Fourier index was 505, since we believe that this is the region of most relevant interest to our work. Note that Figure 3.1 which is the first term of equation (2.85) is the harmonic analysis of the velocity of the CWE and it does not contain the Fourier index ($\beta = 0$) while the other term of (2.85) is the nth term of the velocity component of the CWE.

It should be observed that Figure 3.3 is a continuation of 3.2. We only separated them in order to unveil the embedded velocity waveform if it is jointly plotted. The velocity waveform of the CWE is somewhat sinusoidal and regular with attractive (constructive) and repulsive (destructive) phases of the minimum velocity. Consequently, positive velocity means attraction and hence constructive interference between the two interfering waves while negative velocity means repulsion and hence destructive interference between the two interfering waves. The minimum velocity attained by the propagation of the CWE attenuates to zero after about 300000s or 83 hrs (3.4 days).

In the absence of the 'parasitic wave' in which case the multiplier $\lambda = 0$ the resulting fundamental minimum velocity of the CWE are clearly shown in Figures 3.4 – 3.6. As shown in Figure 3.4 the fundamental minimum velocity does not change with time and hence the acceleration is zero. The graphs of the minimum velocity of the CWE are shown in Figure 3.5 and 3.6. These figures depict the propagation of only the 'host wave' ($\lambda = 0$). It is clearly revealed in these figures that the bandwidths of the spectrum of the minimum velocity of the CWE are wider than those of minimum velocity when the influence of the 'parasitic wave' is considered. This result can be compared with the graphs of Figure 3.2 and 3.3. The frequency of the minimum velocity attains by the CWE decreases after 400000s of 111 hrs (4.6 days).

The graphs of the energy possess by the CWE which is given by (2.98) as it propagates in the viscous fluid under the influence of the 'parasitic wave' ($\lambda = 0 - 505$) are shown in Figures 3.7 – 3.9. It is shown in Figure 3.7 that the decay process of the fundamental energy of the CWE is exponential and it is brought to rest after about 50000s (14 hrs). We should also know that Figure 3.9 is a continuation of Figure 3.8. The spectrum of the energy attenuation is sinusoidal and initially the decay frequency is very high which decreases as the time progresses. The energy of the CWE is finally brought to rest after about 200000s or 56 hrs (2.3 days).

The graphs of the energy of the CWE as it propagates in the viscous fluid when the ‘parasitic wave’ is not considered, in which case $\lambda = 0$, are represented by Figures 3.10 – 3.12. The fundamental energy ($\lambda = 0$) is constant with time and so the fundamental energy does not change as the ‘host wave’ propagates with time. This information is shown in fig. 3.10. The energy spectrum of the CWE ($\lambda = 0$) initially has a very high frequency of propagation as shown in Figure 3.11. However, from Figure 3.12 the frequency of the energy spectrum reduces after about 400000s or 111 hrs (4.6 days). The energy of the CWE finally attenuates to zero after 892180s or 248 hrs (10.3 days).

Thus generally, in the absence of the ‘parasitic wave’ ($\lambda = 0$) the energy of the CWE was able to propagate for a period of 892180s when the Fourier index $\beta = 505$ before it is finally attenuated to zero. However, when the influence of the ‘parasitic wave’ is considered ($\lambda = 0 - 505$) the energy of the CWE was able to propagate for a period of about 180000s when the Fourier index $\beta = 505$ before it finally goes to zero. This information is made clear when we compare Figure 3.9 and Figure 3.12. Hence the interference of the ‘parasitic wave’ with the ‘host wave’ resulted to a drastic reduction in the energy propagation time of the CWE by 712180s which is about 80% reduction.

5 Conclusion

We have shown in this work that the process of energy attenuation in most physical systems does not obviously begin immediately. The characteristics of the ‘host wave’ which defines the activity and performance of most physical system is guided by some factors which enables it to resist any internal or external interfering wave of a destructive tendency. The unsteady decay behaviour exhibited by the carrier wave equation during the energy damping process is due to the resistance pose by the characteristics of the ‘host wave’ in an attempt to annul the destructive effects of the interfering ‘parasitic wave’. It is evident from this work that when a carrier wave is undergoing energy attenuation, it does not steadily or consistently come to rest; rather it shows some resistance at some point in time during the damping process, before the carrier wave equation finally comes to rest.

5.1 Suggestions for further work

This study in theory and practice can be extended to investigate wave interference and propagation in two- and three- dimensional systems. The carrier wave equation we developed in this work can be utilized in the deductive and predictive study of wave attenuation in exploration geophysics and telecommunication engineering.

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Appendix

The following is the list of some useful identities which we implemented in the study.

(1) $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

(2) $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

(3) $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

(4) $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

(5) $2\sin x\cos y = \sin(x+y) + \sin(x-y)$

(6) $2\cos x\sin y = \sin(x+y) - \sin(x-y)$

(7) $2\cos x\cos y = \cos(x+y) + \cos(x-y)$

(8) $2\sin x\sin y = \cos(x-y) - \cos(x+y)$

(9) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

(10) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

(11) $\sin 2x = 2\sin x\cos x$

(12) $\sin(-x) = -\sin x$ (odd and antisymmetric function)

(13) $\cos(-x) = \cos x$ (even and symmetric function)