Journal of Computations & Modelling, vol.4, no.4, 2014, 53-70
ISSN: 1792-7625 (print), 1792-8850 (online)
Scienpress Ltd, 2014

Limits on the use of Qualitative stability for Non-Linear Systems

A.S. White[1]

Abstract

This paper describes the use of qualitative stability applied to nonlinear systems with reference to ecological models. The principles of qualitative stability are outlined for linearised systems. The requirements necessary for this special condition are given with an example. A modified Jacobian matrix is evaluated using a Hessian expansion for the nonlinear terms following the method of Saleh and Davidsen. Some observations are made about the possibility of achieving Qualitative stability for the nonlinear case.

A predator prey example is examined showing that although the linearised system is qualitatively stable the nonlinear case is not necessarily stable away from the equilibrium. The time varying eigenvalues show periods of both instability and stability which converge to stable behaviour as the initial conditions are made closer to the equilibrium values. Analysis displayed here illustrates the possibility of finding a region of the nonlinear system that is qualitatively stable.

Mathematics Subject Classification: 93; 37

[1] School of Science and Technology, Middlesex University, UK ,
E-mail: a.white@mdx.ac.uk

Article Info: *Received* : September 2, 2014. *Revised* : October 11, 2014.
Published online : November 15, 2014.

Keywords: nonlinear systems; Qualitative stability; sign stability; predator-prey model; MATLAB

1 Introduction

In engineering and science significant use is made of models to enable performance predictions of both human made and natural systems. Studies of physical and human systems are based both on experiments and analysis incorporating the decision processes used to control them. The behaviour of these systems depends on the initial conditions and system parameters. It is clear, from the study of chaos theory, that small variations in the initial conditions can make substantial differences to the overall behaviour of the systems. Thompson and Stewart [1] state that chaotic behaviour is unpredictable over long time scales because any two trajectories starting close to a chaotic attractor will separate as they progress in time. This separation rate depends on the largest Lyapunov exponent (Kapitaniak [2]) related to the system eigenvalues. If the system parameters change or are not those of the nominal value then we cannot guarantee how close the system will come to a chaotic state. Traditionally, due to lack of analytical and computer tools, linear systems were the only systems whose response could be computed. Stability of such linear systems was treated as a convenient way of assessing the behaviour of a given system due to the amount of effort required to obtain the system response. As outlined by Bennett [3], such stability criteria were developed by Routh [4], Hurwitz [5] and justified by Lyapunov's [6] first theorem which stated that "in a small neighbourhood of the origin where the origin of a perturbed system is a point of equilibrium, then a nonlinear system can be approximated by its' linearization about the origin". This principle of assessing the stability of nonlinear systems around the locality of equilibrium point using the Jacobian (Siljak [7]), (Khalil [8]) is a technique that has served analysis very well. An important practical reason for stability analysis is the desire of engineers to design systems that can be both predicted and controlled. However in ecological and economic systems nonlinear sets of equations cannot be “designed out”. Methods were developed by May [9] and others to determine a set of criteria that could allow stability of equilibriums for a wide range of parameters. An important set of criteria have been developed for linear systems to facilitate estimations of system stability, an essential part of control.

The purpose of this paper is to see what modification is possible to extend the principle of qualitative stability to nonlinear systems. A short review of the Qualitative stability follows with an extension to nonlinear systems via a higher order expansion to the Jacobian. An example of the wider sign stable region found for a predator prey problem is then described.

2 Qualitative Stability

The concept of qualitative stability and the problems of stability of large chains and networks of elements in systems has had many contributors since the 1960’s including significantly Lancaster [10], Quirk & Ruppert [11], Levins [12], Maybee & Quirk [13], by the seminal paper of Gardner and Ashby [14], Quirk [15], May [16], [17] & [9] and Jeffries [18]. The work of Gardner and Ashby showed that there was a limit to the number of interactions that could be allowed in any system whilst maintaining stability. It was not until the paper of May [16] that the significance of the application of these methods to ecological systems and similar became clear. The conditions he described were necessary but not sufficient. This was later remedied by Jeffries [18] by inventing the “colour” tests that provided the sufficient conditions for guaranteed sign stability. Jeffries showed that the five conditions M1-5 could not distinguish between neutral and asymptotic stability. This was achieved with the colour test.

These authors were concerned with both ecological and economic systems where precise observations were lacking or almost impossible to obtain. The suggestion presented here is that for ecosystems the population of species i is affected, increased, unaffected or decreased by species j. In this case the conventional approach of examining the system matrix eigenvalues becomes impracticable. May [16] and Jeffries [18], developed with others the concept of qualitative stability. In this approach the stability of the system matrix is determined from the sign characteristics of the elements. The first order set of differential equations is represented by equation 1.

$$\dot{\boldsymbol{x}}\boldsymbol{(t)} = \mathbf{A}\boldsymbol{x}\boldsymbol{(t)} \tag{1}$$

Matrix **A** is said to be *Qualitatively Stable* if a matrix **B** of the same sign pattern as **A**

($sgn\ (b_{ij}) = sgn\ (a_{ij})\ \forall\ i, j$) is Hurwitz stable for any magnitude of $b_{i,j}$.

Thus the concept of “qualitative” or “sign Stability” implies stability as obtained from Routh-Hurwitz conditions (D’Azzo & Houpis [19], Ogata [20]) with negative real parts to the eigenvalues.

Examples of a system where these factors can be represented qualitatively are:

$$\mathbf{A_1} = \begin{bmatrix} - & + & + & + \\ - & - & 0 & 0 \\ - & 0 & - & 0 \\ - & 0 & 0 & - \end{bmatrix}, \qquad \mathbf{A_2} = \begin{bmatrix} - & + & + & + \\ - & - & + & + \\ - & - & - & + \\ - & - & - & - \end{bmatrix}$$

Elements of the matrices are represented by positive, negative signs or zeros. These conditions were couched in terms appropriate for ecological systems and it wasn’t until Yedavalli [21] put these conditions into a form that scientists and engineers could readily use.

The following restated from Deverakonda [22], constitute the requirements for qualitative or “sign” stability for a system of state equations:

$M1. a_{ii} \leq 0 \; for \; all \; i$

$M2. a_{ii} < 0$ *for at least one i*

$M3. a_{ij}a_{ji} \leq 0 \; for \;\; all \; i \neq j$

$M4. a_{ij}a_{jk} \ldots a_{qr}a_{ri}$

$= 0 \; for \; any \; sequence \; of \; three \; or \; more \; distinct \; indices \; i, j, k, \ldots q, r$

$M5. det(A) \neq 0$

$M6.$ *The Matrix must* ***Fail*** *the colour test*

The colour test is described by Yedavalli [21] as:

"ct1. Each (i,i) element that is negative is a *black* node denoted $a_{bi,bi}$.

ct2. Each (i,i) element that is zero is a *white* node and denoted $a_{wi,wi}$. To pass this condition there must be at least one white node. No zero diagonal elements means the colour test is failed.

ct3. Form all the products $a_{wi,wj}\, a_{wj,wi}$. Passing this condition implies that there is at least one of these products that is negative. If there is only one white node in which case there is no indicated possible product implies that this condition is failed.

ct4. Form all the products $a_{bj,wi}\, a_{wi,bj}$. passing this condition implies that if, for each fixed *bj* black node, the product $a_{bj,wi}\, a_{wi,bj}$ is negative, then another product $a_{bj,wk}\, a_{wk,bj}$ is also negative for some $wk \neq wi$. If there is only one product possible, then passing this condition implies that this product is negative. If the products formed under this condition are *all* zero or *all* negative, it implies passing this condition."

In the case of the two systems above, matrix 1 is qualitatively stable and matrix 2 is not. Yedavalli [21] has applied these concepts to the design of robust spacecraft control systems.

Ecological and linear control systems are often interpreted in terms of signed digraphs as in Figure 1. This system can be shown to be qualitatively stable. To illustrate the methodology conditions M1-6 will be applied to system 1 in Figure 1.

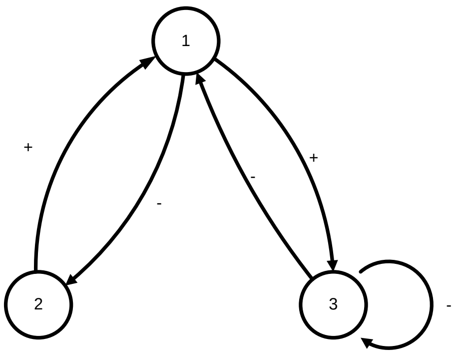

Figure 1: Digraph of System 1

The sign matrix **Q** for this system is:

$$\mathbf{Q} = sign\,\mathbf{J} = \begin{bmatrix} 0 & + & + \\ - & 0 & 0 \\ - & 0 & - \end{bmatrix} \quad (2)$$

For ecological systems:

*M*1. This corresponds to there being no positive loops on any state (species), i.e. no positive feedback.

*M*2. At least one negative loop for some state (species) in the graph.

*M*3. No pair of like arrows connecting a pair of states (species)

*M*4. No cycles connecting three or more states.

*M*5. No node devoid of inputs.

This matrix example satisfies all these 5 conditions.

The matrix, in equ. 2, *fails* ct4 and therefore the system it represents is qualitatively stable.

Recent developments by Allesina and Pascual [23] and Pawar [24] building on the work of Fox [25] have examined the nearness to sign stability in concepts labelled quasi sign stability by determining the sign of the dominant eigenvalues and the interaction strengths.

2.1. Nonlinear Eigenvalue analysis

Saleh and Davidsen [26] have extended the use of eigenvalue analysis to nonlinear models. They used a higher order Taylor expansion to allow a better approximation to be achieved as follows:

$$\dot{\boldsymbol{x}}(r) = \dot{\boldsymbol{x}}_{\mathbf{0}}(r) + grad_r^T(\boldsymbol{x} - \boldsymbol{x}_{\mathbf{0}}) + 0.5(\boldsymbol{x} - \boldsymbol{x}_{\mathbf{0}})^T \mathbf{H}_{\boldsymbol{r}}(\boldsymbol{x} - \boldsymbol{x}_{\mathbf{0}}) \quad (3)$$

Where *r is the state or active node, r=1 to n.*

Hence

$$\dot{\boldsymbol{y}}(r) = grad_r^T \boldsymbol{y} + 0.5\boldsymbol{y}^{\boldsymbol{T}} \mathbf{H}_{\boldsymbol{r}} \boldsymbol{y} \quad (4)$$

$$\dot{\boldsymbol{y}} = \mathbf{J}^* \boldsymbol{y} \quad (5)$$

$$\boldsymbol{J}^*(r,:) = \boldsymbol{grad}_{\boldsymbol{r}}^{\boldsymbol{T}} + \boldsymbol{y}^{\boldsymbol{T}} \mathbf{H}_{\boldsymbol{r}} \quad (6)$$

$$\boldsymbol{y} = \left(\sum_1^n \boldsymbol{v}_r e^{\lambda_i} \boldsymbol{w}_r^T\, \boldsymbol{x}(0)\right) \quad (7)$$

$\mathbf{H}_{\boldsymbol{r}}$ is a Hessian Matrix.

A Hessian Matrix is given by:

$$\mathbf{H}_r = \begin{bmatrix} \frac{\partial^2 f_r}{\partial x_1^2} & \frac{\partial^2 f_r}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f_r}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f_r}{\partial x_2 \partial x_1} & \frac{\partial^2 f_r}{\partial x_2^2} & \cdots & \frac{\partial^2 f_r}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f_r}{\partial x_n \partial x_1} & \frac{\partial^2 f_r}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f_r}{\partial x_n^2} \end{bmatrix} \quad (8)$$

The eigenvalues $\boldsymbol{\lambda}$ of the nonlinear system are solutions of:

$$|\mathbf{J}^* - \boldsymbol{\lambda}\mathbf{I}| = 0 \quad (9)$$

2.1.1 Application to Qualitative Stability

We can now create the total modified Jacobian $\mathbf{J}^*$ in order to examine what changes are made to the stability criteria for sign stability.

$$\mathbf{J}^* = \begin{bmatrix} \mathbf{J_e}(1,:) + 0.5\boldsymbol{y}^{\boldsymbol{T}}\mathbf{H_1} \\ \mathbf{J_e}(2,:) + 0.5\boldsymbol{y}^{\boldsymbol{T}}\mathbf{H_2} \\ \vdots \\ \mathbf{J_e}(n,:) + 0.5\boldsymbol{y}^{\boldsymbol{T}}\mathbf{H}_{\boldsymbol{n}} \end{bmatrix} \quad (10)$$

$$\boldsymbol{y}^{\boldsymbol{T}} = [y_1, \quad \cdots, \quad y_r, \quad \cdots, \quad y_n] \quad (11)$$

$$\mathbf{J}^* = \begin{bmatrix} \boldsymbol{a_{11}} + \sum_{k=1}^{n} \boldsymbol{y_k h_1}(k,1) & \cdots & \boldsymbol{a_{1r}} + \sum_{k=1}^{n} \boldsymbol{y_k h_1}(k,r) & \cdots & \boldsymbol{a_{1n}} + \sum_{k=1}^{n} \boldsymbol{y_k h_1}(k,n) \\ \vdots & \ddots & \boldsymbol{a_{kr}} + \sum_{k=1}^{n} \boldsymbol{y_k h_r}(k,r) & \ddots & \vdots \\ \boldsymbol{a_{n1}} + \sum_{k=1}^{n} \boldsymbol{y_k h_n}(k,1) & \cdots & \boldsymbol{a_{nr}} + \sum_{k=1}^{n} \boldsymbol{y_k h_n}(k,r) & \cdots & \boldsymbol{a_{nn}} + \sum_{k=1}^{n} \boldsymbol{y_k h_n}(k,n) \end{bmatrix} \quad (12)$$

For qualitative stability to be maintained then $\mathbf{J}^*$must continue to satisfy conditions M1 to M6. The terms a_{kr} in M1 to M6 are replaced by a_{kr}^*. since the elements a_{kr}^* now contain terms in y_r it is more difficult to guarantee compliance. However if $h_r(k,r)=0\ \forall r$ then the original conformation to the conditions M1 to M6 would be preserved.

In general, terms such as $\sin x$, x^3, $x_1 x_2^3$ can yield zero Hessian components if the equilibrium point is the origin. However terms such as $\cos x$ will not give this result. Even if sign stability is not achievable, it is still possible that the eigenvalues of $\mathbf{J}^*$ satisfy the Hurwitz conditions for stability numerically.

A further possibility is to use the terms in equation 12 to give a range for which the qualitative stability criterion applies in the nonlinear case, as shown in the following example. In this case we can consider the condition where the addition of the nonlinear terms does not change the sign condition of each element of the modified Jacobian

It may be possible in some systems to use the robust controller techniques proposed by Yedavalli [21] to create a feedback controller (biological or otherwise) to feedback nonlinear terms to cancel the Hessian components.

3 Case study

A predator-prey example is taken from Edelstein-Keshet [27], where the predator is x_1 and x_2, x_3 are the prey. In this example the predator dies out in absence of prey and the first prey x_2 grows at an exponential rate in the absence of a predator. The second prey x_3 grows logistically in the absence of its predator. This problem has a set of nonlinear equations given by:

$$\dot{x} = \begin{bmatrix} ax_1x_3 + bx_1x_2 - cx_1 \\ dx_2 - ex_1x_2 \\ fx_3(h - x_3) - gx_1x_3 \end{bmatrix} \tag{13}$$

A non-zero equilibrium condition exists where:

$$\bar{x}_1 = \frac{d}{e}, \bar{x}_2 = c - a\bar{x}_3, \bar{x}_3 = h - \frac{g}{f}\bar{x}_1 \tag{14}$$

This leads to the Jacobian at the equilibrium point:

$$\mathbf{J} = \begin{bmatrix} 0 & b\bar{x}_1 & a\bar{x}_1 \\ -e\bar{x}_2 & 0 & 0 \\ -g\bar{x}_3 & 0 & -f\bar{x}_3 \end{bmatrix} \tag{15}$$

If the sign matrix is written it can be seen to be the same as equation 2. Hence the system is qualitatively stable around the equilibrium position.

The nonlinear Jacobian can also be obtained:

$$\mathbf{J}^* = \begin{bmatrix} \frac{b}{2}(x_2 - \bar{x}_2) + \frac{a}{2}(x_1 - \bar{x}_1) & \frac{b}{2}(\bar{x}_1 + x_1) & \frac{a}{2}(\bar{x}_1 + x_1) \\ -\frac{e}{2}(\bar{x}_2 + x_2) & \frac{e}{2}(\bar{x}_1 - x_1) & 0 \\ -\frac{g}{2}(\bar{x}_3 + x_3) & 0 & -fx_3 - \frac{g}{2}(x_1 - \bar{x}_1) \end{bmatrix} \tag{16}$$

It can be seen that this reduces back to equation 15 at the equilibrium point. If we apply conditions M1-M6, to the matrix in equ. 16, then we can see that a_{11} fails to satisfy M1 as $x_1 \& x_2$ would both have to be less than the equilibrium values, which cannot be true for all the range of values. The system, for states not close to the equilibrium, is not qualitatively stable.

An example of data relevant to this case was found in the paper by Fay and Greeff [28] relating to the population of Lions, Wildebeest and Zebra in the Kruger National park. Although they did not use the precisely same model, this model has similar features.

The data for the model is:

a=0.125, *b*=0.125, *c*=1.5, *d*=0.4005, *e*=0.81, *f*=0.0283, *g*=0.02, *h*=12, per thousand animals. The equilibrium conditions are: $\bar{x}_1 = 0.5, \bar{x}_2 = 0.0441, \bar{x}_3 = 11.65$

A simulation using MATLAB is shown here for initial conditions ic#1 and ic#2. For ic#1 the initial conditions were far from the equilibrium values whereas for ic#2 they were much closer.

The computed results for ic#1= (0.5, 10.6, 10.6) are shown in Figures 2 & 3. The number of lions rises to a peak and then stabilises. One species of prey falls steadily while the other recovers. Figure 3 illustrates that the real part of the nonlinear eigenvalue does not stay stable although they are stable for a region close to the equilibrium values. In Figures 2 & 3 at the equilibrium sig1 has a value of +0.02 indicating a slight divergence. The other two eigenvalues indicate subsidence responses. The eigenvalue behaviour in nonlinear systems varies with time and the modes change drastically over time, some merging and then diverging again.

These results indicate that this system, which is qualitatively stable analytically at the equilibrium point, does *not* have stable eigenvalues outside the equilibrium unless the initial condition is close to that equilibrium. .

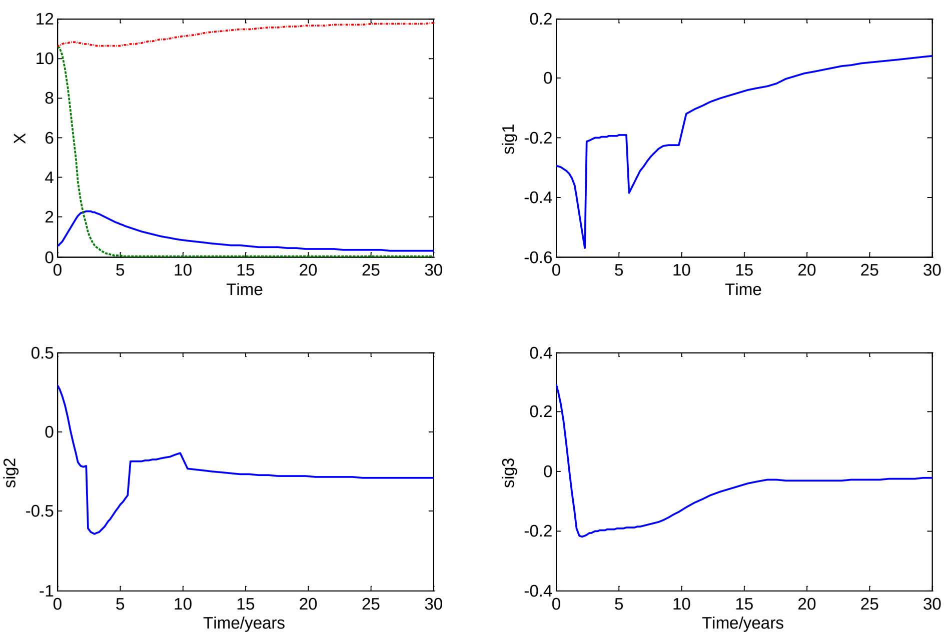

Figure 2: Plot of real components of eigenvalues for ic#1

Figure 4 evaluated for ic#2= (0.3, 1, 11.6) shows that for a large proportion of the time one of the roots is real, especially close to the equilibrium. The imaginary components of roots 1 & 3 are initially complex conjugates about this pattern changes at around 20 years and roots 2 & 3 then become the complex conjugates. The system is also unstable close to the initial conditions and later before 50 years have elapsed. However in Figures 4 & 5 the oscillations die down and the behaviour is stable after 50 years. The nearer we start the computation to the equilibrium values the more linear the behaviour looks and the better approximation the qualitative stability analysis is to the whole system. But when the system starts far from the equilibrium values then the eigenvalues diverge from our linearised system at any time in the period chosen.

We can use the expression for the nonlinear Jacobian to examine what happens if we try to keep the sign conditions of the linear Jacobian the same in the nonlinear case.

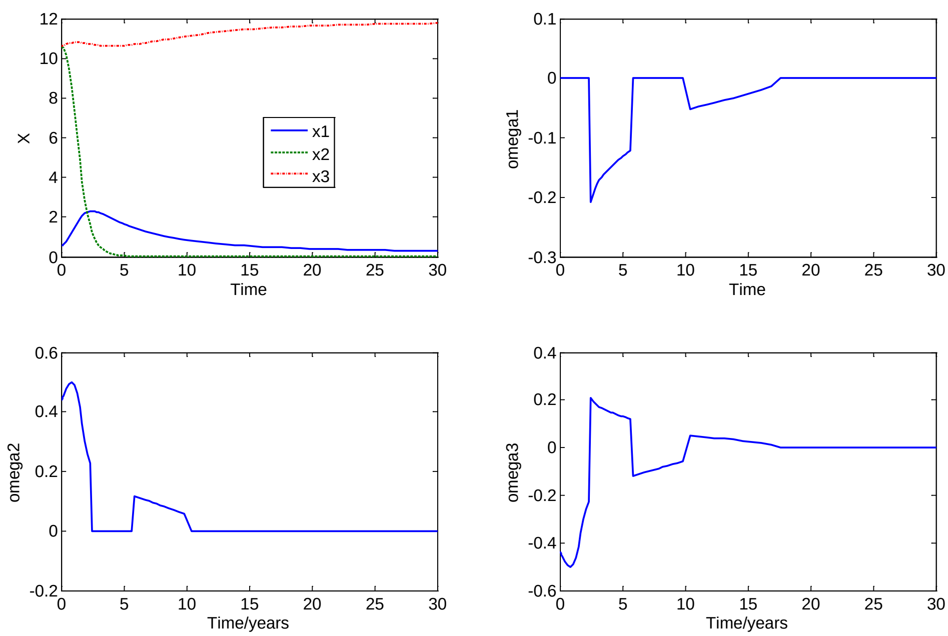

Figure 3: Plot of imaginary components of eigenvalues

Can we find a region where the system is still qualitatively stable?

Examining the Jacobian term by term we obtain:

$$a_{11}^* = 0 \xrightarrow{yields} bx_2 + ax_1 = b\bar{x}_2 + a\bar{x}_1 \quad (17)$$

$$a_{12}^* > 0 \xrightarrow{yields} (\bar{x}_1 + x_1) > 0 \quad (18)$$

$$a_{13}^* > 0 \xrightarrow{yields} \bar{x}_1 + x_1 > 0 \quad (19)$$

$$a_{21}^* < 0 \xrightarrow{yields} \bar{x}_2 + x_2 > 0 \quad (20)$$

$$a_{22}^* = 0 \xrightarrow{yields} \bar{x}_1 = x_1 \quad (21)$$

$$a_{31}^* < 0 \xrightarrow{yields} \bar{x}_3 + x_3 > 0 \quad (22)$$

$$a_{33}^* < 0 \xrightarrow{yields} fx_3 + \frac{g}{2}(x_1 - \bar{x}_1) > 0 \quad (23)$$

From equation (21) we obtain $x_1 = \bar{x}_1$

This leads to $x_2 = \bar{x}_2$ from equation (17) and from equation (23) we have: $x_3 > 0$.

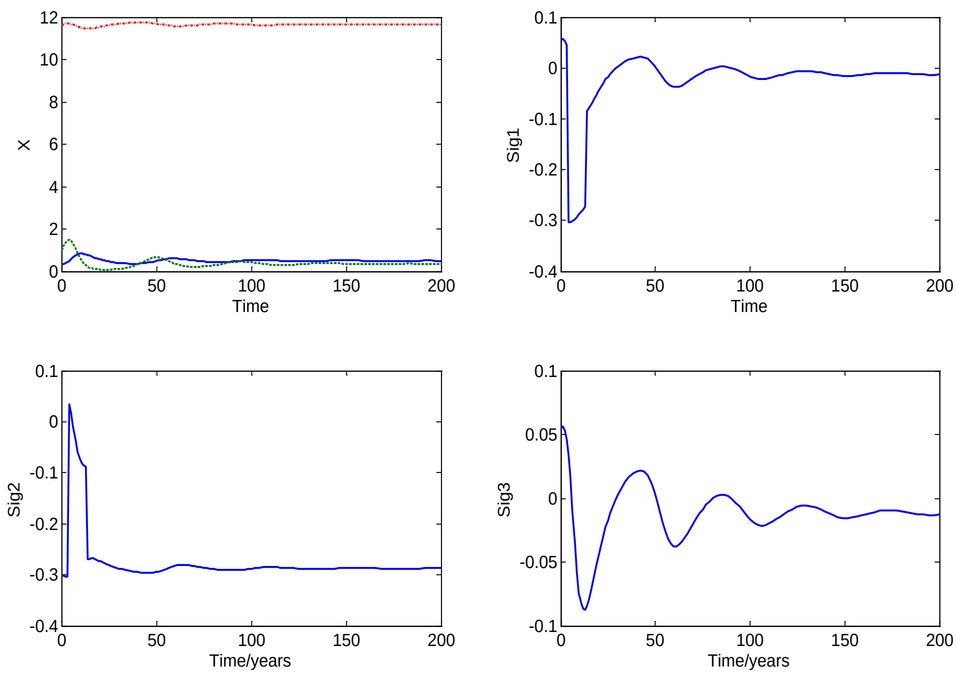

Figure 4: Real values of eigenvalues for ic#2

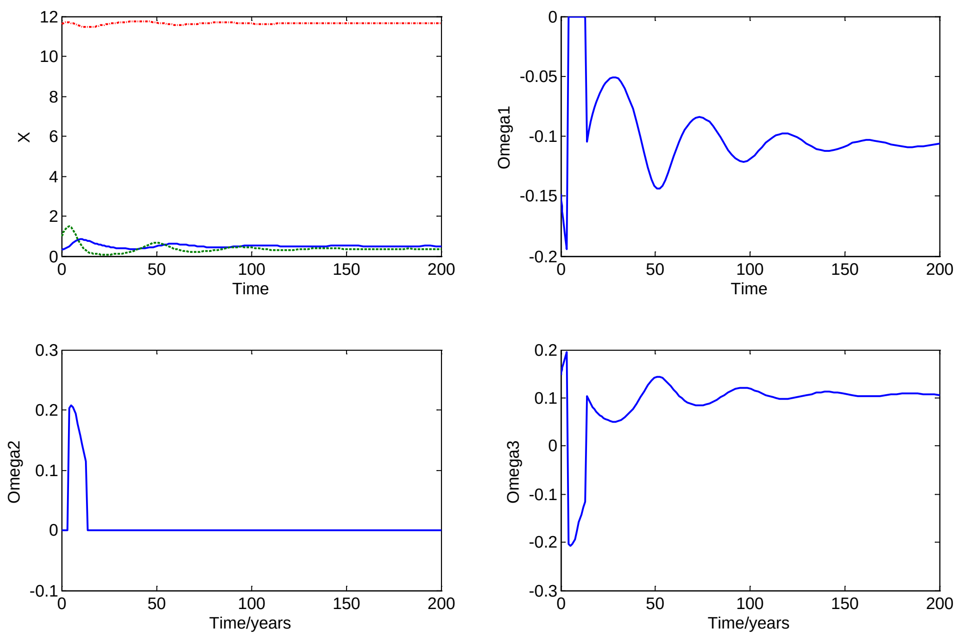

Figure 5: Imaginary values of eigenvalues for ic#2

This means that the nonlinear system has a region where it is sign stable at values of $x_1 = \bar{x}_1 \; and \; x_2 = \bar{x}_2$ for a large range of x_3, subject to satisfying equation (3).

This is illustrated by the data in Table 1 for the eigenvalues of the nonlinear system.

Here it can be seen that the eigenvalues are stable for a range of parameter values at up to 5 times the equilibrium value of x_3. Thus for fixed equilibrium of predator + prey 1 the condition of sign stability applies for a wide range of numbers of prey 2. To improve modelling we need only to concentrate on evaluating the equilibrium values of the predator and prey1.

Table 1: Eigenvalues vs State and Parameter values

Parameters \ X3	*50* λ	*2*Nominal* λ	*Nominal* λ	*0.5*Nominal* λ
2*Nominal	-0.0567±0.0781i -2.7167	-1.174 -0.0742±0.0675i	-0.0699 -0.3463 -0.2432	-0.0342 -0.1478±0.2537i
Nominal	-0.0283±0.039i -1.3583	-0.0371±0.0338i -0.5852	-0.035 -0.1731 -0.1216	-0.0171 -0.739±0.1269i
0.5*Nominal	-0.0142±0.0195i -0.6792	-0.0185±0.0169i -0.2926	-0.0175 -0.0866 -0.0608	-0.0085 -0.0369±0.0634i

4 Conclusion

This paper outlines the arguments for qualitative stability, listing the conditions to be satisfied. An analysis using nonlinear time varying eigenvalues has been made to determine the possibility of retaining qualitative stability for nonlinear equations sets. The results presented here indicate that it is possible to have qualitative stability over a larger range of state values than just close to the equilibrium but it is only likely to apply globally to some nonlinear systems, in particular:

a) Those with equilibrium at the origin of the coordinate system and/or
b) Where the Hessian Matrix elements have zero values.

A third possibility is that:

c) It is possible to determine a region where sign stability is preserved in the nonlinear system

A case study of a nonlinear predator+2 prey set of equations shows that although the system is sign stable at the equilibrium point the nonlinear equation, for the initial conditions chosen, has an unstable eigenvalue outside that point but has a significant region where it is sign stable for a range of values of the second prey numbers illustrating point c above. This would have significance for the design of robust control systems.

5 Symbols

A	state matrix
a	*Constant*
a_{kr}^*	Elements of nonlinear Jacobian
a_{kr}	Element of linear jacobian
B	sample matrix
b	*Constant*
c	*Constant*
d	*Constant*
e	*Constant*
f	*Constant*
g	*Constant*
h	*Constant*
$\mathbf{H_i}$	Hessian matrix for state *i*
$h_r(k,r)$	Hessian elements
i	Suffix, square root of -1
j	Suffix
J	Jacobian
$\mathbf{J}^*$	Nonlinear Jacobian
n	*Constant*
Omega k	*Imaginary component of eigenvalue k*
Q	Sign Matrix
r	*is the state or active node, r=1 to n.*

Sigk	Real part of eigenvalue k
$\boldsymbol{x}$	System states
x_1	Predator numbers
x_2	Prey 1 numbers,
x_3	Prey 2 numbers,
x_0	Equilibrium values
$\boldsymbol{y}$	State Vector
$\boldsymbol{v}$	Right eigenvector
$\boldsymbol{w}$	Left eigenvector
$\boldsymbol{\lambda}$	*eigenvalue*

References

[1] J. Thompson, H. Stewart, *Nonlinear Dynamics and Chaos*, 2nd Ed., Chichester, Wiley, 2002.

[2] T. Kapitaniak, *Chaos for Engineers*, Berlin, Springer, 1998.

[3] S. Bennett, *History of Control Engineering 1800-1930*, London, IET, 1979.

[4] E. J. Routh, *A treatise on the stability of a given state of motion*, London, Macmillan, 1877.

[5] A. Hurwitz, Über die Bedingungen, unter welchen eine Gleichung nur Wurzen mit negativen reelen Teilen bestizt, *Mathematische Annalen*, **46**, (1895), 273-280.

[6] A. Lyapounov, *Le probléme general de la stabilité du movement*, Ann. de la Faculté des Sci. de Toulouse, 9, translation of his 1892 dissertation, 1907.

[7] H. Khalil, *Nonlinear Systems*, New York, Macmillan, 1992.

[8] D. Šiljak, *Nonlinear Systems*, New York, John Wiley, 1969.

[9] R May, Qualitative stability in Model Ecosystems, *Ecol.*, **54**(3), (1974), 638-641.

[10] K. Lancaster, Theory of qualitative linear systems, *Econometrics*, **33**(2), (1965), 395-408.

[11] J. Quirk and R. Ruppert, Qualitative Economics and the stability of equilibrium, *Rev. Econ. Stud*., **3**, (1965), 311-326.

[12] R Levins, *Evolution in changing environments*, Princeton N.J., Princeton University Press, 1968.

[13] J. Maybee and J. Quirk, Qualitative Problems in Matrix Theory, *SIAM Rev*., **11**(1), (1969), 30-51.

[14] M. Gardner and W. Ashby, Connectance of Large Dynamical (cybernetic) systems: critical values for stability, *Nature*, **228**, (1970), 784.

[15] J. Quirk, Complementarity and Stability of Equilibrium, *Am. Econ. Rev*., **60**(3), (1970), 358-63.

[16] R May, Stability in multi-species community models, *Math. Biosc.*, **12**, 91971), 57-79.

[17] R. May, *Stability and Complexity in Model Ecosystems*, Princeton, N.J., Princeton University Press, 1973, 16-34.

[18] C. Jeffries, Qualitative Stability and Digraphs in Model Ecosystems, *Ecol*., **55**, (1974), 1415-1419.

[19] J. D'Azzo and C. Houpis, *Linear Control System Analysis and Design Conventional and Modern*, 3rd Ed, New York, McGraw Hill, 1988.

[20] K. Ogata, *Modern Control Engineering*, 2nd Ed, N.J., Prentice-Hall, 1990.

[21] R.K. Yedavalli, Qualitative Stability Concept from Ecology and its Use in the Robust Control of Engineering Systems, *Proc. of the Am. Cont. Conf*., Minneapolis, June (2006), 5097-5102.

[22] N. Deverakonda, *Eco-inspired robust control design for linear dynamic systems with applications*, PhD Thesis, Ohio State University, Graduate school, 2011.

[23] S. Allesina and M. Pascual, Network structure, predator–prey modules, and stability in large food webs, *Theoret. Ecology*, **1**(1), (2008), 55-64.

[24] S. Pawar, Community assembly, stability and signatures of dynamical constraints on food web structure, *J. of Theoret. Biol.*, **259**(3), (2009), 601-612, DOI - http://dx.doi.org/10.1016/j.jtbi.2009.04.006

[25] J. W. Fox, Current food web models cannot explain the overall topological structure of observed food webs, *Oikos*, **115**(1), (2006), 97-109, DOI - 10.1111/j.2006.0030-1299.14868.x.

[26] M. Saleh and P. Davidsen, Extending eigenvalue analysis to nonlinear models via incorporating higher order terms of Taylor series expansion, *25th Int. Systems Dynamics Conf.*, (2007), available at:
http://www.systemdynamics.org/conferences/2007/.../SALEH425.pdf
accessed on 20/04/2012

[27] L. Edelstein-Keshet, *Mathematical Models in Biology*, New York, Soc. for Ind. Maths, 2005.

[28] T. Fay, and J. Greeff, Lion, wildebeest and zebra: a predator-prey model, *Ecol. Model.*, **196**, (2006), 237-244.