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On Optimal Solution of Boundary and Interior Support Points for Solving Quadratic Programming Problems Through Super Convergent Line Series

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Abstract

An investigation into the support points for which optimal solutions can be got through super convergent line series of quadratic programming problems has been done. The line search algorithm was used to achieve all these. Support points from the response surface were classified into boundary and interior support points. Two illustrative examples of quadratic programming problems were solved using boundary and interior support points. It was verified that support points from the boundary of the response surface yielded optimal solutions that compared favorably with the existing solutions of the illustrative examples. But the solution of the support points from interior of the response surface was far from optimal when compared with existing solutions.

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1 Introduction

Over the years a variety of line search algorithms have been developed in locating the local optimizers of response surfaces. Some of the techniques are: the active set and simplex methods which are available for solving linear programming problems, see for example, [1] and the Wolfe algorithm for solving quadratic programming problems, [2]. Other classical line search algorithms are: the methods of steepest ascent, the Newton's method and the conjugate direction method, see for example, [3]. Recent line search algorithms include line search algorithms for solving large scale unconstrained optimization problems, [4], line search algorithm based on the Majorize minimum principle, [5], a one dimensional search algorithm for solving general high-dimensional optimization problems that uses line search algorithm as sub routine, [6] etc. Another important study on line search algorithm called Super Convergent Line Series (SCLS) is widely discussed by [7], [8] and [9]. Here we shall investigate which of the support points of the boundary and interior that will give optimal solutions of quadratic programming problems through Super Convergent Line Series (SCLS).

2 Preliminaries

We shall define and discuss basic concepts in Super Convergent Line Series.

2.1 Algorithm for Super Convergent Line Series

The line search algorithm called Super Convergent Line Series (SCLS) is a powerful tool for solving different optimization problems that are encountered in such areas as design of experiment with emphasis on incomplete blocking, Mathematical Programming, Stochastic Programming, etc.The line search algorithm, which is built around the concept of Super Convergence have several points of departure from the classical, gradient –based line series. Of course, these gradient-based series do often times fail to converge to the optimum but the Super Convergent Line Series (SCLS) which incidentally are also gradient –based techniques locate the global optimum of response surfaces with certainty, [10].

The algorithm is defined by the following sequence of steps:

(a) Select N_s support points from the kth boundary or interior of the response surface,

Hence make up an N-point design. $\zeta_{\mathrm{N}} = \{X_1, X_2, ..., \underline{X}_n, ..., \underline{X}_N\}, N = \sum_{K}^{s} N_K$

(b) Compute the vectors $\underline{X}^*, \underline{d}^*$ and ρ^* where $\bar{X}^*, \underline{d}^*$ and ρ^* are the optimal starting point, direction vector and optimal step length, respectively.

(c) Move to the point $\underline{X}^* = \overline{\underline{X}}^* - \rho^* \underline{d}^*$

(d) Is $X^* = \underline{X}_f^*$, where $\underline{X}_f^*$ is the maximize of $f(.)$.

(e) Is $N_K \geq n+1 \, \forall \, k = 1, ..., S$?

Yes: go to step (b) above

No: take extra support points so that $N_k \geq$ n+1 and go to step (b) above.

2.2 Mean Square error Matrix

When $f(x)$ is of the regression function, the Mean Square Error (MSE) matrix is used to obtain the matrix, H_k, of the coefficients of convex combination.

Therefore, the mean square error matrix, is defined by

$$\overline{M}(\underline{C}_k) = M^{-1}(\zeta_{nk}) + M^{-1}(\zeta_{nk})X_k{}'X_{BK}\,\underline{g}_2\,\underline{g}_2{}'X_{BK}{}'X_K M^{-1}(\zeta_{nk})$$

$$= M^{-1}(\zeta_{nk}) + \underline{b}_k\underline{b}_K{}'$$

where

$$M^{-1}(\zeta_{nk}) = \left(X_k{}'X_k\right)^{-1},$$

X_k is the design matrix,

$$\underline{b}_k = M^{-1}(\zeta_{nk})\,X_k{}'X_{BK}\,\underline{g}_2,$$

X_{BK} is the coefficients matrix for the biasing effects, and g_2 is the vector of biasing effects.

Therefore, the mean square error matrix with ith row and jth column is given by

$$M(\underline{C}_k) = \begin{bmatrix} \overline{M}(k_{11}) & \overline{M}(k_{12}) & \dots & \overline{M}(k_{in}) \\ \overline{M}(k_{21}) & \overline{M}(k_{22}) & \dots & \overline{M}(k_{2n}) \\ \vdots & \vdots & & \\ \overline{M}(k_{n1}) & \overline{M}(k_{n2}) & \dots & \overline{M}(k_{nn}) \end{bmatrix}$$

for $i, j, = 1, 2, \dots, n$.

The diagonal elements, $\overline{M}(k_{ii})$ or $\overline{M}(k_{nn})$, are the mean square errors while $\overline{M}(k_{1n})$ and $\overline{M}(k_{n1})$ are the off – diagonal elements.

2.3 The Average information matrix and the direction vector

The average information matrix, $M(\zeta_n)$, is the sum of the product of the k information matrices, and the k matrices of the coefficient of convex combinations given by $M(\zeta_n) = \sum_{k=1}^{s} H_k X_k{}' X_k H_k{}'$.

In vector form, $M(\zeta_n) = H\underline{X}^{/}\underline{X}H^{/} \quad HH^{/} = 1$

$$\underline{X}^{/}\underline{X} = diag \quad \{X_1{}^{/}X_1, X_2{}^{/}X_2, ..., X_S{}^{/}X_S\}$$

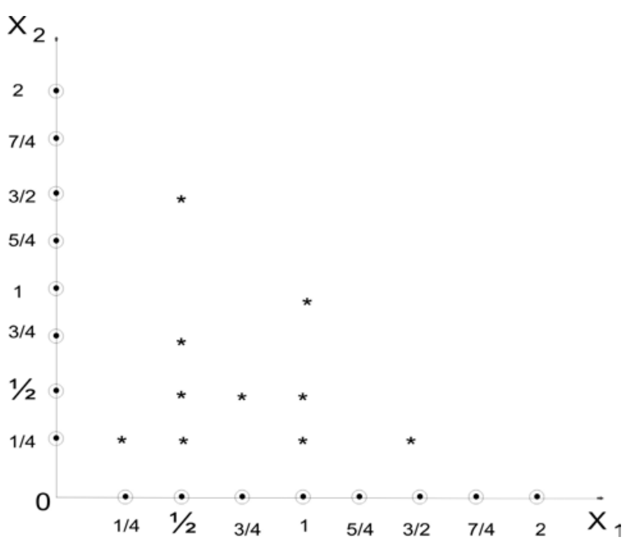
$$= \begin{pmatrix} X_1^{/}X_1 & 0 & \dots & 0 \\ 0 & X_2^{/}X_2 & \vdots & \vdots \\ \vdots & . & & \\ 0 & . & & X_S{}^{/}X_S \end{pmatrix}$$

The direction vector is given by $\hat{d} = M^{-1}(\zeta_n)\cancel{Z}(.)$

where $Z(.) = (\cancel{Z}_0, \cancel{Z}_1, ..., \cancel{Z}_n)$ is an n component vector of responses; $Z_i = f(M_i)$,m_i is the ith row of the information matrix $M(\zeta_n)$.

3 Results and Discussion

3.1 Response Surface (Experimental Area)



The selection of support points for both boundary and interior are governed by the inequalities below.

Boundary support points (illustrative Example 1)

(a) $\left\{X_1, X_2 : 0 \leq X_1, \leq \frac{1}{2}, \quad 0 \leq X_2 \leq \frac{7}{4}\right\}$ for design X_1

(b) $\left\{X_1, X_2 : 0 \leq X_1, \leq \frac{3}{2}, \quad 0 \leq X_2 \leq \frac{7}{4}\right\}$ for design X_2

illustrative Example 2

(a) $\left\{X_1, X_2 : 0 \leq X_1, \leq \frac{3}{4}, \quad 0 \leq X_2 \leq 1\right\}$ for design X_1

(b) $\left\{X_1, X_2 : 0 \leq X_1, \leq \frac{5}{4}, \quad 0 \leq X_2 \leq \frac{3}{4}\right\}$ for design X_2

Interior support points (illustrative Example 1)

(a) $\left\{X_1, X_2 : \frac{1}{2} \leq X_1 \leq \frac{3}{2}, \quad \frac{1}{4} \leq X_2 \leq \frac{3}{4}\right\}$ for design X_1

(b) $\left\{X_1, X_2 : \frac{1}{4} \leq X_1 \leq 1, \quad \frac{1}{4} \leq X_2 \leq 1\right\}$ for design X_2

illustrative Example 2

(a) $\left\{X_1, X_2 : \frac{1}{2} \leq X_1 \leq \frac{3}{2}, \quad \frac{1}{4} \leq X_2 \leq \frac{3}{4}\right\}$ for design X_1

(b) $\left\{X_1, X_2 : \frac{1}{4} \leq X_1 \leq 1, \quad \frac{1}{4} \leq X_2 \leq \frac{1}{2}\right\}$ for design X_2

Note: Points with circle at the boundary of the response surface are the *boundary support points*, while points with asterisks in the interior of the response surface are *interior support points*.

3.2 Illustrative examples

Example 1 ([3], chapter 13, p 601)

Maximize $f(X_1, X_2) = 5X_1 - X_1^{\,2} + 8X_2 - 2X_2^{\,2}$

S.t. $3X_1 + 2X_2 \le 6$

$X_1, X_2 \ge 0$

- Example 1 using boundary support points.

The design and bias matrices are:

$$X_1 = \begin{pmatrix} 1 & 0 & \frac{7}{4} \\ 1 & 0 & \frac{3}{2} \\ 1 & 0 & \frac{3}{4} \\ 1 & \frac{1}{2} & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & \frac{1}{4} \\ 1 & 0 & \frac{7}{4} \\ 1 & \frac{3}{2} & 0 \end{pmatrix}, \quad X_{1B} = \begin{pmatrix} 0 & \frac{49}{16} \\ 0 & \frac{9}{4} \\ 0 & \frac{9}{16} \\ \frac{1}{4} & 0 \end{pmatrix}, \quad X_{2B} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{16} \\ 0 & \frac{49}{16} \\ \frac{9}{4} & 0 \end{pmatrix}.$$

The vector of the biasing parameters is $g_2 = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

The mean square error matrices are:

$$\overline{M}_1 = \begin{pmatrix} 14720 & & SYM \\ 724 & 55 & \\ 18872 & 943 & 24209 \end{pmatrix},$$

$$\overline{M}_2 = \begin{pmatrix} 2.1297 & & SYM \\ -2.8973 & 5.3170 & \\ -4.5805 & 8.9601 & 17.1443 \end{pmatrix}$$

Where **“** *SYM* **”** means that the mean square error matrices $\overline{M}_1$ and $\overline{M}_2$ are symmetric matrices.

The matrices of coefficient of convex combination of the means square error matrices are

$$H_1 = diag \quad \{0.0001 \quad 0.0880 \quad 0.0007\},$$

$$H_2 = diag \quad \{0.9999 \quad 0.9120 \quad 0.9993\}.$$

These matrices are normalized to give:

$$H_1^* = diag \quad \{1.0001 \quad 0.0960 \quad 0.0007\}$$

$$H_2^* = diag \quad \{1.0000 \quad 0.9954 \quad 1.0000\}$$

The direction vector,

$$\underline{d} = \begin{pmatrix} 0.7379 \\ 0.9035 \end{pmatrix}.$$

This is normalized to give

$$\underline{d}^* = \begin{pmatrix} 0.6326 \\ 0.7745 \end{pmatrix}, \quad \overline{\underline{X}}^* = \begin{pmatrix} 0.3107 \\ 0.6224 \end{pmatrix}$$

The step length, $\rho^* = -1.1092$

$$X^* = \begin{pmatrix} 1.01 \\ 1.48 \end{pmatrix}; Max\ f(x) = 11.5$$

This value is approximately the same as the value got by [3], Chapter 13, p 601, which is $Maxf(x) = 11.5$, for $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1.0 \\ 1.5 \end{pmatrix}$.

- Example 1 using interior support points.

The design and bias matrices are:

$$X_1 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{3}{2} & \frac{1}{4} \\ 1 & \frac{1}{2} & \frac{3}{4} \\ 1 & 1 & \frac{1}{2} \end{pmatrix}, \ X_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{4} & \frac{1}{4} \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & \frac{3}{4} & \frac{1}{2} \end{pmatrix}, \ X_{1B} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{9}{4} & \frac{1}{16} \\ \frac{1}{4} & \frac{9}{16} \\ 1 & \frac{1}{4} \end{pmatrix}, X_{2B} = \begin{pmatrix} 1 & 1 \\ \frac{1}{16} & \frac{1}{16} \\ \frac{1}{4} & \frac{1}{16} \\ \frac{9}{16} & \frac{1}{4} \end{pmatrix}.$$

The vector of the biasing parameters is $g_2 = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

The mean square error matrices are:

$$\overline{M}_1 = \begin{pmatrix} 22.5625 & & SYM \\ -12.5 & 9.3333 & \\ -26.5 & 14.6667 & 33.3333 \end{pmatrix},$$

$$\overline{M}_2 = \begin{pmatrix} 2.2227 & & SYM \\ -4.1031 & 19.2225 & \\ -0.4063 & -15.4750 & 28.2500 \end{pmatrix}.$$

The matrices of coefficient of convex combination of the mean square error matrices are:

$$H_1 = diag \quad \{0.0897 \quad 0.6732 \quad 0.4587\}$$

$$H_2 = diag \quad \{0.9103 \quad 0.3268 \quad 0.5413\}.$$

These matrices are normalized to give

$$H_1^* = diag \quad \{0.0981 \quad 0.8996 \quad 0.6465\}$$

$$H_2^* = diag \quad \{0.9952 \quad 0.4367 \quad 0.7629\}.$$

The direction vector,

$$d = \begin{pmatrix} -0.5567 \\ 12.0594 \end{pmatrix}.$$

This is normalized to give

$$d^* = \begin{pmatrix} -0.0461 \\ 0.9989 \end{pmatrix}.$$

The optimal starting point,

$$\underline{\overline{X}}^* = \begin{pmatrix} 0.6350 \\ 0.4594 \end{pmatrix}.$$

The step length, $\rho^* = -1.7081$,

$$X = \begin{pmatrix} 0.56 \\ 2.17 \end{pmatrix} \quad \text{and} \quad Max\ f(x) = 10.42\,.$$

This value is not optimal and does not compare favorably with existing solution got by [3].

Example 2 ([1], chapter 19, p 795).

Maximize $\cancel{Z} = 4X_1 + 6X_2 - 2X_1{}^2 - 2X_1X_2 - 2x_2^2$

S. t $X_1 + 2X_2 \leq 2$

$X_1, X_2 \geq 0$

- Example 2 using boundary support points.

The design and bias matrices are:

$$X_1 = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{3}{4} \\ 1 & 0 & 1 \\ 1 & \frac{3}{4} & 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 1 & \frac{5}{4} & 0 \\ 1 & 0 & \frac{3}{4} \\ 1 & 0 & \frac{1}{2} \\ 1 & \frac{1}{8} & 0 \end{pmatrix},\ X_{1B} = \begin{pmatrix} 0 & 0 & \frac{1}{4} \\ 0 & 0 & \frac{9}{16} \\ 0 & 0 & 1 \\ 0 & \frac{9}{16} & 0 \end{pmatrix},\ X_{2B} = \begin{pmatrix} 0 & \frac{25}{16} & 0 \\ 0 & 0 & \frac{9}{16} \\ 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{64} & 0 \end{pmatrix}$$

The vector of the biasing parameters is $g_2 = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

The mean square error Matrices are:

$$\overline{M}_1 = \begin{pmatrix} 5.9184 & & SYM \\ -9.4537 & 18.7160 & \\ -9.1250 & 16.6667 & 17 \end{pmatrix}$$

$$\overline{M}_2 = \begin{pmatrix} 1.2608 & & SYM \\ -1.9677 & 9.2445 & \\ -2.4120 & 6.7741 & 7.4801 \end{pmatrix}$$

The matrices of coefficient of convex combination of the mean square error matrices are:

$$H_1 = diag \quad \{0.1756 \quad 0.3306 \quad 0.3056\}$$

$$H_2 = diag \quad \{0.8244 \quad 0.6694 \quad 0.6944\}.$$

These matrices are normalized to give

$$H_1^* = diag \quad \{0.2083 \quad 0.4428 \quad 0.4028\}$$

$$H_2^* = diag \quad \{0.9781 \quad 0.8966 \quad 0.9152\}$$

The direction vector,

$$d = \begin{pmatrix} 4.6583 \\ 9.0081 \end{pmatrix}$$

This is normalized to give

$$\underline{d}^r = \begin{pmatrix} 0.4593 \\ 0.8883 \end{pmatrix}.$$

The optimal starting point,

$$\underline{\overline{X}} = \begin{pmatrix} 0.2022 \\ 0.4191 \end{pmatrix}.$$

The step -length, $\rho^* = -0.4292$

$$\underline{X}^t = \begin{pmatrix} 0.399 \\ 0.800 \end{pmatrix}, \quad Max.\cancel{Z} = 4.16$$

This value is very close to the value got by [1], chapter 19, p 795, which is

$Max\ \cancel{Z} = 4.16$, for $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.333 \\ 0.833 \end{pmatrix}$

- Example 2 using interior support points**.**

The design and bias matrices are:

$$X_1 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{3}{2} & \frac{1}{4} \\ 1 & \frac{1}{2} & \frac{3}{4} \\ 1 & 1 & \frac{1}{2} \end{pmatrix}, \ X_2 = \begin{pmatrix} 1 & 1 & \frac{1}{4} \\ 1 & \frac{1}{4} & \frac{1}{4} \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & \frac{3}{4} & \frac{1}{2} \end{pmatrix}, \ X_{1B} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{8} & \frac{9}{4} & \frac{1}{16} \\ \frac{3}{8} & \frac{1}{4} & \frac{9}{4} \\ \frac{1}{2} & 1 & \frac{1}{4} \end{pmatrix}, \ X_{2B} = \begin{pmatrix} \frac{1}{4} & 1 & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{16} \\ \frac{3}{8} & \frac{9}{16} & \frac{1}{4} \end{pmatrix}$$

The vector of the biasing parameters is $g_2 = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}$.

$$\overline{M}_1 = \begin{pmatrix} 102 & & SYM \\ -70.75 & 50.8958 & \\ -152.25 & 106.8542 & 232.395 \end{pmatrix}$$

$$\overline{M}_2 = \begin{pmatrix} 4.0765 & & SYM \\ -4.7194 & 12.8622 & \\ -8.1633 & 4.7347 & 28.0816 \end{pmatrix}.$$

The matrices of coefficient of convex combination of the mean square error matrices are:

$$H_1 = diag \quad \{0.0384 \quad 0.2017 \quad 0.1078\}$$

$$H_2 = diag \quad \{0.9616 \quad 0.7983 \quad 0.8922\}.$$

These matrices are normalized to give

$$H_1^* = diag \quad \{0.0981 \quad 0.8996 \quad 0.6465\}$$

$$H_2^* = diag \quad \{0.9952 \quad 0.4367 \quad 0.7629\}.$$

The direction vector,

$$d = \begin{pmatrix} -4.9480 \\ 88.0971 \end{pmatrix}.$$

This is normalized to give

$$d^* = \begin{pmatrix} -0.0561 \\ 0.9984 \end{pmatrix}.$$

The optimal starting point,

$$\underline{X}^* = \begin{pmatrix} 0.6468 \\ 0.3989 \end{pmatrix}.$$

The step length, $\rho^* = -0.2862$,

$$X^* = \begin{pmatrix} 0.66 \\ 0.11 \end{pmatrix} \text{ and Max. } Z = 2.26$$

This value is not optimal and does not compare favorably with existing solution got by [1].

4 Conclusion

From the foregoing, it is evident that the solution of quadratic programming problems using boundary supports points compared favorably with existing solution given by [3] and [1]. But the solution using interior support points did not yield optimal solution and did not also compared favorably with existing solutions.

It is therefore advisable to use support points from the boundary of the response surface to solve quadratic programming problems.

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